

Recent developments about multivalued weakly Picard operators

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Abstract

In the present paper, considering a recent technique, which is used by Wardowski [24] for fixed points of single-valued mappings, we give a new and general class of multivalued weakly Picard operators on complete metric spaces and show that the class of multivalued almost contractions (both linear and nonlinear cases in the sense of Berinde and Berinde [9]) is a proper subset of this new class. We also give a nontrivial example showing this fact.

1 Introduction and preliminaries

Let (X, d) be a metric space, $P(X)$ be the collection of all nonempty subsets of X , $CB(X)$ be the collection of all nonempty closed and bounded subsets of X and $K(X)$ be the collection of all nonempty compact subsets of X . It is well known that $H : CB(X) \times CB(X) \rightarrow \mathbb{R}$ defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\}$$

is a metric on $CB(X)$, which is called the Pompeiu-Hausdorff metric induced by d , where $D(x, B) = \inf \{d(x, y) : y \in B\}$. We can find detailed information about the Pompeiu-Hausdorff metric in [2, 10, 13]. An element $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \rightarrow P(X)$ if $x \in Tx$. In 1991,

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Rus [19] introduced the concept of multivalued weakly Picard (MWP) operator on a metric space (X, d) as follows. A multivalued mapping $T : X \rightarrow P(X)$ is MWP operator if there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0 , which converges to a fixed point of T (see [20, 21]). We shall denote the class of all MWP operators on X by $\mathcal{M}_{W\mathcal{P}}(X)$.

Let $T : X \rightarrow CB(X)$. T is said to be a multivalued nonlinear almost contraction [9] if there exist a constant $L \geq 0$ and a function $\varphi : [0, \infty) \rightarrow [0, 1)$ satisfying

$$\limsup_{t \rightarrow s^+} \varphi(t) < 1, \quad \forall s \geq 0, \quad (1.1)$$

such that

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + LD(y, Tx) \quad (1.2)$$

for all $x, y \in X$. A function $\varphi : [0, \infty) \rightarrow [0, 1)$ satisfying (1.1) is called Mizoguchi-Takahashi function (\mathcal{MT} -function [11, 12]) in the literature.

Note that, by the symmetry property of the distance, the nonlinear almost contractive condition implicitly includes the following dual one

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + LD(x, Ty) \quad (1.3)$$

for all $x, y \in X$. So, in order to check the nonlinear almost contractiveness of a multivalued mapping T , it is necessary to check both (1.2) and (1.3). Specially, if $\varphi : [0, \infty) \rightarrow [0, 1)$ is a constant, then T is called multivalued almost contraction. If $L = 0$, then (1.2) turns to the famous Mizoguchi-Takahashi [14] contractive condition, which includes the multivalued contractions in sense of Nadler [16]. We denote show the class of all multivalued nonlinear almost contractions on X by $\mathcal{M}_{NA}(X)$.

Berinde and Berinde [9] proved that if (X, d) is a complete metric space and $T : X \rightarrow CB(X)$ is a multivalued nonlinear almost contraction, then T has a fixed point. Analyzing the proof of this result, we can observe that T is MWP operator. Therefore, if (X, d) is complete then $\mathcal{M}_{NA}(X) \subset \mathcal{M}_{W\mathcal{P}}(X)$. We can find some important information about single-valued and multivalued weakly Picard operators on complete metric space in [5, 6, 7, 8].

In this paper, by considering the recent technique of Wardowski [24], we give a new class of MWP operators, which is more general than $\mathcal{M}_{NA}(X)$ on complete metric spaces. First, we recall the Wardowski's technique.

Let $F : (0, \infty) \rightarrow \mathbb{R}$ be a function. Consider the following conditions for $F :$

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,

(F2) For each sequence $\{\alpha_n\}$ of positive numbers

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \text{ if and only if } \lim_{n \rightarrow \infty} F(\alpha_n) = -\infty,$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$,

(F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

By considering the conditions (F1)-(F3), Wardowski [24] introduced the concept of F -contractions, which more general than ordinary contraction, for single valued maps and show that every F -contraction on a complete metric space has a unique fixed point. (See [24] for more detailed information about F -contractions). Then, considering this new concept, some studies have been made.

Remark 1. In [23] Udo-Utun shows that F -contractions are closely related to (δ, k) -weak contractions (almost contractions) in the sense of Berinde. Therefore in Theorem 2.1 (ii) of [23], it has been shown that every F -contraction T is a (δ, k) -weak contraction, where T is a self map of a bounded closed and convex subset of a real Banach space. But we know that this two concepts are independent on a (complete) metric space as shown in the following two examples. We can find them in [24] and in [18], respectively.

In the following example, the mapping T is not (δ, k) -weak contraction, but it is F -contraction on a complete metric space.

Example 1. Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define a mapping $T : X \rightarrow X$ by:

$$Tx = \begin{cases} x_1 & , \quad x = x_1 \\ x_{n-1} & , \quad x = x_n \end{cases} .$$

Then, as shown in Example 2.5 of [24], T is F -contraction with $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$.

On the other hand, since $d(x_{n-1}, Tx_n) = 0$ and

$$\lim_{n \rightarrow \infty} \frac{d(Tx_n, Tx_{n-1})}{d(x_n, x_{n-1})} = \lim_{n \rightarrow \infty} \frac{2n - 2}{2n} = 1,$$

then we can not find $\delta \in (0, 1)$ and $k \geq 0$ satisfying

$$d(Tx, Ty) \leq \delta d(x, y) + kd(y, Tx)$$

for all $x, y \in X$. Therefore, T is not a (δ, k) -weak contraction.

In the following example, the mapping T is (δ, k) -weak contraction, but it is not F -contraction on a complete metric space.

Example 2. Let $X = [0, 2]$ with the usual metric and $T : X \rightarrow X$ be defined by

$$Tx = \begin{cases} \frac{x}{2} & , \quad x \in [0, 1) \\ 2 & , \quad x \in [1, 2] \end{cases} .$$

Then T is a (δ, k) -weak contraction with $\delta = \frac{1}{2}$ and $k = 3$ (Example 1.3.19 of [18]). But, since T is not continuous and so it is not an F -contraction.

We denote by \mathcal{F} and \mathcal{F}_* the set of all functions F satisfying (F1)-(F3) and (F1)-(F4), respectively. It is clear that $\mathcal{F}_* \subset \mathcal{F}$. Some examples of functions belonging \mathcal{F}_* are $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \alpha + \ln \alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln(\alpha^2 + \alpha)$. If we define $F_5(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F_5(\alpha) = 2\alpha$ for $\alpha > 1$, then $F_5 \in \mathcal{F} \setminus \mathcal{F}_*$. If F is nondecreasing, then it satisfies (F4) if and only if it is right continuous.

Later, the technique of Wardowski was used for multivalued mappings by many authors (see [1, 3, 4, 15, 17, 22]). Here, we shall mentioned some important of them.

Theorem 1 ([17]). *Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$. If there exist $F \in \mathcal{F}_*$ and $\tau : (0, \infty) \rightarrow (0, \infty)$ such that*

$$\liminf_{t \rightarrow s^+} \tau(t) > 0, \quad \forall s \geq 0, \quad (1.4)$$

satisfying

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y))$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. Then T has a fixed point in X .

Theorem 2 ([3]). *Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$. If T is a multivalued almost F -contraction with $F \in \mathcal{F}_*$, that is, there exist two constants $\tau > 0$ and $\lambda \geq 0$ satisfying*

$$\tau + F(H(Tx, Ty)) \leq F((d(x, y) + \lambda D(y, Tx)))$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$, then T has a fixed point in X .

If we examine the proofs of Theorem 1 and Theorem 2, we can see that the mentioned maps belong to $\mathcal{M}_{WP}(X)$. In Theorem 1 (or in Theorem 2), the condition (F4) on F cannot be removed as shown in Example 1 of [3]. However, if we take $T : X \rightarrow K(X)$ in these theorems, we can remove the condition (F4) on F .

2 Main result

Our main results are based on the following new definition.

Definition 1. *Let (X, d) be a metric space and $T : X \rightarrow CB(X)$. We say that T is a multivalued nonlinear almost F -contraction with $F \in \mathcal{F}$ if there exist a constant $\lambda \geq 0$ and a function $\tau : (0, \infty) \rightarrow (0, \infty)$ such that*

$$\liminf_{t \rightarrow s^+} \tau(t) > 0, \quad \text{for all } s \geq 0$$

satisfying

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F((d(x, y) + \lambda D(y, Tx))) \quad (2.1)$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

We shall denote the class of all multivalued nonlinear almost F -contractions on X by $\mathcal{M}_{NAF}(X)$.

Remark 2. *Taking $\tau(t) = \tau > 0$ in Definition 1, we deduce that every multivalued almost F -contraction is also multivalued nonlinear almost F -contraction.*

Remark 3. *Every multivalued nonlinear almost contraction is also multivalued nonlinear almost F -contraction with a special F , that is, $\mathcal{M}_{NA}(X) \subset \mathcal{M}_{NAF}(X)$. Indeed, let (X, d) be metric space and $T \in \mathcal{M}_{NA}(X)$. Then, there exist a constant $L \geq 0$ and an \mathcal{MT} -function φ satisfying*

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + LD(y, Tx) \quad (2.2)$$

for all $x, y \in X$. Define $\beta(t) = \frac{1+\varphi(t)}{2}$, then β is also an \mathcal{MT} -function. Therefore from (2.2),

$$\begin{aligned} H(Tx, Ty) &\leq \varphi(d(x, y))d(x, y) + LD(y, Tx) \\ &\leq \frac{1 + \varphi(d(x, y))}{2}d(x, y) + L[1 + \varphi(d(x, y))]D(y, Tx) \\ &= \beta(d(x, y))d(x, y) + 2L\beta(d(x, y))D(y, Tx) \\ &= \beta(d(x, y))[d(x, y) + 2LD(y, Tx)] \end{aligned}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. Thus, we get

$$-\ln(\beta(d(x, y))) + \ln(H(Tx, Ty)) \leq \ln(d(x, y) + 2LD(y, Tx)) \quad (2.3)$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. Now, define $\tau(t) = -\ln \beta(t)$. Since β is an \mathcal{MT} -function, then

$$\liminf_{t \rightarrow s^+} \tau(t) > 0 \text{ for all } s \geq 0.$$

Therefore, from (2.3), T is a multivalued nonlinear almost F -contraction with $F(\alpha) = \ln \alpha$, $\lambda = 2L$ and $\tau(t) = -\ln \left(\frac{1+\varphi(t)}{2} \right)$, that is $T \in \mathcal{M}_{NAF}(X)$.

Theorem 3. Let (X, d) be a complete metric space and $T \in \mathcal{M}_{NAF}(X)$ with $F \in \mathcal{F}_*$, then $T \in \mathcal{M}_{WP}(X)$.

Proof. Let $x_0 \in X$. As Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T . In this case, we construct a sequence $\{x_n\}$ by $x_n = x_1$ for $n \geq 1$, then $x_{n+1} \in Tx_n$ and $\{x_n\}$ converges to a fixed point of T , that is $T \in \mathcal{M}_{WP}(X)$. Now, suppose $x_1 \notin Tx_1$. Then, as Tx_1 is closed, $D(x_1, Tx_1) > 0$. On the other hand, as $D(x_1, Tx_1) \leq H(Tx_0, Tx_1)$, from (F1) we have

$$F(D(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)).$$

Since $T \in \mathcal{M}_{NAF}(X)$, we have

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq F(H(Tx_0, Tx_1)) \\ &\leq F(d(x_1, x_0) + \lambda D(x_1, Tx_0)) - \tau(d(x_1, x_0)) \\ &= F(d(x_1, x_0)) - \tau(d(x_1, x_0)). \end{aligned} \quad (2.4)$$

From (F4) we can write (note that $D(x_1, Tx_1) > 0$)

$$F(D(x_1, Tx_1)) = \inf_{y \in Tx_1} F(d(x_1, y)),$$

and so from (2.4) we have

$$\begin{aligned} \inf_{y \in Tx_1} F(d(x_1, y)) &\leq F(d(x_1, x_0)) - \tau(d(x_1, x_0)) \\ &< F(d(x_1, x_0)) - \frac{\tau(d(x_1, x_0))}{2}. \end{aligned} \quad (2.5)$$

Then, from (2.5) there exists $x_2 \in Tx_1$ such that

$$F(d(x_1, x_2)) \leq F(d(x_1, x_0)) - \frac{\tau(d(x_1, x_0))}{2}.$$

If $x_2 \in Tx_2$ we are finished. Otherwise, by the same way we can find $x_3 \in Tx_2$ such that

$$F(d(x_2, x_3)) \leq F(d(x_2, x_1)) - \frac{\tau(d(x_2, x_1))}{2}.$$

By induction, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ (we may assume $x_n \notin Tx_n$) and

$$F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n-1})) - \frac{\tau(d(x_n, x_{n-1}))}{2} \quad (2.6)$$

for all $n \in \mathbb{N}$. Denote $a_n = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$, then $a_n > 0$ and from (2.6) $\{a_n\}$ is decreasing. Therefore there exists $\delta \geq 0$ such that $\lim_{n \rightarrow \infty} a_n = \delta$. Now let $\delta > 0$. Using (2.6), the following holds:

$$\begin{aligned} F(a_n) &\leq F(a_{n-1}) - \frac{\tau(a_{n-1})}{2} \\ &\leq F(a_{n-2}) - \frac{\tau(a_{n-1})}{2} - \frac{\tau(a_{n-2})}{2} \\ &\quad \vdots \\ &\leq F(a_0) - \frac{\tau(a_{n-1}) + \tau(a_{n-2}) + \cdots + \tau(a_0)}{2}. \end{aligned} \quad (2.7)$$

Let p_n be a greatest number in $\{0, 1, \dots, n-1\}$ such that

$$\tau(a_{p_n}) = \min \{\tau(a_0), \tau(a_1), \dots, \tau(a_{n-1})\}$$

for all $n \in \mathbb{N}$. In this case, $\{p_n\}$ is a nondecreasing sequence. From (2.7), we get

$$F(a_n) \leq F(a_0) - \frac{n}{2} \tau(a_{p_n}). \quad (2.8)$$

Now consider the sequence $\{\tau(a_{p_n})\}$. We distinguish two cases.

Case 1. For each $n \in \mathbb{N}$ there is $m > n$ such that $\tau(a_{p_n}) > \tau(a_{p_m})$. Then we obtain a subsequence $\{a_{p_{n_k}}\}$ of $\{a_{p_n}\}$ with $\tau(a_{p_{n_k}}) > \tau(a_{p_{n_{k+1}}})$ for all k . Since $a_{p_{n_k}} \rightarrow \delta^+$ we deduce that

$$\liminf_{k \rightarrow \infty} \tau(a_{p_{n_k}}) > 0.$$

Hence $F(a_{n_k}) \leq F(a_0) - \frac{n_k}{2} \tau(a_{p_{n_k}})$ for all k . Consequently $\lim_{k \rightarrow \infty} F(a_{n_k}) = -\infty$, and by (F2), $\lim_{k \rightarrow \infty} a_{p_{n_k}} = 0$, which contradicts that $\lim_{n \rightarrow \infty} a_n > 0$.

Case 2. There is $n_0 \in \mathbb{N}$ such that $\tau(a_{p_{n_0}}) = \tau(a_{p_m})$ for all $m > n_0$. Then $F(a_m) \leq F(a_0) - \frac{m}{2} \tau(a_{p_{n_0}})$ for all $m > n_0$. Hence $\lim_{m \rightarrow \infty} F(a_m) = -\infty$, so $\lim_{m \rightarrow \infty} a_m = 0$, which contradicts that $\lim_{m \rightarrow \infty} a_m > 0$.

Thus $\lim_{n \rightarrow \infty} a_n = 0$. From (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} a_n^k F(a_n) = 0.$$

By (2.7), the following holds for all $n \in \mathbb{N}$

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k \frac{n}{2} \tau(a_{p_n}) \leq 0,$$

that is,

$$0 \leq a_n^k \frac{n}{2} \tau(a_{p_n}) \leq a_n^k F(a_0) - a_n^k F(a_n). \quad (2.9)$$

Letting $n \rightarrow \infty$ in (2.9), we obtain that

$$\lim_{n \rightarrow \infty} a_n^k \frac{n}{2} \tau(a_{p_n}) = 0. \quad (2.10)$$

Since

$$\alpha := \liminf_{n \rightarrow \infty} \tau(a_{p_n}) > 0,$$

then there exists $n_0 \in \mathbb{N}$ such that $\tau(a_{p_n}) > \frac{\alpha}{2}$ for all $n \geq n_0$. Thus

$$a_n^k \frac{n\alpha}{4} \leq a_n^k \frac{n}{2} \tau(a_{p_n}) \quad (2.11)$$

for all $n \geq n_0$. Letting $n \rightarrow \infty$ in (2.11), we obtain that

$$0 \leq \lim_{n \rightarrow \infty} a_n^k \frac{n\alpha}{4} \leq \lim_{n \rightarrow \infty} a_n^k \frac{n}{2} \tau(a_{p_n}) = 0,$$

that is

$$\lim_{n \rightarrow \infty} n a_n^k = 0. \quad (2.12)$$

From (2.12), there exists $n_1 \in \mathbb{N}$ such that $n a_n^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$a_n \leq \frac{1}{n^{\frac{1}{k}}}. \quad (2.13)$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.13), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \leq \sum_{i=n}^{\infty} a_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$, passing to limit $n \rightarrow \infty$, we get $d(x_n, x_m) \rightarrow 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d) . Since (X, d) is a complete metric space, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. From (2.1), for all $x, y \in X$ with $H(Tx, Ty) > 0$, we get

$$H(Tx, Ty) < d(x, y) + \lambda D(y, Tx)$$

and so

$$H(Tx, Ty) \leq d(x, y) + \lambda D(y, Tx)$$

for all $x, y \in X$. Then

$$\begin{aligned} D(x_{n+1}, Tz) &\leq H(Tx_n, Tz) \\ &\leq d(x_n, z) + \lambda D(z, Tx_n) \\ &\leq d(x_n, z) + \lambda d(z, x_{n+1}) \end{aligned}$$

Passing to limit $n \rightarrow \infty$, we obtain $D(z, Tz) = 0$. Thus, we get $z \in \overline{Tz} = Tz$. Therefore $T \in \mathcal{M}_{WP}(X)$. \blacksquare

Remark 4. If we take $T : X \rightarrow K(X)$ in Theorem 3, we can remove the condition (F4) on F . Indeed, let $x_0 \in X$ and $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then the proof is complete. Let $x_1 \notin Tx_1$. Then, as Tx_1 is closed, $D(x_1, Tx_1) > 0$. On the other hand, as $D(x_1, Tx_1) \leq H(Tx_0, Tx_1)$, from (F1) we have

$$F(D(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)).$$

From (2.1), we can write that

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq F(H(Tx_0, Tx_1)) \\ &\leq F(d(x_1, x_0) + \lambda D(x_1, Tx_0)) - \tau(d(x_1, x_0)) \\ &= F(d(x_1, x_0)) - \tau(d(x_1, x_0)). \end{aligned} \quad (2.14)$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = D(x_1, Tx_1)$. Then from (2.14) we have

$$F(d(x_1, x_2)) \leq F(d(x_1, x_0)) - \tau(d(x_1, x_0)).$$

The rest of the proof can be completed as in the proof of Theorem 3.

Example 3. Consider the complete metric space (X, d) , where $X = \{\frac{1}{n^2} : n \in \mathbb{N}, n \geq 2\} \cup \{0\}$ and $d : X \times X \rightarrow [0, \infty)$ is given by $d(x, y) = |x - y|$. Define $T : X \rightarrow CB(X)$ and $F : (0, \infty) \rightarrow \mathbb{R}$ by

$$Tx = \begin{cases} \left\{0, \frac{1}{(n+1)^2}\right\} & , \quad x = \frac{1}{n^2}, n > 2 \\ \{x\} & , \quad x \in \left\{0, \frac{1}{4}\right\} \end{cases}$$

and

$$F(\alpha) = \begin{cases} \frac{\ln \alpha}{\sqrt{\alpha}} & , \quad 0 < \alpha < e^2 \\ \frac{2\alpha}{e^3} & , \quad \alpha \geq e^2 \end{cases} ,$$

respectively. We can see that $F \in \mathcal{F}_*$ and $\sup_{x, y \in X} d(x, y) = \frac{1}{4} < e^2$.

Since $H(T0, T\frac{1}{4}) = \frac{1}{4} = d(0, \frac{1}{4})$, then for all $F \in \mathcal{F}_*$ and $\tau : (0, \infty) \rightarrow (0, \infty)$ satisfying inequality (1.4), we have

$$\tau(d(0, \frac{1}{4})) + F(H(T0, T\frac{1}{4})) > F(d(0, \frac{1}{4})).$$

Therefore Theorem 1 can not be applied to this example.

Now we show that T is not a multivalued nonlinear almost contraction. Indeed, suppose that there exist a constant $L \geq 0$ and a \mathcal{MT} -function φ satisfying (1.2). Therefore, for $x = \frac{1}{n^2}$ and $y = \frac{1}{(n+1)^2}$, then $D(y, Tx) = 0$,

$$H(Tx, Ty) = \frac{2n+3}{(n+1)^2(n+2)^2} \text{ and } d(x, y) = \frac{2n+1}{n^2(n+1)^2}.$$

Thus

$$\begin{aligned} H(Tx, Ty) &\leq \varphi(d(x, y))d(x, y) + LD(y, Tx) \\ \Leftrightarrow \frac{2n+3}{(n+1)^2(n+2)^2} &\leq \varphi\left(\frac{2n+1}{n^2(n+1)^2}\right)\frac{2n+1}{n^2(n+1)^2} \\ \Leftrightarrow \frac{(2n+3)n^2}{(2n+1)(n+2)^2} &\leq \varphi\left(\frac{2n+1}{n^2(n+1)^2}\right). \end{aligned}$$

Taking limit supremum as $n \rightarrow \infty$, we have

$$1 \leq \limsup_{n \rightarrow \infty} \varphi\left(\frac{2n+1}{n^2(n+1)^2}\right) \leq \limsup_{t \rightarrow 0^+} \varphi(t) < 1,$$

which is a contradiction. Therefore T is not multivalued nonlinear almost contraction and so $T \notin \mathcal{M}_{NA}(X)$.

On the other hand, $T \in \mathcal{M}_{NAF}(X)$ with $\lambda = 1$ and $\tau = \ln \frac{100}{81}$. To see this have to show that

$$\ln \frac{100}{81} + F(H(Tx, Ty)) \leq F(d(x, y) + \min \{D(y, Tx), D(x, Ty)\}), \quad (2.15)$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. First, observe that if $H(Tx, Ty) > 0$, then $x \neq y$.

Case 1: For $x = \frac{1}{n^2}$ and $y = \frac{1}{m^2}$ with $m > n > 2$, we obtain

$$\begin{aligned} H(Tx, Ty) &\frac{1}{\sqrt{H(Tx, Ty)}} d(x, y)^{-\frac{1}{\sqrt{d(x, y)}}} \\ &= \left(\frac{1}{(n+1)^2} - \frac{1}{(m+1)^2} \right) \frac{1}{\sqrt{\frac{1}{(n+1)^2} - \frac{1}{(m+1)^2}}} \left(\frac{1}{n^2} - \frac{1}{m^2} \right)^{-\frac{1}{\sqrt{\frac{1}{n^2} - \frac{1}{m^2}}}} \\ &= \left(\frac{(m+1)^2 - (n+1)^2}{(n+1)^2(m+1)^2} \right) \frac{(n+1)(m+1)}{\sqrt{(m+1)^2 - (n+1)^2}} \left(\frac{m^2 - n^2}{n^2 m^2} \right)^{-\frac{nm}{\sqrt{m^2 - n^2}}} \\ &= \left(\frac{(m+1)^2 - (n+1)^2}{(n+1)^2(m+1)^2} \right) \frac{(n+1)(m+1)}{\sqrt{(m+1)^2 - (n+1)^2}} \frac{nm}{\sqrt{m^2 - n^2}} \left(\frac{(m+n+2)n^2 m^2}{(m+n)(n+1)^2(m+1)^2} \right)^{\frac{nm}{\sqrt{m^2 - n^2}}}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \frac{(m+1)^2 - (n+1)^2}{(n+1)^2(m+1)^2} &\leq \frac{1}{2}, \\ \frac{(n+1)(m+1)}{\sqrt{(m+1)^2 - (n+1)^2}} - \frac{nm}{\sqrt{m^2 - n^2}} &\geq 1, \\ \frac{(m+n+2)n^2 m^2}{(m+n)(n+1)^2(m+1)^2} &< 1 \end{aligned}$$

and

$$\frac{nm}{\sqrt{m^2 - n^2}} > 1,$$

then we have

$$H(Tx, Ty) \frac{1}{\sqrt{H(Tx, Ty)}} d(x, y)^{-\frac{1}{\sqrt{d(x, y)}}} \leq \frac{1}{2} < \frac{81}{100}.$$

Thus we obtain

$$\begin{aligned} \ln \frac{100}{81} + F(H(Tx, Ty)) &\leq F(d(x, y)) \\ &\leq F(d(x, y) + \min \{D(y, Tx), D(x, Ty)\}), \end{aligned}$$

that is (2.15) is satisfied.

Case 2. For $x = \frac{1}{n^2}$, $n > 2$ and $y = 0$, we obtain

$$\begin{aligned} H(Tx, Ty) \frac{1}{\sqrt{H(Tx, Ty)}} d(x, y)^{-\frac{1}{\sqrt{d(x, y)}}} &= \left(\frac{1}{(n+1)^2} \right)^{\frac{1}{\sqrt{(n+1)^2}}} \left(\frac{1}{n^2} \right)^{-\frac{1}{\sqrt{\frac{1}{n^2}}}} \\ &= \frac{n^{2n}}{(n+1)^{2(n+1)}} \\ &= \left(\frac{n}{n+1} \right)^{2n} \left(\frac{1}{n+1} \right)^2 \\ &< \frac{1}{2} \end{aligned}$$

and so (2.15) is satisfied.

Case 3. For $x = \frac{1}{4}$ and $y = 0$, since $H(Tx, Ty) = d(x, y) = \min \{D(y, Tx), D(x, Ty)\} = \frac{1}{4}$, we obtain

$$\begin{aligned} H(Tx, Ty) \frac{1}{\sqrt{H(Tx, Ty)}} [d(x, y) + \min \{d(y, Tx), d(x, Ty)\}]^{-\frac{1}{\sqrt{d(x, y) + \min \{d(y, Tx), d(x, Ty)\}}}} \\ = \left(\frac{1}{4} \right)^2 \left(\frac{1}{2} \right)^{-\sqrt{2}} < \frac{1}{16} \cdot 4 = \frac{1}{4} < \frac{1}{2}, \end{aligned}$$

that is (2.15) is satisfied.

Case 4. For $x = \frac{1}{n^2}$, $n > 2$ and $y = \frac{1}{4}$, since $H(Tx, Ty) = \frac{1}{4}$ and

$$d(x, y) = \min \{D(y, Tx), D(x, Ty)\} = \frac{1}{4} - \frac{1}{n^2},$$

since

$$\frac{1}{2} - \frac{2}{n^2} \geq \frac{1}{2} - \frac{2}{9} = \frac{5}{18}$$

for $n \geq 3$, we obtain

$$\begin{aligned} H(Tx, Ty) \frac{1}{\sqrt{H(Tx, Ty)}} [d(x, y) + \min \{d(y, Tx), d(x, Ty)\}]^{-\frac{1}{\sqrt{d(x, y) + \min \{d(y, Tx), d(x, Ty)\}}}} \\ = \left(\frac{1}{4} \right)^2 \left(\frac{1}{2} - \frac{2}{n^2} \right)^{-\frac{1}{\sqrt{\frac{1}{2} - \frac{2}{n^2}}}} \\ \leq \frac{1}{16} \left(\frac{18}{5} \right) \sqrt{\frac{18}{5}} \\ < \frac{1}{16} \left(\frac{18}{5} \right)^2 \\ = \frac{81}{100}. \end{aligned}$$

This shows that all conditions of Theorem 3 are satisfied and so $T \in \mathcal{M}_{WP}(X)$.

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