Normal forms and functional completeness for four-valued languages

Vincent Degauquier

Abstract

Based on the semantic concepts developed by M. Dunn and N. Belnap, a four-valued language containing only two logical symbols is proposed. We show that this language is functionally complete with regard to the given semantics. Specifically, we prove that every truth-function is expressed by a formula of the language. To do this, we define two concepts akin to the disjunctive and conjunctive normal forms. Using these concepts, we establish that every truth-function for a four-valued semantics can be represented by a formula in a disjunctive form or in a conjunctive form.

1 Introduction

The functional completeness of four-valued languages is a well known topic that has been extensively discussed. However, it appears that some results have yet to be shown. In particular, the fact that two logical connectives are enough to define a functionally complete four-valued language does not seem to have been proven.

Using the semantic concepts developed by M. Dunn (see [3]) and N. Belnap (see [2]), this article intends to show that a four-valued language containing only one unary logical symbol and one binary logical symbol is expressive enough to represent any truth-function. In other words, we show that such a language is functionally complete with regard to the four-valued semantics described hereafter.

Received by the editors in February 2015 - In revised form in April 2015.
Communicated by F. Point.
2010 *Mathematics Subject Classification* : Primary 03B50 ; Secondary 03A99, 03B53, 03G10.
Key words and phrases : four-valued logic, functional completeness, disjunctive normal form, conjunctive normal form.

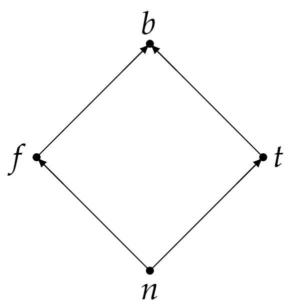
Although the functional completeness of the set of connectives that we have identified can be verified on the basis of previous results, especially those obtained by R. Muskens (see [5]) and A. Avron (see [1]), we provide here an original and self-contained proof of this result. This proof consists in showing that any function defined on a set of four values, regarded as subsets of the set of classical truth-values (see [6]), can be represented by a formula having a structure akin to the disjunctive and conjunctive normal forms.

In short, the aim of this paper is twofold. It consists, firstly, in showing that two logical connectives are enough to define a functionally complete four-valued language and, secondly, in proposing a disjunctive normal form and a conjunctive normal form suitable for the four-valued semantics.

2 Language and semantics

A *language* $\mathcal{L}$ is composed of a countable set of propositional symbols plus the unary logical symbol $-$ and the binary logical symbol $\ominus$. As for the syntax, the concept of formula is defined inductively in the usual way.

Four values are identified using the two classical truth-values 1 and 0, where 1 denotes the truth-value truth and 0 denotes the truth-value falsehood. Assuming that truth and falsehood are neither exhaustive nor exclusive, four values are defined in terms of sets of truth-values: $b = \{0, 1\}$ (which stands for 'both true and false'), $t = \{1\}$ (which stands for 'true'), $f = \{0\}$ (which stands for 'false'), and $n = \emptyset$ (which stands for 'neither true nor false'). Also, the structure of these values is given by the lattice formed by the power set of $\{0, 1\}$ under the partial ordering by inclusion.



A *model* $\mathcal{M}$ for a language $\mathcal{L}$ is a function from the set of propositional symbols of $\mathcal{L}$ to the power set of $\{0, 1\}$. The value assigned to a propositional symbol P in the model $\mathcal{M}$ is denoted by $\mathcal{M}[P]$. This definition is extended inductively to all formulas of the language as follows:

$1 \in \mathcal{M}[-A]$ if and only if $0 \notin \mathcal{M}[A]$
$0 \in \mathcal{M}[-A]$ if and only if $1 \notin \mathcal{M}[A]$
$1 \in \mathcal{M}[(A \ominus B)]$ if and only if $1 \in \mathcal{M}[A]$ or $1 \in \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \ominus B)]$ if and only if $0 \notin \mathcal{M}[A]$ and $0 \notin \mathcal{M}[B]$

The unary logical symbol $-$ is interpreted as a dualisation connective that inverts truth and non-falsehood on the one hand and falsehood and non-truth on the other hand. In this way, this logical connective corresponding to what M. Fitting calls conflation (see [4]) highlights the symmetry between the approaches dealing with truth-value gaps and those dealing with truth-value gluts. As for the binary logical symbol, it will be called alternative disjunction and denoted $\ominus$. Its semantic interpretation can be regarded as a combination of two well-known classical connectives, namely the inclusive disjunction (with regard to the definition of truth) and the Sheffer stroke (with regard to the definition of falsehood). The truth tables for $-$ and $\ominus$ are as follows:

$-$	
t	t
f	f
n	b
b	n

$\ominus$	t	f	n	b
t	b	t	b	t
f	t	n	n	t
n	b	n	f	t
b	t	t	t	t

To simplify the notation, some abbreviations are introduced. Through the semantic interpretation of the logical symbols, it is easy to verify the following properties for these abbreviations:

$$\neg A \quad =_{def} \quad (-(A \ominus A) \ominus -(A \ominus A))$$

$1 \in \mathcal{M}[\neg A]$ if and only if $0 \in \mathcal{M}[A]$
$0 \in \mathcal{M}[\neg A]$ if and only if $1 \in \mathcal{M}[A]$

$$\triangleleft A \quad =_{def} \quad (A \ominus A)$$

$1 \in \mathcal{M}[\triangleleft A]$ if and only if $1 \in \mathcal{M}[A]$
$0 \in \mathcal{M}[\triangleleft A]$ if and only if $0 \notin \mathcal{M}[A]$

$$\triangleright A \quad =_{def} \quad -(- A \ominus - A)$$

$1 \in \mathcal{M}[\triangleright A]$ if and only if $1 \notin \mathcal{M}[A]$
$0 \in \mathcal{M}[\triangleright A]$ if and only if $0 \in \mathcal{M}[A]$

$$(A \vee B) \quad =_{def} \quad ((A \ominus A) \ominus (B \ominus B))$$

$1 \in \mathcal{M}[(A \vee B)]$ if and only if $1 \in \mathcal{M}[A]$ or $1 \in \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \vee B)]$ if and only if $0 \in \mathcal{M}[A]$ and $0 \in \mathcal{M}[B]$

$$(A \wedge B) \quad =_{def} \quad \neg(\neg A \vee \neg B)$$

$1 \in \mathcal{M}[(A \wedge B)]$ if and only if $1 \in \mathcal{M}[A]$ and $1 \in \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \wedge B)]$ if and only if $0 \in \mathcal{M}[A]$ or $0 \in \mathcal{M}[B]$

$$\circ A \quad =_{def} \quad (A \wedge - A)$$

$1 \in \mathcal{M}[\circ A]$ if and only if $1 \in \mathcal{M}[A]$ and $0 \notin \mathcal{M}[A]$
$0 \in \mathcal{M}[\circ A]$ if and only if $1 \notin \mathcal{M}[A]$ or $0 \in \mathcal{M}[A]$

$$\bullet A \quad =_{def} \quad (A \vee - A)$$

$1 \in \mathcal{M}[\bullet A]$ if and only if $1 \in \mathcal{M}[A]$ or $0 \notin \mathcal{M}[A]$
$0 \in \mathcal{M}[\bullet A]$ if and only if $1 \notin \mathcal{M}[A]$ and $0 \in \mathcal{M}[A]$

$$(A \mathbin{@} B) \quad =_{def} \quad ((A \vee \lhd A) \wedge (B \vee \rhd B))$$

$1 \in \mathcal{M}[(A \mathbin{@} B)]$ if and only if $1 \in \mathcal{M}[A]$
$0 \in \mathcal{M}[(A \mathbin{@} B)]$ if and only if $0 \in \mathcal{M}[B]$

These properties are reflected in the following truth tables:

$\neg$	
t	f
f	t
n	n
b	b

$\lhd$	
t	b
f	n
n	f
b	t

$\rhd$	
t	n
f	b
n	t
b	f

$\circ$	
t	t
f	f
n	f
b	f

$\bullet$	
t	t
f	f
n	t
b	t

$\vee$	t	f	n	b
t	t	t	t	t
f	t	f	n	b
n	t	n	n	t
b	t	b	t	b

$\wedge$	t	f	n	b
t	t	f	n	b
f	f	f	f	f
n	n	f	n	f
b	b	f	f	b

$@$	t	f	n	b
t	t	b	t	b
f	n	f	n	f
n	n	f	n	f
b	t	b	t	b

The abbreviations denoted by $\neg A$, $(A \vee B)$, and $(A \wedge B)$ correspond respectively to the notions of negation, disjunction, and conjunction as defined in Dunn-Belnap's four-valued logic (see [3] and [2]). Abbreviations $\lhd A$ and $\rhd A$ can be thought of as reflecting a swap between what N. Belnap calls the approximation lattice and the logical lattice (see [2]). The expression $\lhd A$ is both true and false if and only if A is (only) true, and it is neither true nor false if and only if A is (only) false. On the other hand, the expression $\rhd A$ is neither true nor false if and only if A is (only) true, and it is both true and false if and only if A is (only) false. As for the abbreviations $\circ A$ and $\bullet A$, their value is always either t or f. More precisely, $\circ A$ is assigned the value t if A is both true and not false (one could say 'when A is strongly acceptable'), otherwise its value is f. Symmetrically, $\bullet A$ is assigned the value t if A is either true or not false (one could say 'when A is weakly acceptable'), otherwise its value is f. Finally, the last abbreviation $(A \mathbin{@} B)$ is particularly useful and plays a crucial role in the definition of the adapted disjunctive and conjunctive normal forms (see [5] for a similar use). Indeed, the truth of this expression is only determined by the truth of A and its falsehood is only determined by the falsehood of B. In this way, this abbreviation will allow us to deal with the truth and falsehood of a formula separately.

3 Functional completeness

Let $g : \{t, f, n, b\}^n \to \{t, f, n, b\}$ be an n-ary truth-function and let A be a formula of a language $\mathcal{L}$ in which exactly the propositional symbols $P_1, \ldots, P_n$ occur. Then, g is *expressed by A* if $g(\mathcal{M}[P_1], \ldots, \mathcal{M}[P_n]) = \mathcal{M}[A]$, for every model $\mathcal{M}$ for $\mathcal{L}$. A language $\mathcal{L}$ is *functionally complete* if every n-ary truth-function is expressed by a formula of $\mathcal{L}$.

Theorem 1. *For every n-ary truth-function g, there is a formula in a disjunctive normal form A^δ of $\mathcal{L}$ such that g is expressed by A^δ.*

Proof. Let $P_1, \ldots, P_n$ be a finite sequence of distinct propositional symbols of a language $\mathcal{L}$ and let $\mathcal{L}(P_1, \ldots, P_n)$ denote the sublanguage of $\mathcal{L}$ whose propositional symbols are exactly those occurring in the sequence. Also, let $g : \{t, f, n, b\}^n \to \{t, f, n, b\}$ be an n-ary truth-function.

The proof amounts to showing that there exists a formula A^δ of $\mathcal{L}$ such that P_1, $\ldots$, P_n are exactly the atomic formulas occurring in A^δ and such that $\mathcal{M}[A^\delta] = g(\mathcal{M}[P_1], \ldots, \mathcal{M}[P_n])$, for every model $\mathcal{M}$ for $\mathcal{L}(P_1, \ldots, P_n)$. To do this, let us define such a formula A^δ and show that it satisfies the desired property.

Given that there are 4^n models for $\mathcal{L}(P_1, \ldots, P_n)$, denoted $\mathcal{M}_1, \ldots, \mathcal{M}_{4^n}$, the formula A^δ is defined as follows. For every i such that $1 \leq i \leq 4^n$, let C_i be the formula $\circ(P_1^i \wedge \cdots \wedge P_n^i)$ where, for all j such that $1 \leq j \leq n$:

P_j^i is the formula P_j if and only if $\mathcal{M}_i[P_j] = t$
P_j^i is the formula $\neg P_j$ if and only if $\mathcal{M}_i[P_j] = f$
P_j^i is the formula $\rhd P_j$ if and only if $\mathcal{M}_i[P_j] = n$
P_j^i is the formula $\lhd P_j$ if and only if $\mathcal{M}_i[P_j] = b$

The formula A^δ is defined as the formula $(D_1 \mathbin{@} D_2)$ where D_1 is the disjunction of the C_i formulas such that $1 \in g(\mathcal{M}_i[P_1], \ldots, \mathcal{M}_i[P_n])$ (i.e. such that $g(\mathcal{M}_i[P_1], \ldots, \mathcal{M}_i[P_n])$ equals either t or b) and where D_2 is the disjunction of the C_i formulas such that $0 \notin g(\mathcal{M}_i[P_1], \ldots, \mathcal{M}_i[P_n])$ (i.e. such that $g(\mathcal{M}_i[P_1], \ldots, \mathcal{M}_i[P_n])$ equals either t or n), for all i such that $1 \leq i \leq 4^n$.

If there is no model $\mathcal{M}$ for $\mathcal{L}(P_1, \ldots, P_n)$ such that $g(\mathcal{M}[P_1], \ldots, \mathcal{M}[P_n])$ equals either t or b, then D_1 is equated to the formula $\circ(P_1 \wedge \cdots \wedge P_n \wedge \neg P_n)$, and if there is no model $\mathcal{M}$ for $\mathcal{L}(P_1, \ldots, P_n)$ such that $g(\mathcal{M}[P_1], \ldots, \mathcal{M}[P_n])$ equals either t or n, then D_2 is equated to the formula $\circ(P_1 \wedge \cdots \wedge P_n \wedge \neg P_n)$.

To verify that the truth-function g is indeed expressed by the formula A^δ, we first show that $1 \in \mathcal{M}_i[C_k]$ if and only if $i = k$ and $0 \in \mathcal{M}_i[C_k]$ if and only if $i \neq k$, for all i and k such that $1 \leq i, k \leq 4^n$.

Suppose that $i = k$. Then, by the definition of the formula C_k, $1 \in \mathcal{M}_i[P_j^k]$ and $0 \notin \mathcal{M}_i[P_j^k]$, for all j such that $1 \leq j \leq n$. From the semantic definition of the logical symbols, it follows that $1 \in \mathcal{M}_i[C_k]$ and $0 \notin \mathcal{M}_i[C_k]$.

Suppose that $i \neq k$. Then, the formulas C_k and C_i differ in that there is a j such that $1 \leq j \leq n$ and P_j^i is distinct from P_j^k. By inspecting the possible cases, it can be shown that $1 \notin \mathcal{M}_i[C_k]$ and $0 \in \mathcal{M}_i[C_k]$.

We now prove that, for all i such that $1 \leq i \leq 4^n$, $1 \in \mathcal{M}_i[D_1]$ if and only if $g(\mathcal{M}_i[P_1], \dots, \mathcal{M}_i[P_n])$ equals either t or b. A similar argument can be made to establish that $0 \in \mathcal{M}_i[D_2]$ if and only if $g(\mathcal{M}_i[P_1], \dots, \mathcal{M}_i[P_n])$ equals either f or b.

Suppose that $\mathcal{M}_i$ is a model such that $g(\mathcal{M}_i[P_1], \dots, \mathcal{M}_i[P_n])$ equals either t or b. Then, by the definition of D_1, the formula C_i is one of the disjuncts of D_1. From the foregoing result, it follows that $1 \in \mathcal{M}_i[C_i]$. Therefore, by the semantic definition of the logical symbols, $1 \in \mathcal{M}_i[D_1]$.

Suppose that $\mathcal{M}_i$ is a model such that $1 \in \mathcal{M}_i[D_1]$. Then, by the definition of the formula D_1 and the semantic definition of the logical symbols, there is a formula C_k such that C_k is one of the disjuncts of D_1 and $1 \in \mathcal{M}_i[C_k]$, where $1 \leq k \leq 4^n$. From the foregoing result, it follows that $i = k$. In other words, this means that C_i is a disjunct of D_1. Hence, by the definition of D_1, $g(\mathcal{M}_i[P_1], \dots, \mathcal{M}_i[P_n])$ equals either t or b.

Finally, we conclude the proof by observing that, for every model $\mathcal{M}$ for $\mathcal{L}(P_1, \dots, P_n)$, $1 \in g(\mathcal{M}[P_1], \dots, \mathcal{M}[P_n])$ if and only if $1 \in \mathcal{M}[D_1]$ if and only if $1 \in \mathcal{M}[A^\delta]$ on the one hand, and $0 \in g(\mathcal{M}[P_1], \dots, \mathcal{M}[P_n])$ if and only if $0 \in \mathcal{M}[D_2]$ if and only if $0 \in \mathcal{M}[A^\delta]$ on the other hand. ■

Theorem 2. *For every n-ary truth-function g, there is a formula in a conjunctive normal form A^γ of $\mathcal{L}$ such that g is expressed by A^γ.*

Proof. This proof is very similar to that of Theorem 1. Let $\mathcal{L}$ be a language and g be an n-ary truth-function such that $g : \{t, f, n, b\}^n \to \{t, f, n, b\}$. Given that there are 4^n models for $\mathcal{L}(P_1, \dots, P_n)$, denoted $\mathcal{M}_1, \dots, \mathcal{M}_{4^n}$, the formula A^γ is defined as follows. For every i such that $1 \leq i \leq 4^n$, let D_i be the formula $\bullet(P_1^i \vee \cdots \vee P_n^i)$ where, for all j such that $1 \leq j \leq n$:

P_j^i is the formula P_j if and only if $\mathcal{M}_i[P_j] = f$
P_j^i is the formula $\neg P_j$ if and only if $\mathcal{M}_i[P_j] = t$
P_j^i is the formula $\triangleright P_j$ if and only if $\mathcal{M}_i[P_j] = b$
P_j^i is the formula $\triangleleft P_j$ if and only if $\mathcal{M}_i[P_j] = n$

The formula A^γ is then defined as the formula $(C_1 \mathbin{@} C_2)$ where C_1 is the conjunction of the D_i formulas such that $1 \notin g(\mathcal{M}_i[P_1], \dots, \mathcal{M}_i[P_n])$ (i.e. such that $g(\mathcal{M}_i[P_1], \dots, \mathcal{M}_i[P_n])$ equals either f or n) and where C_2 is the conjunction of the D_i formulas such that $0 \in g(\mathcal{M}_i[P_1], \dots, \mathcal{M}_i[P_n])$ (i.e. such that $g(\mathcal{M}_i[P_1], \dots, \mathcal{M}_i[P_n])$ equals either f or b), for all i such that $1 \leq i \leq 4^n$.

If there is no model $\mathcal{M}$ for $\mathcal{L}(P_1, \dots, P_n)$ such that $g(\mathcal{M}[P_1], \dots, \mathcal{M}[P_n])$ equals either f or n, then C_1 is equated to the formula $\bullet(P_1 \vee \cdots \vee P_n \vee \neg P_n)$, and if there is no model $\mathcal{M}$ for $\mathcal{L}(P_1, \dots, P_n)$ such that $g(\mathcal{M}[P_1], \dots, \mathcal{M}[P_n])$ equals either f or b, then C_2 is equated to the formula $\bullet(P_1 \vee \cdots \vee P_n \vee \neg P_n)$.

It can be verified that the truth-function g is expressed by the formula A^γ as follows. We first show that $1 \in \mathcal{M}_i[D_k]$ if and only if $i \neq k$ and $0 \in \mathcal{M}_i[D_k]$ if and only if $i = k$, for all i and k such that $1 \leq i, k \leq 4^n$. Then, we prove that, for all i such that $1 \leq i \leq 4^n$, $1 \in \mathcal{M}_i[C_1]$ if and only if $g(\mathcal{M}_i[P_1], \ldots, \mathcal{M}_i[P_n])$ equals either t or b and that $0 \in \mathcal{M}_i[C_2]$ if and only if $g(\mathcal{M}_i[P_1], \ldots, \mathcal{M}_i[P_n])$ equals either f or b. The desired result follows from the definition of A^γ. ■

It is worth noting that Theorem 2 can be deduced from Theorem 1 by using the fact that $\mathcal{M}[\neg(A \mathbin{@} B)] = \mathcal{M}[(\neg B \mathbin{@} \neg A)]$, for every model $\mathcal{M}$ for $\mathcal{L}$. However, such a proof is not as straightforward as in classical logic and provides no real insight into the concept of conjunctive normal form.

4 Sheffer stroke and Peirce arrow

Alternative disjunction actually belongs to a family of eight binary connectives, denoted $\natural_i$ ($1 \leq i \leq 8$), whose semantic interpretation can be obtained by combining those of the inclusive disjunction and the Sheffer stroke, or those of the conjunction and the Peirce arrow.

Let $\mathcal{L}'$ be the extension of a language $\mathcal{L}$ by these logical symbols (where $\natural_3$ is equated with $\ominus$) and the unary logical symbol $\neg$ (which is taken as primitive). The definition of model is extended inductively to all formulas of $\mathcal{L}'$ as follows:

$1 \in \mathcal{M}[(A \natural_1 B)]$ if and only if $1 \in \mathcal{M}[A]$ and $1 \in \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \natural_1 B)]$ if and only if $0 \notin \mathcal{M}[A]$ or $0 \notin \mathcal{M}[B]$
$1 \in \mathcal{M}[(A \natural_2 B)]$ if and only if $0 \in \mathcal{M}[A]$ and $0 \in \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \natural_2 B)]$ if and only if $1 \notin \mathcal{M}[A]$ or $1 \notin \mathcal{M}[B]$
$1 \in \mathcal{M}[(A \natural_3 B)]$ if and only if $1 \in \mathcal{M}[A]$ or $1 \in \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \natural_3 B)]$ if and only if $0 \notin \mathcal{M}[A]$ and $0 \notin \mathcal{M}[B]$
$1 \in \mathcal{M}[(A \natural_4 B)]$ if and only if $0 \in \mathcal{M}[A]$ or $0 \in \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \natural_4 B)]$ if and only if $1 \notin \mathcal{M}[A]$ and $1 \notin \mathcal{M}[B]$
$1 \in \mathcal{M}[(A \natural_5 B)]$ if and only if $1 \notin \mathcal{M}[A]$ and $1 \notin \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \natural_5 B)]$ if and only if $0 \in \mathcal{M}[A]$ or $0 \in \mathcal{M}[B]$
$1 \in \mathcal{M}[(A \natural_6 B)]$ if and only if $0 \notin \mathcal{M}[A]$ and $0 \notin \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \natural_6 B)]$ if and only if $1 \in \mathcal{M}[A]$ or $1 \in \mathcal{M}[B]$
$1 \in \mathcal{M}[(A \natural_7 B)]$ if and only if $1 \notin \mathcal{M}[A]$ or $1 \notin \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \natural_7 B)]$ if and only if $0 \in \mathcal{M}[A]$ and $0 \in \mathcal{M}[B]$
$1 \in \mathcal{M}[(A \natural_8 B)]$ if and only if $0 \notin \mathcal{M}[A]$ or $0 \notin \mathcal{M}[B]$
$0 \in \mathcal{M}[(A \natural_8 B)]$ if and only if $1 \in \mathcal{M}[A]$ and $1 \in \mathcal{M}[B]$

The idea behind the connectives $\natural_3$, $\natural_4$, $\natural_7$, and $\natural_8$ is that they result from combinations of the dual version of the disjunction and the Sheffer stroke, or the disjunction and the dual version of the Sheffer stroke. On the other hand, the connectives $\natural_1$, $\natural_2$, $\natural_5$, and $\natural_6$ can be understood as combinations of the dual version of the conjunction and the Peirce arrow, or the conjunction and the dual version of the Peirce arrow. This view is not only suggested by the semantic definitions but is also supported by the following properties for every model $\mathcal{M}$ for $\mathcal{L}'$:

$\mathcal{M}[(A \natural_1 B)] = \mathcal{M}[((A \wedge B) \mathbin{@} (\neg - A \wedge \neg - B))]$
$\mathcal{M}[(A \natural_2 B)] = \mathcal{M}[((\neg A \wedge \neg B) \mathbin{@} (- A \wedge - B))]$

$$\begin{aligned}
\mathcal{M}[(A \natural_3 B)] &= \mathcal{M}[((A \vee B) @ (\neg - A \vee \neg - B))] \\
\mathcal{M}[(A \natural_4 B)] &= \mathcal{M}[((\neg A \vee \neg B) @ (- A \vee - B))] \\
\mathcal{M}[(A \natural_5 B)] &= \mathcal{M}[((\neg - A \wedge \neg - B) @ (A \wedge B))] \\
\mathcal{M}[(A \natural_6 B)] &= \mathcal{M}[((- A \wedge - B) @ (\neg A \wedge \neg B))] \\
\mathcal{M}[(A \natural_7 B)] &= \mathcal{M}[((\neg - A \vee \neg - B) @ (A \vee B))] \\
\mathcal{M}[(A \natural_8 B)] &= \mathcal{M}[((- A \vee - B) @ (\neg A \vee \neg B))]
\end{aligned}$$

Let S_1 and S_2 be two sets of logical symbols of $\mathcal{L}'$. Let $\mathcal{F}^{S_1}$ and $\mathcal{F}^{S_2}$ be the sets of formulas of $\mathcal{L}'$ such that all their logical symbols belong to S_1 and S_2, respectively. Then, S_1 and S_2 have *the same expressive power* if for every model $\mathcal{M}$ for $\mathcal{L}'$: (1) for every formula A in $\mathcal{F}^{S_1}$ there is a formula B in $\mathcal{F}^{S_2}$ such that $\mathcal{M}[A] = \mathcal{M}[B]$ and (2) for every formula B in $\mathcal{F}^{S_2}$ there is a formula A in $\mathcal{F}^{S_1}$ such that $\mathcal{M}[A] = \mathcal{M}[B]$.

Theorem 3. *Let $\{\natural, *\}$ and $\{\natural', *'\}$ be two sets of logical symbols of $\mathcal{L}'$ where $\natural$ and $\natural'$ belong to $\{\natural_i \mid 1 \leq i \leq 8\}$ and where $*$ and $*'$ belong to $\{\neg, -\}$. Then, $\{\natural, *\}$ and $\{\natural', *'\}$ have the same expressive power.*

Proof. The proof proceeds by induction on the complexity of the formulas. The basis case is trivial and the induction case can be established by means of the following properties. For every model $\mathcal{M}$ for $\mathcal{L}'$ and every i such that $1 \leq i \leq 8$:

1. $\mathcal{M}[(A \natural_i B)] = \mathcal{M}[\neg(A \natural_{9-i} B)]$
2. $\mathcal{M}[(A \natural_i B)] = \mathcal{M}[-(A \natural_{i+1} B)]$, if i is odd
3. $\mathcal{M}[(A \natural_i B)] = \mathcal{M}[-(A \natural_{i-1} B)]$, if i is even
4. $\mathcal{M}[\neg A] = \mathcal{M}[(-(A \natural_i A) \natural_i -(A \natural_i A))]$, if i is odd
5. $\mathcal{M}[\neg A] = \mathcal{M}[-((A \natural_i A) \natural_i (A \natural_i A))]$, if i is even
6. $\mathcal{M}[- A] = \mathcal{M}[(\neg(A \natural_i A) \natural_i \neg(A \natural_i A))]$, if i is odd
7. $\mathcal{M}[- A] = \mathcal{M}[\neg((A \natural_i A) \natural_i (A \natural_i A))]$, if i is even

In order to complete the proof it remains to note that $\mathcal{M}[(A \natural_3 B)] = \mathcal{M}[\neg(- A \natural_1 - B)]$ and $\mathcal{M}[(A \natural_1 B)] = \mathcal{M}[\neg(- A \natural_3 - B)]$, for every model $\mathcal{M}$ for $\mathcal{L}'$. Then, by the fourth property, it follows, first, that for every formula of the form $(A \natural_3 B)$ there is a formula C whose only logical symbols belong to $\{\natural_1, -\}$ and such that $\mathcal{M}[(A \natural_3 B)] = \mathcal{M}[C]$ and, second, that for every formula of the form $(A \natural_1 B)$ there is a formula C whose only logical symbols belong to $\{\natural_3, -\}$ and such that $\mathcal{M}[(A \natural_1 B)] = \mathcal{M}[C]$. ∎

From Theorems 1–3, we conclude that any four-valued language containing a symbol $\natural_i$ ($1 \leq i \leq 8$) plus the conflation or the negation is functionally complete.

5 Concluding remarks

Two remarks can be made on the basis of Theorems 1 and 2. The first point concerns the minimal functionally complete sets of four-valued connectives and the second concerns the concepts of disjunctive and conjunctive normal forms introduced in the proof of these theorems.

In his article [1], A. Avron defines a functionally complete set of four-valued connectives consisting of two unary and two binary connectives and shows that this set is minimal in the sense that 'no proper subset of it is functionally complete'. In this connection, we show that a set consisting of only one unary connective and one binary connective can still be functionally complete. In addition, after a quick review of the truth tables for $-$ and $\ominus$, it appears that neither $\{-\}$ nor $\{\ominus\}$ is functionally complete and therefore that $\{-, \ominus\}$ is minimal. Indeed, no formula whose only logical symbol is $\ominus$ will have a value in $\{t, b\}$ or $\{f, n\}$ if all its propositional symbols are assigned a value from $\{f, n\}$ or $\{t, b\}$, respectively.

Moreover, the proofs presented in this article are based on the view that the four values are actually combinations of two primary truth-values, namely, truth and falsehood. The originality of these proofs is also mainly due to this aspect of our approach. A good example of this is the use made of the abbreviation $(A @ B)$, which plays a crucial role in the definition of the adapted disjunctive and conjunctive normal forms. Using this abbreviation, the revised concept of the disjunctive normal form, A^{δ}, consists in combining two disjunctive forms such that the first determines the truth of the formula and the second determines its falsehood, and similarly for the revised concept of the conjunctive normal form, A^{γ}.

Acknowledgements

We are grateful to Thierry Lucas for numerous discussions on the functional completeness of four-valued languages. We also thank an anonymous reviewer for suggesting many improvements.

References

[1] A. Avron, *On the expressive power of three-valued and four-valued languages*, Journal of Logic and Computation **9** (1999), no. 6, 977–994.

[2] N. D. Belnap, *A useful four-valued logic*, Modern uses of multiple-valued logic (J. M. Dunn and G. Epstein, eds.), Reidel Publishing Company, Dordrecht, 1977, pp. 8–37.

[3] J. M. Dunn, *Intuitive semantics for first-degree entailments and 'coupled trees'*, Philosophical Studies **29** (1976), no. 3, 149–168.

[4] M. Fitting, *Bilattices and the semantics of logic programming*, The Journal of Logic Programming **11** (1991), no. 2, 91–116.

[5] R. Muskens, *Meaning and partiality*, Studies in Logic, Language and Information Publications, Stanford, 1995.

[6] A. Visser, *Four valued semantics and the liar*, Journal of Philosophical Logic **13** (1984), no. 2, 181–212.

Département Sciences, Philosophies, Sociétés
Université de Namur
Rue de Bruxelles 61
5000 Namur - Belgium
email: vincent.degauquier@gmail.com