On the Fekete-Szegö problem for classes of bi-univalent functions

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Abstract

In this paper we obtain the Fekete-Szegö inequalities for the classes $\mathcal{H}_\sigma(\varphi)$, $\mathcal{ST}_\sigma(\alpha,\varphi)$ and $\mathcal{M}_\sigma(\alpha,\varphi)$ of bi-univalent functions defined in terms of subordination. These inequalities result in the bounds of the third coefficient which improve many known results concerning different classes of bi-univalent functions.

1 Introduction

Let $\mathcal{A}$ be the class of all functions f in the unit disk $\mathbb{D} \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ normalized by the conditions $f(0) = f'(0) - 1 = 0$ and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. For every $f \in \mathcal{S}$ there exists an inverse function f^{-1} which is defined in some neighbourhood of the origin. According to the Koebe one-quarter theorem f^{-1} is defined in some disk containing the disk $|w| < 1/4$. In some cases this inverse function can be extended to whole $\mathbb{D}$. Clearly, f^{-1} is also univalent. This is the reason of discussing so called bi-univalent functions.

A function $f \in \mathcal{A}$ is called bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Following Lewin, we denote the class of bi-univalent functions by σ.

Observe that for $f \in \sigma$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

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the inverse function f^{-1} has the Taylor-Maclaurin series expansion

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \dots \tag{2}$$

Lewin gave the first estimate of coefficients in σ. Namely, he proved that $|a_2| < 1.51$. On the other hand, Styer and Wright showed that $|a_2| > 4/3$ for some function in σ. The problem of estimating coefficients $|a_n|$, $n \geq 2$ is still open. However, a lot of results for a_2, a_3 and a_4 were proved for some subclasses of σ. Unfortunatelly, they are not sharp.

In the recent paper Ali *et al.* obtained results in classes defined in terms of subordination. Among others they discussed classes $\mathcal{H}_\sigma(\varphi)$, $\mathcal{ST}_\sigma(\alpha,\varphi)$ and $\mathcal{M}_\sigma(\alpha,\varphi)$. In the definitions of the three classes a function φ appears. In all cases it is assumed that φ is an analytic function in $\mathbb{D}$ with positive real part. Moreover, it has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \dots \tag{3}$$

where all coefficients are real and $B_1 > 0$.

Now we can formulate the definitions of the classes mentioned above.

Definition 1. *A function $f \in \sigma$ is in $\mathcal{H}_\sigma(\varphi)$ if*

$$f'(z) \prec \varphi(z) \quad \text{and} \quad g'(w) \prec \varphi(w) \quad , \quad g = f^{-1}.$$

Definition 2. *A function $f \in \sigma$ is in $\mathcal{ST}_\sigma(\alpha,\varphi)$ if*

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \alpha\frac{w^2 g''(w)}{g(w)} \prec \varphi(w) \quad , \quad g = f^{-1}.$$

Definition 3. *A function $f \in \sigma$ is in $\mathcal{M}_\sigma(\alpha,\varphi)$ if*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \quad \text{and}$$
$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w) \quad , \quad g = f^{-1}.$$

Observe that the class $\mathcal{ST}_\sigma(0,\varphi) = \mathcal{M}_\sigma(0,\varphi)$ is known as the class of Ma-Minda bi-starlike functions and the class $\mathcal{M}_\sigma(1,\varphi)$ is known as the class of Ma-Minda bi-convex functions. These particular classes are denoted by $\mathcal{ST}_\sigma(\varphi)$ and $\mathcal{CV}_\sigma(\varphi)$ respectively.

In this paper we shall obtain the Fekete-Szegö inequalities for $\mathcal{H}_\sigma(\varphi)$, $\mathcal{ST}_\sigma(\alpha,\varphi)$ and $\mathcal{M}_\sigma(\alpha,\varphi)$. These inequalities will result in bounds of the third coefficient which are, in some cases, better then these obtained in [1], [2], [6].

2 Main results

At the beginning, observe that the conditions in all three definitions can be written as follows:

$$F(z) \prec \varphi(z) \quad \text{and} \quad G(w) \prec \varphi(w) \tag{4}$$

where

$$F(z) = f'(z)\,,\ G(w) = g'(w) \qquad \text{for} \quad \mathcal{H}_\sigma(\varphi)\,,$$

$$F(z) = \frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}\,,\ G(w) = \frac{wg'(w)}{g(w)} + \alpha\frac{w^2 g''(w)}{g(w)} \qquad \text{for} \quad \mathcal{ST}_\sigma(\alpha,\varphi)\,,$$

$$F(z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\,,$$

$$G(w) = (1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1+\frac{wg''(w)}{g'(w)}\right) \qquad \text{for} \quad \mathcal{M}_\sigma(\alpha,\varphi)\,.$$

The conditions (4) are equivalent to

$$F(z) = \varphi(u(z)) \quad \text{and} \quad G(w) = \varphi(v(w))\,. \tag{5}$$

Here, functions u and v are analytic in $\mathbb{D}$, $u(0) = v(0) = 0$, and $|u(z)| < 1$, $|v(z)| < 1$ for all $z \in \mathbb{D}$.

We apply the same technique as in [1]. Assume that

$$p(z) \equiv \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \dots \tag{6}$$

and

$$q(z) \equiv \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \dots\,. \tag{7}$$

Clearly, $\operatorname{Re} p(z) > 0$ and $\operatorname{Re} q(z) > 0$. From (6), (7) one can derive

$$u(z) = \tfrac{1}{2}p_1 z + \tfrac{1}{2}(p_2 - \tfrac{1}{2}{p_1}^2)z^2 + \dots \tag{8}$$

and

$$v(z) = \tfrac{1}{2}q_1 z + \tfrac{1}{2}(q_2 - \tfrac{1}{2}{q_1}^2)z^2 + \dots\,. \tag{9}$$

Combining (3), (5), (8) and (9),

$$F(z) = 1 + \tfrac{1}{2}B_1 p_1 z + \left(\tfrac{1}{4}B_2 {p_1}^2 + \tfrac{1}{2}B_1(p_2 - \tfrac{1}{2}{p_1}^2)\right)z^2 + \dots \tag{10}$$

and

$$G(z) = 1 + \tfrac{1}{2}B_1 q_1 z + \left(\tfrac{1}{4}B_2 {q_1}^2 + \tfrac{1}{2}B_1(q_2 - \tfrac{1}{2}{q_1}^2)\right)z^2 + \dots\,. \tag{11}$$

From (10) and (11) and the series expansions of F and G, it follows that

$$A_1(F) = \tfrac{1}{2}B_1 p_1 \tag{12}$$

$$A_2(F) = \tfrac{1}{4}B_2 {p_1}^2 + \tfrac{1}{2}B_1(p_2 - \tfrac{1}{2}{p_1}^2) \tag{13}$$

$$A_1(G) = \tfrac{1}{2}B_1 q_1 \tag{14}$$

$$A_2(G) = \tfrac{1}{4}B_2 {q_1}^2 + \tfrac{1}{2}B_1(q_2 - \tfrac{1}{2}{q_1}^2)\,, \tag{15}$$

where $A_j(h)$ stands for j-th coefficient of a function h.

Now we can establish our main results.

Theorem 1. *Let f of the form (1) be in $\mathcal{H}_\sigma(\varphi)$ and $\mu \in \mathbb{R}$. Then*

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \frac{1}{3}B_1 & \text{for} \quad |\mu - 1| \leq \left|1 + \frac{4}{3}\frac{B_1 - B_2}{{B_1}^2}\right| \\ \frac{{B_1}^3|\mu-1|}{|3{B_1}^2 + 4(B_1-B_2)|} & \text{for} \quad |\mu - 1| \geq \left|1 + \frac{4}{3}\frac{B_1 - B_2}{{B_1}^2}\right| \end{cases} .$$

Theorem 2. *Let f of the form (1) be in $\mathcal{ST}_\sigma(\alpha, \varphi)$ and $\mu \in \mathbb{R}$. Then*

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \frac{1}{2(1+3\alpha)}B_1 \\ \qquad \text{for } |\mu - 1| \leq \frac{1}{2(1+3\alpha)}\left|1 + 4\alpha + (1+2\alpha)^2\frac{B_1 - B_2}{{B_1}^2}\right| \\ \frac{{B_1}^3|\mu-1|}{|(1+4\alpha){B_1}^2 + (1+2\alpha)^2(B_1-B_2)|} \\ \qquad \text{for } |\mu - 1| \geq \frac{1}{2(1+3\alpha)}\left|1 + 4\alpha + (1+2\alpha)^2\frac{B_1 - B_2}{{B_1}^2}\right| \end{cases} .$$

Theorem 3. *Let f of the form (1) be in $\mathcal{M}_\sigma(\alpha, \varphi)$ and $\mu \in \mathbb{R}$. Then*

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \frac{1}{2(1+2\alpha)}B_1 & \text{for} \quad |\mu - 1| \leq \frac{1+\alpha}{2(1+2\alpha)}\left|1 + (1+\alpha)\frac{B_1 - B_2}{{B_1}^2}\right| \\ \frac{{B_1}^3|\mu-1|}{|(1+\alpha){B_1}^2 + (1+\alpha)^2(B_1-B_2)|} & \text{for} \quad |\mu - 1| \geq \frac{1+\alpha}{2(1+2\alpha)}\left|1 + (1+\alpha)\frac{B_1 - B_2}{{B_1}^2}\right| \end{cases} .$$

Taking various real numbers μ, α and functions φ one can obtain many results. Some of them improve earlier results published in [1], [2] or [6].

We begin with the class $\mathcal{H}_\sigma(\varphi)$. Taking $\mu = 1$ or $\mu = 0$ we get

Corollary 1. *If $f \in \mathcal{H}_\sigma(\varphi)$ then*

$$|a_3 - {a_2}^2| \leq \frac{1}{3}B_1 .$$

Corollary 2. *If $f \in \mathcal{H}_\sigma(\varphi)$ then*

$$|a_3| \leq \begin{cases} \frac{1}{3}B_1 & \text{for} \quad \frac{B_1 - B_2}{{B_1}^2} \in (-\infty, -\frac{3}{2}] \cup [0, \infty) , \\ \frac{{B_1}^3}{|3{B_1}^2 + 4(B_1-B_2)|} & \text{for} \quad \frac{B_1 - B_2}{{B_1}^2} \in [-\frac{3}{2}, -\frac{3}{4}) \cup (-\frac{3}{4}, 0] . \end{cases}$$

Remark. It is easily seen that if $|\frac{4}{3}\frac{B_1-B_2}{{B_1}^2} + 1| \geq 1$ then the bound $\frac{1}{3}B_1$ in Corollary 2 is better than the bound in [1], Theorem 2.1. The detailed discussion of the case $0 < |\frac{4}{3}\frac{B_1-B_2}{{B_1}^2} + 1| < 1$ shows that for some choices of B_1, B_2 the bound in Corollary 2 is better then the bound in Theorem 2.1 of [1], but for other choices the result of Ali *et al.* is better than this in Corollary 2. The situation is the same while comparing results of Corollary 7 with Theorem 2.2, and Corollary 14 with Theorem 2.3 in [1].

In papers [1], [5] some special choices of φ were considered. Namely,

$$\varphi_1(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + \dots \quad , \gamma \in (0, 1] \tag{16}$$

and

$$\varphi_2(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \dots \quad , \beta \in [0, 1) . \tag{17}$$

Certainly, for suitably taken γ or β we get

$$\varphi_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots \quad . \tag{18}$$

The choice of φ_0 leads to the classes: $\mathcal{H}_\sigma(\varphi_0)$ of bi-univalent functions with bounded turning and $\mathcal{M}_\sigma(\alpha, \varphi_0)$ of bi-Mocanu convex functions.

For φ_j, $j = 0, 1, 2$ we obtain the following conclusions.

Corollary 3. *If* $f \in \mathcal{H}_\sigma(\varphi_1)$ *then*

$$a) \quad |a_3| \le \tfrac{2}{3}\gamma \,,$$

$$b) \quad |a_3 - a_2{}^2| \le \tfrac{2}{3}\gamma \,.$$

Corollary 4. *If* $f \in \mathcal{H}_\sigma(\varphi_2)$ *then*

$$a) \quad |a_3| \le \tfrac{2}{3}(1-\beta) \,,$$

$$b) \quad |a_3 - a_2{}^2| \le \tfrac{2}{3}(1-\beta) \,.$$

Corollary 5. *If* $f \in \mathcal{H}_\sigma(\varphi_0)$ *then*

$$a) \quad |a_3| \le \tfrac{2}{3} \,,$$

$$b) \quad |a_3 - a_2{}^2| \le \tfrac{2}{3} \,.$$

Now we can turn to $\mathcal{ST}_\sigma(\alpha, \varphi)$. For $\mu = 1$ or $\mu = 0$ we get

Corollary 6. *If* $f \in \mathcal{ST}_\sigma(\alpha, \varphi)$ *then*

$$|a_3 - a_2{}^2| \le \frac{1}{2(1+3\alpha)} B_1 \,.$$

Corollary 7. *If* $f \in \mathcal{ST}_\sigma(\alpha, \varphi)$ *then*

$$|a_3| \le \begin{cases} \frac{1}{2(1+3\alpha)} B_1 & \text{for } \frac{B_1 - B_2}{B_1{}^2} \in \left(-\infty, -\frac{3+10\alpha}{(1+2\alpha)^2}\right] \cup \left[\frac{1}{1+2\alpha}, \infty\right) , \\ \frac{B_1{}^3}{|(1+4\alpha)B_1{}^2 + (1+2\alpha)^2(B_1 - B_2)|} & \text{for } \frac{B_1 - B_2}{B_1{}^2} \in \left[-\frac{3+10\alpha}{(1+2\alpha)^2}, -\frac{1+4\alpha}{(1+2\alpha)^2}\right) \cup \left(-\frac{1+4\alpha}{(1+2\alpha)^2}, \frac{1}{1+2\alpha}\right] . \end{cases}$$

For φ_1, φ_2, φ_0 we conclude

Corollary 8. *If* $f \in \mathcal{ST}_\sigma(\alpha, \varphi_1)$ *then*

$$a) \quad |a_3| \le \begin{cases} \frac{\gamma}{1+3\alpha} & \text{for} \quad \gamma \le \frac{1+2\alpha}{3+2\alpha} \\ \frac{4\gamma^2}{(1+2\alpha)^2 + \gamma(1+4\alpha-4\alpha^2)} & \text{for} \quad \gamma \ge \frac{1+2\alpha}{3+2\alpha} \,, \end{cases}$$

$$b) \quad |a_3 - a_2{}^2| \le \tfrac{\gamma}{1+3\alpha} \,.$$

Corollary 9. *If $f \in \mathcal{ST}_\sigma(\alpha, \varphi_2)$ then*

$$a) \quad |a_3| \le \tfrac{2(1-\beta)}{1+4\alpha}\,,$$

$$b) \quad |a_3 - {a_2}^2| \le \tfrac{1-\beta}{1+3\alpha}\,.$$

Corollary 10. *If $f \in \mathcal{ST}_\sigma(\alpha, \varphi_0)$ then*

$$a) \quad |a_3| \le \tfrac{2}{1+4\alpha}\,,$$

$$b) \quad |a_3 - {a_2}^2| \le \tfrac{1}{1+3\alpha}\,.$$

As it was said, the class $\mathcal{ST}_\sigma(0, \varphi)$ coincides with the class $\mathcal{ST}_\sigma(\varphi)$ of Ma-Minda bi-starlike functions. All the corollaries 6 - 10 can be rewritten for $\alpha = 0$. It is worth writting explicitly only the estimates of the third coefficient. From Corollary 7 it follows that

Corollary 11. *If $f \in \mathcal{ST}_\sigma(\varphi)$ then*

$$|a_3| \le \begin{cases} \tfrac{1}{2}B_1 & \text{for} \quad \tfrac{B_1-B_2}{{B_1}^2} \in (-\infty, -3] \cup [1, \infty)\,, \\ \tfrac{{B_1}^3}{|{B_1}^2+B_1-B_2|} & \text{for} \quad \tfrac{B_1-B_2}{{B_1}^2} \in [-3, -1) \cup (-1, 1]\,. \end{cases}$$

This yields

Corollary 12.

a) If $f \in \mathcal{ST}_\sigma(\varphi_1)$ then $|a_3| \le \begin{cases} \gamma & \text{for} \quad \gamma \le \tfrac{1}{3} \\ \tfrac{4\gamma^2}{1+\gamma} & \text{for} \quad \gamma \ge \tfrac{1}{3}\,, \end{cases}$

b) If $f \in \mathcal{ST}_\sigma(\varphi_2)$ then $|a_3| \le 2(1-\beta)$,

c) If $f \in \mathcal{ST}_\sigma(\varphi_0)$ then $|a_3| \le 2$.

The estimate in Corollary 12 point a) is better than the result given in Theorem 3.1 in [2] or in Corollary 2.1 (or Remark 2.2) in [1].

Similar conclusions can be obtained for the class $\mathcal{CV}_\sigma(\varphi)$ of Ma-Minda bi-convex functions, as a special case of $\mathcal{M}_\sigma(\alpha, \varphi)$. This part of conclusions we start with more general corollaries.

Corollary 13. *If $f \in \mathcal{M}_\sigma(\alpha, \varphi)$ then*

$$|a_3 - {a_2}^2| \le \frac{1}{2(1+2\alpha)} B_1\,.$$

Corollary 14. *If $f \in \mathcal{M}_\sigma(\alpha, \varphi)$ then*

$$|a_3| \le \begin{cases} \tfrac{1}{2(1+2\alpha)}B_1 & \text{for} \quad \tfrac{B_1-B_2}{{B_1}^2} \in (-\infty, -\tfrac{3+5\alpha}{(1+\alpha)^2}] \cup [\tfrac{1+3\alpha}{(1+\alpha)^2}, \infty)\,, \\ \tfrac{{B_1}^3}{(1+\alpha)|{B_1}^2+(1+\alpha)(B_1-B_2)|} & \text{for} \quad \tfrac{B_1-B_2}{{B_1}^2} \in [-\tfrac{3+5\alpha}{(1+\alpha)^2}, -\tfrac{1}{1+\alpha}) \cup (-\tfrac{1}{1+\alpha}, \tfrac{1+3\alpha}{(1+\alpha)^2}]\,. \end{cases}$$

As particular cases we get

Corollary 15. *If $f \in \mathcal{M}_\sigma(\alpha, \varphi_1)$ then*

$$a)\quad |a_3| \le \begin{cases} \frac{\gamma}{1+2\alpha} & \text{for} \quad \gamma \le \frac{(1+\alpha)^2}{3+8\alpha+\alpha^2} \\ \frac{4\gamma^2}{(1+\alpha)[(1+\alpha)+\gamma(1-\alpha)]} & \text{for} \quad \gamma \ge \frac{(1+\alpha)^2}{3+8\alpha+\alpha^2}, \end{cases}$$

$$b)\quad |a_3 - a_2{}^2| \le \frac{\gamma}{1+2\alpha}.$$

Corollary 16. *If $f \in \mathcal{M}_\sigma(\alpha, \varphi_2)$ then*

$$a)\quad |a_3| \le \frac{2(1-\beta)}{1+\alpha},$$

$$b)\quad |a_3 - a_2{}^2| \le \frac{1-\beta}{1+2\alpha}.$$

Corollary 17. *If $f \in \mathcal{M}_\sigma(\alpha, \varphi_0)$ then*

$$a)\quad |a_3| \le \frac{2}{1+\alpha},$$

$$b)\quad |a_3 - a_2{}^2| \le \frac{1}{1+2\alpha}.$$

Hence, for $\mathcal{CV}_\sigma(\varphi)$ the following corollaries hold.

Corollary 18. *If $f \in \mathcal{CV}_\sigma(\varphi)$ then*

$$|a_3| \le \begin{cases} \frac{1}{6}B_1 & \text{for} \quad \frac{B_1-B_2}{B_1{}^2} \in (-\infty, -2] \cup [1, \infty), \\ \frac{B_1{}^3}{2|B_1{}^2+2(B_1-B_2)|} & \text{for} \quad \frac{B_1-B_2}{B_1{}^2} \in [-2, -\frac{1}{2}) \cup (-\frac{1}{2}, 1]. \end{cases}$$

Corollary 19.

a) If $f \in \mathcal{CV}_\sigma(\varphi_1)$ then $|a_3| \le \begin{cases} \frac{1}{3}\gamma & \text{for} \quad \gamma \le \frac{1}{3} \\ \gamma^2 & \text{for} \quad \gamma \ge \frac{1}{3}, \end{cases}$

b) If $f \in \mathcal{CV}_\sigma(\varphi_2)$ then $|a_3| \le 1 - \beta$,

c) If $f \in \mathcal{CV}_\sigma(\varphi_0)$ then $|a_3| \le 1$.

Observe that the bound in Corollary 19 point a) improves the known result for $\mathcal{CV}_\sigma(\varphi_1)$ (see, Theorem 5.1 in [2] or in Corollary 2.2 in [1]).

3 Proofs of Theorems

Proof of Theorem 1. Since $F = f'$ and $G = g'$, from (13)-(15) it follows that

$$\begin{aligned} 2a_2 &= \tfrac{1}{2}B_1 p_1 && (19)\\ 3a_3 &= \tfrac{1}{4}B_2 {p_1}^2 + \tfrac{1}{2}B_1(p_2 - \tfrac{1}{2}{p_1}^2) && (20)\\ -2a_2 &= \tfrac{1}{2}B_1 q_1 && (21)\\ 3(2{a_2}^2 - a_3) &= \tfrac{1}{4}B_2 {q_1}^2 + \tfrac{1}{2}B_1(q_2 - \tfrac{1}{2}{q_1}^2)\ . && (22) \end{aligned}$$

From (19) and (21)

$$p_1 = -q_1\ . \tag{23}$$

Subtracking (22) from (20) and applying (23) we have

$$a_3 = {a_2}^2 + \frac{1}{12}B_1(p_2 - q_2)\ . \tag{24}$$

On the other hand, summing (20) and (22) results in

$$6{a_2}^2 = \frac{1}{2}B_1(p_2 + q_2) - \frac{1}{4}(B_1 - B_2)({p_1}^2 + {q_1}^2)\ .$$

Combining this with (19) and (21) leads to

$${a_2}^2 = \frac{{B_1}^3(p_2 + q_2)}{4[3{B_1}^2 + 4(B_1 - B_2)]}\ . \tag{25}$$

From (24) and (25) it follows that

$$a_3 - \mu {a_2}^2 = \frac{B_1}{12}\left[(h(\mu) + 1)p_2 + (h(\mu) - 1)q_2\right]\ ,$$

where

$$h(\mu) = \frac{3{B_1}^2(1 - \mu)}{3{B_1}^2 + 4(B_1 - B_2)}\ .$$

Since all B_j are real and $B_1 > 0$, we conclude that

$$\left|a_3 - \mu {a_2}^2\right| \le \begin{cases} \frac{B_1}{3}|h(\mu)| & \text{for} \quad |h(\mu)| \ge 1 \\ \frac{B_1}{3} & \text{for} \quad 0 \le |h(\mu)| \le 1\ , \end{cases}$$

which completes the proof. ■

Proof of Theorem 2. For the class $\mathcal{ST}_\sigma(\alpha, \varphi)$ the functions in (4) are of the form $F(z) = \frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}$ and $G(w) = \frac{wg'(w)}{g(w)} + \alpha\frac{w^2 g''(w)}{g(w)}$. Hence

$$\begin{aligned} (1 + 2\alpha)a_2 &= \tfrac{1}{2}B_1 p_1 && (26)\\ 2(1 + 3\alpha)a_3 - (1 + 2\alpha){a_2}^2 &= \tfrac{1}{4}B_2 {p_1}^2 + \tfrac{1}{2}B_1(p_2 - \tfrac{1}{2}{p_1}^2) && (27)\\ -(1 + 2\alpha)a_2 &= \tfrac{1}{2}B_1 q_1 && (28)\\ (3 + 10\alpha){a_2}^2 - 2(1 + 3\alpha)a_3 &= \tfrac{1}{4}B_2 {q_1}^2 + \tfrac{1}{2}B_1(q_2 - \tfrac{1}{2}{q_1}^2)\ . && (29) \end{aligned}$$

From (26) and (28) there is

$$p_1 = -q_1 \,. \tag{30}$$

Subtracking (29) from (27) and applying (30) we get

$$a_3 = {a_2}^2 + \frac{1}{8(1+3\alpha)} B_1 (p_2 - q_2) \,. \tag{31}$$

Now, summing (27) and (29) leads to

$$2(1+4\alpha){a_2}^2 = \frac{1}{2} B_1 (p_2 + q_2) - \frac{1}{4} (B_1 - B_2)({p_1}^2 + {q_1}^2) \,.$$

This equality and (26), (28) result in

$${a_2}^2 = \frac{{B_1}^3 (p_2 + q_2)}{4[(1+4\alpha){B_1}^2 + (1+2\alpha)^2 (B_1 - B_2)]} \,. \tag{32}$$

From (31) and (32) it follows that

$$a_3 - \mu {a_2}^2 = B_1 \left[\left(h(\mu) + \frac{1}{8(1+3\alpha)} \right) p_2 + \left(h(\mu) - \frac{1}{8(1+3\alpha)} \right) q_2 \right] ,$$

where

$$h(\mu) = \frac{{B_1}^2 (1-\mu)}{4[(1+4\alpha){B_1}^2 + (1+2\alpha)^2 (B_1 - B_2)]} \,.$$

Therefore

$$\left| a_3 - \mu {a_2}^2 \right| \le \begin{cases} 4B_1 |h(\mu)| & \text{for} \quad |h(\mu)| \ge \frac{1}{8(1+3\alpha)} \\ \frac{B_1}{2(1+3\alpha)} & \text{for} \quad 0 \le |h(\mu)| \le \frac{1}{8(1+3\alpha)} \,. \end{cases}$$

The proof is completed. ∎

The functions F and G for the class $\mathcal{M}_\sigma(\alpha, \varphi)$ are following:

$$F(z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right)$$

and

$$G(w) = (1-\alpha)\frac{wg'(w)}{g(w)} + \alpha \left(1 + \frac{wg''(w)}{g'(w)} \right) \,.$$

The relations (13)-(15) take form

$$(1+\alpha)a_2 = \tfrac{1}{2} B_1 p_1 \tag{33}$$

$$2(1+2\alpha)a_3 - (1+3\alpha){a_2}^2 = \tfrac{1}{4} B_2 {p_1}^2 + \tfrac{1}{2} B_1 (p_2 - \tfrac{1}{2}{p_1}^2) \tag{34}$$

$$-(1+\alpha)a_2 = \tfrac{1}{2} B_1 q_1 \tag{35}$$

$$(3+5\alpha){a_2}^2 - 2(1+2\alpha)a_3 = \tfrac{1}{4} B_2 {q_1}^2 + \tfrac{1}{2} B_1 (q_2 - \tfrac{1}{2}{q_1}^2) \,. \tag{36}$$

All the details of the proof of Theorem 3 are quite similar to those in the proofs given above and will be omitted.

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