The reflexive and Hermitian reflexive solutions of the generalized Sylvester-conjugate matrix equation

Masoud Hajarian* Mehdi Dehghan

Abstract

The main purpose of this correspondence is to establish two gradient based iterative (GI) methods extending the Jacobi and Gauss-Seidel iterations for solving the generalized Sylvester-conjugate matrix equation

$$A_1XB_1 + A_2\overline{X}B_2 + C_1YD_1 + C_2\overline{Y}D_2 = E,$$

over reflexive and Hermitian reflexive matrices. It is shown that the iterative methods, respectively, converge to the reflexive and Hermitian reflexive solutions for any initial reflexive and Hermitian reflexive matrices. We report numerical tests to show the effectiveness of the proposed approaches.

1 Introduction

In this paper, we denote the set of all $m \times n$ complex matrices by $\mathcal{C}^{m\times n}$ and the identity matrix with the appropriate size by I. The symbols A^T, $\overline{A}$, A^H and $\mathrm{tr}(A)$ mean the transpose, conjugate, conjugate transpose and trace of a matrix A, respectively. $\mathrm{Re}(a)$ denotes the real part of number a. The inner product $\langle .,. \rangle_r$ in $\mathcal{C}^{m\times n}$ over the field $\mathcal{R}$ is defined as follows:

$$\langle A, B\rangle_r = \mathrm{Re}(\mathrm{tr}(B^H A)) \quad \text{for} \quad A, B \in \mathcal{C}^{m\times n},$$

*Corresponding author.

Received by the editors in July 2011 - In revised form in June 2012.

Communicated by K. In't Hout.

2010 *Mathematics Subject Classification :* 15A24; 65F10; 65F30.

Key words and phrases : The generalized Sylvester-conjugate matrix equations; Reflexive solution pair; Hermitian reflexive solution pair; Iterative method.

that is $\langle A, B\rangle_r$ is the real part of the trace of $B^H A$. It is can be shown that $(\mathcal{C}^{m\times n}, \mathcal{R}, \langle.,.\rangle_r)$ is a Hilbert inner product space. The induced matrix norm is $||A|| = \sqrt{\langle A, A\rangle_r} = \sqrt{\mathrm{Re}(\mathrm{tr}(A^H A))}$, which is the Frobenius norm [1, 47, 48].
A matrix $P \in \mathcal{C}^{n\times n}$ is called generalized reflection matrix if $P = P^H$ and $PP^H = I$. Throughout, we always suppose that P and Q are a given generalized reflection matrices. If $A = PAP$ ($A = A^H = PAP$) then A is called a reflexive (Hermitian reflexive) matrix with respect to P. $\mathcal{C}_r^{n\times n}(P)$ ($\mathcal{HC}_r^{n\times n}(P)$) denotes the set of order n reflexive (Hermitian reflexive) matrices with respect to P. Due to that I is a generalized reflection matrix, any $n \times n$ complex (Hermitian) matrix is also a reflexive (Hermitian reflexive) matrix with respect to I. The reflexive matrices (namely the generalized centro-symmetric matrices) have practical applications in many areas such as the numerical solution of certain differential equations [2, 5], pattern recognition [7], Markov processes [43], various physical and engineering problems [8, 6] and so on (e.g. [3, 27, 45, 46]). Chen [4] proposed three applications of reflexive matrices obtained from the altitude estimation of a level network, an electric network and structural analysis of trusses. The symmetric Toeplitz matrices, an important subclass of the class of Hermitian reflexive matrices, appear naturally in digital signal processing applications and other areas [18].
The problem of finding solutions of linear matrix equations arises in a variety of engineering, mathematics and physics problems [9, 15, 30, 31, 32, 33, 35]. Therefore, in recent years much attention has focused on studying the solutions of linear matrix equations. In [36, 37, 38, 39, 40, 42] several quaternion matrix equations were studied. In [19, 21, 23], by extending the well-known Jacobi and Gauss-Seidel iterations, Ding and Chen presented some efficient iterative algorithms based on the hierarchical identification principle [20, 22] for solving the generalized Sylvester matrix equations and general coupled matrix equations. In [44, 24], this efficient approach was also applied for other general matrix equations. In [49, 50], Zhou and Duan studied the solution of the generalized Sylvester matrix equations. In [51], Zhou et al. analyzed the computational complexity of the Smith iteration and its variations for solving the Stein matrix equation $X - AXB = C$. By extending the idea of conjugate gradient (CG) method, Dehghan and Hajarian proposed some efficient iterative methods to solve Sylvester and Lyapunov matrix equations [17, 10, 11, 12, 13, 14]. Thiran et al. [34] investigated a rational approximation problem in connection with the convergence analysis of the ADI iterative method applied to the Stein matrix equation. Jiang and Wei [28] obtained explicit solutions of the Stein matrix equation and Stein-conjugate matrix equation $X - A\overline{X}B = C$ by the method of characteristic polynomial and a method of real representation of a complex matrix respectively.
It is known that solving complex matrix equations can be very difficult and it is sufficiently complicated. This paper is concerned with the reflexive (Hermitian reflexive) solution pair $[X, Y]$ of the generalized Sylvester-conjugate matrix equation

$$A_1XB_1 + A_2\overline{X}B_2 + C_1YD_1 + C_2\overline{Y}D_2 = E, \tag{1.1}$$

where $A_1, A_2 \in \mathcal{C}^{p\times n}$, $B_1, B_2 \in \mathcal{C}^{n\times q}$, $C_1, C_2 \in \mathcal{C}^{p\times m}$, $D_1, D_2 \in \mathcal{C}^{m\times q}$, $E \in \mathcal{C}^{p\times q}$ are known matrices and $X \in \mathcal{C}_r^{n\times n}(P)$, $Y \in \mathcal{C}_r^{m\times m}(Q)$ ($X \in \mathcal{HC}_r^{n\times n}(P)$, $Y \in \mathcal{HC}_r^{m\times m}(Q)$) are unknown matrices to be determined. The reflexive and Hermitian reflexive solutions of the matrix equation (1.1) have not been dealt with yet. This matrix equation includes various linear matrix equations such as Lyapunov, Sylvester, Stein, Yakubovich, Kalman-Yakubovich, homogeneous (nonhomogeneous) Yakubovich-conjugate matrix equations as special cases. Hence the generalized Sylvester-conjugate matrix equation (1.1) can play an important role in control theory and can be used to achieve pole assignment, robust pole assignment and observer design for descriptor linear systems [26]. The rest of the paper is organized as follows. In Section 2, by extending the Jacobi and Gauss-Seidel iterations we propose two GI methods to solve (1.1) over reflexive and Hermitian reflexive matrices. Theoretical analysis shows that the proposed methods converge to the reflexive and Hermitian reflexive solutions of (1.1) for any initial reflexive and Hermitian reflexive matrices, respectively. Finally, two numerical examples are given to illustrate the effectiveness of the proposed methods in Section 3.

2 Main results

In this section we propose two iterative methods for finding the reflexive and Hermitian reflexive solutions of (1.1) respectively and their convergence analysis is also given.

In [21, 29], some iterative methods were presented to solve Sylvester matrix equations over real matrices. In the methods [21, 29], matrix inversion is required in the first iteration. These methods may turn out to be numerically expensive and are not practical for equations of large systems. The purpose in this paper is to obtain two iterative methods without any inverse for solving the linear matrix equation (1.1) over the reflexive and Hermitian reflexive matrices.

First, it is known that the solvability of linear matrix equation (1.1) over the reflexive (Hermitian reflexive) matrix pair $[X, Y]$ is equivalent to the following system of matrix equations:

$$\begin{cases} A_1XB_1 + A_2\overline{X}B_2 + C_1YD_1 + C_2\overline{Y}D_2 = E, \\ A_1PXPB_1 + A_2P\overline{X}PB_2 + C_1QYQD_1 + C_2Q\overline{Y}QD_2 = E, \end{cases} \tag{2.1}$$

$$\left(\begin{cases} A_1XB_1 + A_2\overline{X}B_2 + C_1YD_1 + C_2\overline{Y}D_2 = E, \\ A_1PXPB_1 + A_2P\overline{X}PB_2 + C_1QYQD_1 + C_2Q\overline{Y}QD_2 = E, \\ B_1^HXA_1^H + B_2^H\overline{X}A_2^H + D_1^HYC_1^H + D_2^H\overline{Y}C_2^H = E^H, \\ B_1^HPXPA_1^H + B_2^HP\overline{X}PA_2^H + D_1^HQYQC_1^H + D_2^HQ\overline{Y}QC_2^H = E^H, \end{cases} \right). \tag{2.2}$$

Now consider a linear algebraic system of equations

$$Ax = b, \tag{2.3}$$

which A is the coefficient matrix; and b and x are, respectively, the known right hand side and the solution to be sought. Also suppose that

$$A = D - E - F, \tag{2.4}$$

in which D is the diagonal of A, $-E$ its strict lower part, and $-F$ its strict upper part. It is always assumed that the diagonal entries of A are all nonzero. To solve the linear system (2.3), the Jacobi and the Gauss-Seidel iterations are both of the form

$$Mx(k+1) = Nx(k) + b = (M - A)x(k) + b, \tag{2.5}$$

in which

$$A = M - N, \tag{2.6}$$

is a splitting of A, with $M = D$ for Jacobi, $M = D - E$ for forward Gauss-Seidel, and $M = D - F$ for backward Gauss-Seidel. Here by extending the Jacobi and the Gauss-Seidel iterations (2.5) and by applying the hierarchical identification principle for (2.1) and (2.2), respectively, we can obtain the GI methods described in the following:

Algorithm 1. *To solve (1.1) over reflexive matrix pair* $[X, Y]$*:*
Given an initial reflexive matrix pair $[X(1), Y(1)]$ *with* $X(1) \in \mathcal{C}_r^{n\times n}(P)$ *and* $Y(1) \in \mathcal{C}_r^{m\times m}(Q)$*;*
For k:=1,2,.... until convergence do;

$$R(k) = E - A_1X(k)B_1 - A_2\overline{X(k)}B_2 - C_1Y(k)D_1 - C_2\overline{Y(k)}D_2;$$

$$X(k+1) = X(k) + \frac{\mu}{4}\Big[A_1^HR(k)B_1^H + A_2^T\overline{R(k)}\,B_2^T + PA_1^HR(k)B_1^HP + PA_2^T\overline{R(k)}\,B_2^TP\Big];$$

$$Y(k+1) = Y(k) + \frac{\mu}{4}\Big[C_1^HR(k)D_1^H + C_2^T\overline{R(k)}\,D_2^T + QC_1^HR(k)D_1^HQ + QC_2^T\overline{R(k)}\,D_2^TQ\Big];$$

$$0 < \mu < \frac{2}{||A_1||^2||B_1||^2 + ||A_2||^2||B_2||^2 + ||C_1||^2||D_1||^2 + ||C_2||^2||D_2||^2}.$$

Algorithm 2. *To solve (1.1) over Hermitian reflexive matrix pair* $[X, Y]$*:*
Given an initial Hermitian reflexive matrix pair $[X(1), Y(1)]$ *with* $X(1) \in \mathcal{HC}_r^{n\times n}(P)$ *and* $Y(1) \in \mathcal{HC}_r^{m\times m}(Q)$*;*
For k:=1,2,.... until convergence do;

$$R(k) = E - A_1X(k)B_1 - A_2\overline{X(k)}B_2 - C_1Y(k)D_1 - C_2\overline{Y(k)}D_2;$$

$$X(k+1) = X(k) + \frac{\mu}{8}\Big[A_1^HR(k)B_1^H + A_2^T\overline{R(k)}\,B_2^T + B_1R(k)^HA_1 + \overline{B_2}R(k)^T\,\overline{A_2}$$
$$+PA_1^HR(k)B_1^HP + PA_2^T\overline{R(k)}\,B_2^TP + PB_1R(k)^HA_1P + P\overline{B_2}R(k)^T\,\overline{A_2}P\Big];$$

$$Y(k+1) = Y(k) + \frac{\mu}{8}\Big[C_1^HR(k)D_1^H + C_2^T\overline{R(k)}\,D_2^T + D_1R(k)^HC_1 + \overline{D_2}R(k)^T\,\overline{C_2}$$
$$+QC_1^HR(k)D_1^HQ + QC_2^T\overline{R(k)}\,D_2^TQ + QD_1R(k)^HC_1Q + Q\overline{D_2}R(k)^T\,\overline{C_2}Q\Big];$$

$$0 < \mu < \frac{2}{||A_1||^2||B_1||^2 + ||A_2||^2||B_2||^2 + ||C_1||^2||D_1||^2 + ||C_2||^2||D_2||^2}.$$

From Algorithm 1 (2), we can see that $X(k) \in \mathcal{C}_r^{n\times n}(P)$ and $Y(k) \in \mathcal{C}_r^{m\times m}(Q)$ ($X(k) \in \mathcal{HC}_r^{n\times n}(P)$ and $Y(k) \in \mathcal{HC}_r^{m\times m}(Q)$) for $k = 1, 2, ...$
Now we present the main results of this paper.

Theorem 2.1. *If the generalized Sylvester-conjugate matrix equation (1.1) has a unique reflexive solution pair $[X^*, Y^*]$, then the iterative solution pair $[X(k), Y(k)]$ given by the Algorithm 1 converges to $[X^*, Y^*]$, i.e.,*

$$\lim_{k\to\infty} X(k) = X^* \qquad \textit{and} \qquad \lim_{k\to\infty} Y(k) = Y^*,$$

for any initial reflexive matrix pair $[X(1), Y(1)]$.

Proof. First we define the estimation error matrices as:

$$\xi_1(k) = X(k) - X^* \qquad \text{and} \qquad \xi_2(k) = Y(k) - Y^*.$$

It is obvious that $\xi_1(k) \in \mathcal{C}_r^{n\times n}(P)$ and $\xi_2(k) \in \mathcal{C}_r^{m\times m}(Q)$ for $k = 1, 2, ...$ By using the above error matrices and Algorithm 1, we can obtain

$$R(k) = -A_1\xi_1(k)B_1 - A_2\overline{\xi_1(k)}B_2 - C_1\xi_2(k)D_1 - C_2\overline{\xi_2(k)}D_2,$$

$$\begin{aligned}
&\xi_1(k+1) = \\
&\quad \xi_1(k) - \frac{\mu}{4}\Big\{A_1^H\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]B_1^H \\
&\qquad + A_2^T\Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]B_2^T \\
&\qquad + PA_1^H\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]B_1^H P \\
&\qquad + PA_2^T\Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]B_2^T P\Big\},
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\xi_2(k+1) = \\
&\quad \xi_2(k) - \frac{\mu}{4}\Big\{C_1^H\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]D_1^H \\
&\qquad + C_2^T\Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]D_2^T \\
&\qquad + QC_1^H\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]D_1^H Q \\
&\qquad + QC_2^T\Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]D_2^T Q\Big\}.
\end{aligned} \tag{2.8}$$

By taking the norm of both sides of (2.7) and using following facts for two square complex matrices A, B and the generalized reflection matrix P

$$\begin{cases}
\mathrm{tr}(AB) = \mathrm{tr}(BA), \\
\langle A, B\rangle_r = \langle B, A\rangle_r, \\
||A+B|| \le ||A|| + ||B||, \\
||PAP|| = ||A||,
\end{cases} \tag{2.9}$$

we have

$$\begin{aligned}
||\xi_1(k+1)||^2 &= \Big(\mathrm{tr}\Big(\xi_1(k+1)^H\xi_1(k+1)\Big)\Big) = \Big(\mathrm{tr}\Big(\xi_1(k)^H\xi_1(k)\Big)\Big) \\
&- \frac{\mu}{2}\Big(\mathrm{tr}\Big(\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]^H A_1\xi_1(k)B_1 \\
&+ \Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]^H \overline{A_2}\xi_1(k)\overline{B_2} \\
&+ \Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]^H A_1P\xi_1(k)PB_1 \\
&+ \Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]^H \overline{A_2}P\xi_1(k)P\overline{B_2}\Big)\Big) \\
&+ \frac{\mu^2}{16}\Big|\Big|A_1^H\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]B_1^H \\
&+ A_2^T\Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]B_2^T \\
&+ PA_1^H\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]B_1^HP \\
&+ PA_2^T\Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]B_2^TP\Big|\Big|^2 \\
&\le ||\xi_1(k)||^2 - \mu\Big(\mathrm{tr}\Big(\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]^H A_1\xi_1(k)B_1 \\
&+ \Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]^H A_2\overline{\xi_1(k)}B_2\Big)\Big) \\
&+ \frac{\mu^2}{4}\Big|\Big|A_1^H\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]B_1^H\Big|\Big|^2 \\
&+ \Big|\Big|A_2^T\Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]B_2^T\Big|\Big|^2 \\
&+ \Big|\Big|PA_1^H\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]B_1^HP\Big|\Big|^2 \\
&+ \Big|\Big|PA_2^T\Big[\overline{A_1}\,\overline{\xi_1(k)}\,\overline{B_1} + \overline{A_2}\,\xi_1(k)\,\overline{B_2} + \overline{C_1}\,\overline{\xi_2(k)}\,\overline{D_1} + \overline{C_2}\,\xi_2(k)\,\overline{D_2}\Big]B_2^TP\Big|\Big|^2 \\
&\le ||\xi_1(k)||^2 - \mu\Big(\mathrm{tr}\Big(\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]^H \\
&\times \Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2\Big]\Big)\Big) + \frac{\mu^2}{2}\Big[\Big|\Big|A_1\Big|\Big|^2\Big|\Big|B_1\Big|\Big|^2 + \Big|\Big|A_2\Big|\Big|^2\Big|\Big|B_2\Big|\Big|^2\Big] \\
&\times \Big|\Big|A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big|\Big|^2.
\end{aligned}$$

Similarly to the above, we can write

$$\begin{aligned}
&||\xi_2(k+1)||^2 \le \\
&||\xi_2(k)||^2 - \mu\Big(\mathrm{tr}\Big(\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]^H \\
&\times \Big[C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]\Big)\Big) + \frac{\mu^2}{2}\Big[\Big|\Big|C_1\Big|\Big|^2\Big|\Big|D_1\Big|\Big|^2 + \Big|\Big|C_2\Big|\Big|^2\Big|\Big|D_2\Big|\Big|^2\Big] \\
&\times \Big|\Big|A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big|\Big|^2.
\end{aligned}$$

Define the nonnegative definite function $\xi(k)$ by:

$$\xi(k) = ||\xi_1(k)||^2 + ||\xi_2(k)||^2.$$

By the previous results, this function can be computed as

$$\begin{aligned}
&\xi(k+1) = ||\xi_1(k+1)||^2 + ||\xi_2(k+1)||^2 \\
&\leq ||\xi_1(k)||^2 + ||\xi_2(k)||^2 - \mu\Big(\mathrm{tr}\Big(\Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]^H \\
&\qquad \times \Big[A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big]\Big)\Big) \\
&\quad + \frac{\mu^2}{2}\Big[\Big|\Big|A_1\Big|\Big|^2\Big|\Big|B_1\Big|\Big|^2 + \Big|\Big|A_2\Big|\Big|^2\Big|\Big|B_2\Big|\Big|^2 + \Big|\Big|C_1\Big|\Big|^2\Big|\Big|D_1\Big|\Big|^2 + \Big|\Big|C_2\Big|\Big|^2\Big|\Big|D_2\Big|\Big|^2\Big] \\
&\qquad \times \Big|\Big|A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big|\Big|^2 \\
&\quad = \xi(k) - \mu\Big|\Big|A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big|\Big|^2 \\
&\quad + \frac{\mu^2}{2}\Big[\Big|\Big|A_1\Big|\Big|^2\Big|\Big|B_1\Big|\Big|^2 + \Big|\Big|A_2\Big|\Big|^2\Big|\Big|B_2\Big|\Big|^2 + \Big|\Big|C_1\Big|\Big|^2\Big|\Big|D_1\Big|\Big|^2 + \Big|\Big|C_2\Big|\Big|^2\Big|\Big|D_2\Big|\Big|^2\Big] \\
&\qquad \times \Big|\Big|A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big|\Big|^2 \\
&= \xi(k) - \mu\Big\{1 - \frac{\mu}{2}\Big[\Big|\Big|A_1\Big|\Big|^2\Big|\Big|B_1\Big|\Big|^2 + \Big|\Big|A_2\Big|\Big|^2\Big|\Big|B_2\Big|\Big|^2 + \Big|\Big|C_1\Big|\Big|^2\Big|\Big|D_1\Big|\Big|^2 + \Big|\Big|C_2\Big|\Big|^2\Big|\Big|D_2\Big|\Big|^2\Big]\Big\} \\
&\qquad \times \Big|\Big|A_1\xi_1(k)B_1 + A_2\overline{\xi_1(k)}B_2 + C_1\xi_2(k)D_1 + C_2\overline{\xi_2(k)}D_2\Big|\Big|^2 \\
&\leq \xi(1) - \mu\Big\{1 - \frac{\mu}{2}\Big[\Big|\Big|A_1\Big|\Big|^2\Big|\Big|B_1\Big|\Big|^2 + \Big|\Big|A_2\Big|\Big|^2\Big|\Big|B_2\Big|\Big|^2 + \Big|\Big|C_1\Big|\Big|^2\Big|\Big|D_1\Big|\Big|^2 + \Big|\Big|C_2\Big|\Big|^2\Big|\Big|D_2\Big|\Big|^2\Big]\Big\} \\
&\qquad\qquad \times \sum_{i=1}^{k-1}\Big|\Big|A_1\xi_1(i)B_1 + A_2\overline{\xi_1(i)}B_2 + C_1\xi_2(i)D_1 + C_2\overline{\xi_2(i)}D_2\Big|\Big|^2.
\end{aligned}$$

If the convergence factor μ is chosen to satisfy

$$0 < \mu < \frac{2}{||A_1||^2||B_1||^2 + ||A_2||^2||B_2||^2 + ||C_1||^2||D_1||^2 + ||C_2||^2||D_2||^2}, \tag{2.10}$$

then we can conclude that

$$\sum_{i=1}^{\infty}\Big|\Big|A_1\xi_1(i)B_1 + A_2\overline{\xi_1(i)}B_2 + C_1\xi_2(i)D_1 + C_2\overline{\xi_2(i)}D_2\Big|\Big|^2 < \infty. \tag{2.11}$$

Because if

$$\sum_{i=1}^{\infty}\Big|\Big|A_1\xi_1(i)B_1 + A_2\overline{\xi_1(i)}B_2 + C_1\xi_2(i)D_1 + C_2\overline{\xi_2(i)}D_2\Big|\Big|^2 = \infty,$$

then by considering (2.10) we have

$$\xi(k+1) \leq \xi(1) - \mu\Big\{1 - \frac{\mu}{2}\Big[\Big|\Big|A_1\Big|\Big|^2\Big|\Big|B_1\Big|\Big|^2 + \Big|\Big|A_2\Big|\Big|^2\Big|\Big|B_2\Big|\Big|^2 + \Big|\Big|C_1\Big|\Big|^2\Big|\Big|D_1\Big|\Big|^2 + \Big|\Big|C_2\Big|\Big|^2\Big|\Big|D_2\Big|\Big|^2\Big]\Big\} \times \sum_{i=1}^{k-1} \Big|\Big|A_1\xi_1(i)B_1 + A_2\overline{\xi_1(i)}B_2 + C_1\xi_2(i)D_1 + C_2\overline{\xi_2(i)}D_2\Big|\Big|^2 \leq -\infty \quad (2.12)$$

Now it is obvious that (2.12) contradicts $\xi(k) \geq 0$.
The necessary condition of the series convergence (2.11) implies that

$$\lim_{i\to\infty}\Big[A_1\xi_1(i)B_1 + A_2\overline{\xi_1(i)}B_2 + C_1\xi_2(i)D_1 + C_2\overline{\xi_2(i)}D_2\Big] = 0,$$

or

$$A_1(\lim_{i\to\infty}\xi_1(i))B_1 + A_2(\lim_{i\to\infty}\overline{\xi_1(i)})B_2 + C_1(\lim_{i\to\infty}\xi_2(i))D_1 + C_2(\lim_{i\to\infty}\overline{\xi_2(i)})D_2 = 0,$$

If we define matrices $M_1 := \lim_{i\to\infty}\xi_1(i)$ and $M_2 := \lim_{i\to\infty}\xi_2(i)$ then the above relation can be written by

$$A_1M_1B_1 + A_2\overline{M_1}B_2 + C_1M_2D_1 + C_2\overline{M_2}D_2 = 0. \quad (2.13)$$

It follows from (1.1) (or (2.13)) has a unique reflexive solution pair that

$$M_1 = \lim_{i\to\infty}\xi_1(i) = 0 \quad \text{and} \quad M_2 = \lim_{i\to\infty}\xi_2(i) = 0,$$

or

$$\lim_{i\to\infty} X(i) = X^* \quad \text{and} \quad \lim_{i\to\infty} Y(i) = Y^*.$$

The proof is finished. ■

Similarly to the proof of Theorem 2.1, we can prove the following theorem.

Theorem 2.2. *If the generalized Sylvester-conjugate matrix equation (1.1) has a unique Hermitian reflexive solution pair $[X^*, Y^*]$, then the iterative solution pair $[X(k), Y(k)]$ given by the Algorithm 2 converges to $[X^*, Y^*]$, i.e.,*

$$\lim_{k\to\infty} X(k) = X^* \quad \textit{and} \quad \lim_{k\to\infty} Y(k) = Y^*,$$

for any initial Hermitian reflexive matrix pair $[X(1), Y(1)]$.

3 Numerical experiments

This section gives two numerical experiments to illustrate the convergence behaviors of both Algorithms 1 and 2. All codes were written in Matlab. All the experiments were performed on a PC of Intel Pentium 2.0 GHz.

Example 3.1. In this example we consider the matrix equation

$$X + A\overline{X}B = C, \tag{3.1}$$

with the following parameters

$$A = \begin{pmatrix} 0-4.2028i & 0.7621 & 0.6154 & 0.4057 & 0.0579 \\ 0 & 0-4.7468i & 0.7919 & 0.9355 & 0.3529 \\ 0 & 0 & 0-4.5252i & 0.9169 & 0.8132 \\ 0 & 0 & 0 & 0-4.3795i & 0.0099 \\ 0 & 0 & 0 & 0 & 0-4.1897i \end{pmatrix},$$

$$B = \begin{pmatrix} 4.7027-0.1934i & 0-0.6979i & 0 & 0 & 0 \\ 0-0.6822i & 4.9568-0.3784i & 0-0.8998i & 0 & 0 \\ 0-0.3028i & 0-0.8600i & 4.2523-0.8216i & 0-0.2897i & 0 \\ 0-0.5417i & 0-0.8537i & 0-0.6449i & 4.1991-0.3412i & 0-0.5681i \\ 0-0.1509i & 0-0.5936i & 0-0.8180i & 0-0.5341i & 4.9883-0.3704i \end{pmatrix},$$

$$C = \begin{pmatrix} 2.1724-50.7332i & -2.2213-7.1968i & -0.5299-33.5896i & 7.0098-2.5987i & 8.1422-37.9450i \\ -6.1068-4.7690i & -0.2975-63.4945i & -3.4524-9.1137i & 7.8320-56.2526i & 8.8178-2.4665i \\ 0.5522-34.9945i & -11.9091-7.1195i & -9.1670-67.7309i & -1.2783-2.7177i & 10.6543-62.4720i \\ -13.5538-0.0158i & -10.7554-57.5376i & -17.1337-0.0464i & -1.1919-44.3710i & -5.8483-0.0112i \\ -5.0081-33.4271i & -22.2234 & -17.0505-47.5846i & -10.1156 & -1.6953-64.1988i \end{pmatrix}.$$

It can be verified that this matrix equation is consistent over reflexive matrices and has the reflexive solution $X^* \in \mathcal{C}_r^{5\times 5}(P)$, that is

$$X^* = 10^2 \begin{pmatrix} 0.0117+1.8194i & 0 & 0.0042 & 0 & 0.0083 \\ 0 & 0.0116+1.8129i & 0 & 0.0159 & 0 \\ 0.0103 & 0 & 0.0157+1.8054i & 0 & 0.0175 \\ 0 & 0.0106 & 0 & 0.0121+1.8126i & 0 \\ 0.0087 & 0 & 0.0092 & 0 & 0.0154+1.8115i \end{pmatrix},$$

where

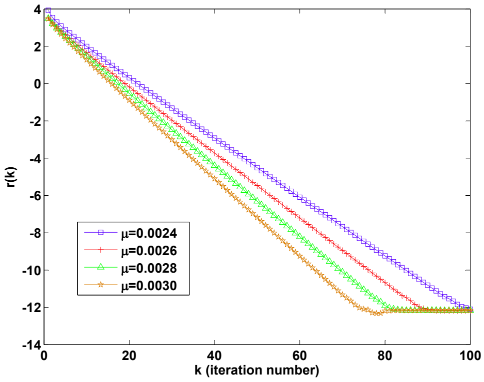
$$P = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

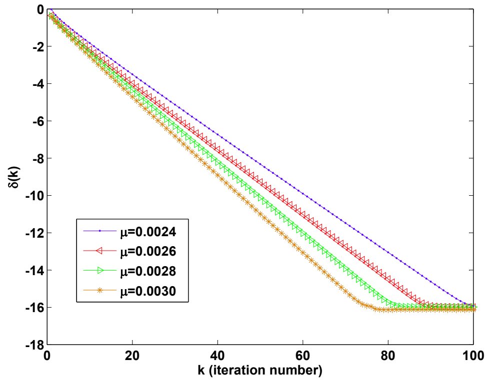
We apply Algorithm 1 with the initial reflexive matrix $X(1) = 0$ to calculate $\{X(k)\}$. The derived results are displayed in Figure 1, where

$$r(k) = \log_{10} ||R(k)|| \ \text{(residual)} \ \text{ and } \ \delta(k) = \log_{10} \frac{||X(k) - X^*||}{||X^*||} \ \text{(relative error)}.$$

We can easily see that $r(k), \delta(k)$ decrease and converge to zero as k increases. In [28], the solution of complex matrix equation (3.1) was obtained by the method of characteristic polynomial and a method of real representation of a complex matrix respectively. Because of characteristic polynomial, the method [28] may turn out to be numerically expensive and is not practical for equations of large systems.

Figure 1: The results obtained for Example 3.1.





Example 3.2. Consider the matrix equation

$$AX + YB = C, \tag{3.2}$$

where

$$A = \begin{pmatrix} 0+3.2028i & 0.7621 & 0.6154 & 0.4057 & 0.0579 \\ 0 & 0+3.7468i & 0.7919 & 0.9355 & 0.3529 \\ 0 & 0 & 0+3.5252i & 0.9169 & 0.8132 \\ 0 & 0 & 0 & 0+3.3795i & 0.0099 \\ 0 & 0 & 0 & 0 & 0+3.1897i \end{pmatrix},$$

$$B = \begin{pmatrix} 3.7027+0.1934i & 0+0.6979i & 0 & 0 & 0 \\ 0+0.6822i & 3.9568+0.3784i & 0+0.8998i & 0 & 0 \\ 0+0.3028i & 0+0.8600i & 3.2523+0.8216i & 0+0.2897i & 0 \\ 0+0.5417i & 0+0.8537i & 0+0.6449i & 3.1991+0.3412i & 0+0.5681i \\ 0+0.1509i & 0+0.5936i & 0+0.8180i & 0+0.5341i & 3.9883+0.3704i \end{pmatrix},$$

$$C = \begin{pmatrix} 0.9901+7.4662i & 2.8418 & 2.0830+4.6413i & 2.9964 & 1.8216+5.4337i \\ 1.7463 & 2.4772+8.6896i & 3.4239 & 2.2559+9.9217i & 3.1991 \\ 1.3796+5.1085i & 2.4280 & 2.1719+11.0454i & 2.2111 & 2.4979+9.4154i \\ 0.0167 & 0+8.9491i & 0.0263 & 0+8.1495i & 0.0303 \\ 0+5.4114i & 0 & 0+8.5193i & 0 & 0+9.7980i \end{pmatrix}.$$

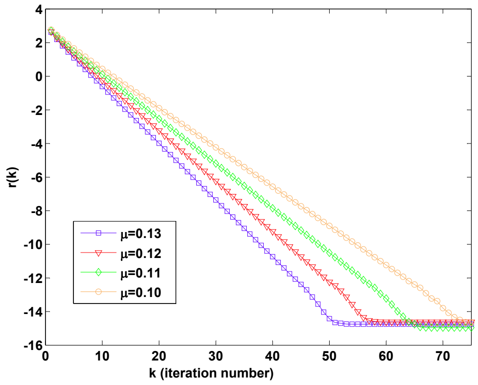
We can verify that the matrix equation (3.2) is consistent over Hermitian reflexive matrices and has Hermitian reflexive solution pair $[X^*, Y^*]$ with $X^*, Y^* \in \mathcal{HC}_r^{5\times5}(P)$ as follows:

$$X^* = \begin{pmatrix} 2.3312 & 0 & 1.4492 & 0 & 1.6966 \\ 0 & 2.3192 & 0 & 2.6481 & 0 \\ 1.4492 & 0 & 3.1333 & 0 & 2.6709 \\ 0 & 2.6481 & 0 & 2.4115 & 0 \\ 1.6966 & 0 & 2.6709 & 0 & 3.0718 \end{pmatrix},$$

$$Y^* = \begin{pmatrix} 1.8057 & 0 & 1.2715 & 0 & 0.1939 \\ 0 & 2.7325 & 0 & 0.8058 & 0 \\ 1.2715 & 0 & 0.0654 & 0 & 2.0575 \\ 0 & 0.8058 & 0 & 2.8705 & 0 \\ 0.1939 & 0 & 2.0575 & 0 & 0.6144 \end{pmatrix}.$$

By Algorithm 2 for (3.2) with the the initial matrix pair $[X(1), Y(1)] = [0, 0]$ we obtain the sequences $\{X(k)\}$ and $\{Y(k)\}$. The obtained results are presented in Figure 2. From Figure 2, we can see that Algorithm 2 is effective.

Figure 2: The results obtained for Example 3.2.



4 Conclusions

In this work, we have proposed Algorithms 1 and 2, respectively, for computing the reflexive and Hermitian reflexive solutions of (1.1). We have proven that Algorithms 1 and 2 always converge to the reflexive and Hermitian reflexive solutions for any initial reflexive and Hermitian reflexive matrices, respectively. Moreover, we have presented two numerical examples to test the performance of the proposed algorithms.

Acknowledgments

The authors would like to express their heartfelt thanks to the anonymous referee for his\her useful comments and constructive suggestions which substantially improved the quality and presentation of this paper. The authors are also very much indebted to Professor Karel in 't Hout (Editor) for his valuable suggestions, generous concern and encouragement during the review process of this paper.

References

[1] A.G. Wu, L.L. Lv, M.Z. Hou, Finite iterative algorithms for a common solution to a group of complex matrix equations, Appl. Math. Comput. 218 (2011) 1191-1202.

[2] A. Andrew, Eigenvectors of certain matrices, Linear Algebra Appl. 7 (1973) 157-162.

[3] Z.J. Bai, The inverse eigenproblem of centrosymmetric matrices with a submatrix constraint and its approximation, SIAM J. Matrix Anal. Appl. 26 (2005) 1100-1114.

[4] H.C. Chen, Generalized reflexive matrices: special properties and applications, SIAM J. Matrix Anal. Appl. 19 (1998) 140-153.

[5] W. Chen, X. Wang, T. Zhong, The structure of weighting coefficient matrices of Harmonic differential quadrature and its application, Comm. Numer. Methods Engrg. 12 (1996) 455-460.

[6] J. Delmas, On Adaptive EVD asymptotic distribution of centro-symmetric covariance matrices, IEEE Trans. Signal Process. 47 (1999) 1402-1406.

[7] L. Datta, S. Morgera, Some results on matrix symmetries and a pattern recognition application, IEEE Trans. Signal Process. 34 (1986) 992-994.

[8] L. Datta, S. Morgera, On the reducibility of centrosymmetric matrices-applications in engineering problems, Circuits Systems Signal Process., 8 (1989) 71-96.

[9] M. Dehghan, M. Hajarian, The reflexive and anti-reflexive solutions of a linear matrix equation and systems of matrix equations, Rocky Mountain J. Math. 40 (2010) 1-23.

[10] M. Dehghan, M. Hajarian, On the generalized bisymmetric and skew-symmetric solutions of the system of generalized Sylvester matrix equations, Linear and Multilinear Algebra, 59 (2011) 1281-1309.

[11] M. Hajarian, M. Dehghan, The generalized centro-symmetric and least squares generalized centro-symmetric solutions of the matrix equation $AYB + CY^T D = E$, Mathematical Methods in the Applied Sciences, 34 (2011) 1562-1579.

[12] M. Dehghan, M. Hajarian, Solving the generalized Sylvester matrix equation $\sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{q} C_j Y D_j = E$ over reflexive and anti-reflexive matrices, International Journal of Control, Automation and Systems, 9 (2011) 118-124.

[13] M. Dehghan, M. Hajarian, Analysis of an iterative algorithm to solve the generalized coupled Sylvester matrix equations, Applied Mathematical Modelling, 35 (2011) 3285-3300.

[14] M. Dehghan, M. Hajarian, Two algorithms for the Hermitian reflexive and skew-Hermitian solutions of Sylvester matrix equations, Applied Mathematics Letters, 24 (2011) 444-449.

[15] M. Dehghan, M. Hajarian, The (R,S)-symmetric and (R,S)-skew symmetric solutions of the pair of matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$, Bulletin of the Iranian Mathematical Society, 37 (2011) 269-279.

[16] M. Dehghan, M. Hajarian, SSHI methods for solving general linear matrix equations, Engineering Computations, 28 (2012) 1028-1043.

[17] M. Dehghan, M. Hajarian, The generalized Sylvester matrix equations over the generalized bisymmetric and skew-symmetric matrices, International Journal of Systems Science, 43 (2012) 1580-1590.

[18] P. Delsarte, Y. Genin, Spectral properties of finite Toeplitz matrices, in Proceedings of the 1983 International Symposium on Mathematical Theory of Networks and Systems, Beer Sheva, Israel, 1983, Springer- Verlag, Berlin, New York, 1984, 194-213.

[19] F. Ding, T. Chen, Gradient based iterative algorithms for solving a class of matrix equations, IEEE Trans. Autom. Contr. 50 (2005) 1216-1221.

[20] F. Ding, T. Chen, Hierarchical gradient-based identification of multivariable discrete-time systems, Automatica, 41 (2005) 315-325.

[21] F. Ding, T. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, Systems Control Lett. 54 (2005) 95-107.

[22] F. Ding, T. Chen, Hierarchical least squares identification methods for multivariable systems, IEEE Trans. Autom. Contr. 50 (2005) 397-402.

[23] F. Ding, T. Chen, On iterative solutions of general coupled matrix equations, SIAM J. Control Optim. 44 (2006) 2269-2284.

[24] F. Ding, P.X. Liu, J. Ding, Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle, Appl. Math. Comput. 197 (2008) 41-50.

[25] S.W. Director, Circuit Theory: A Computational Approach, John Wiley, New York, 1975.

[26] G.R. Duan, On solution to matrix equation $AV + BW + EVF$, IEEE Transactions on Automatic Control AC-4 (4) (1996) 612-614.

[27] G.H. Golub, C.F. Van Loan, Matrix computations, third ed., The Johns Hopkins University Press, Baltimore and London, 1996.

[28] T. Jiang, M. Wei, On Solutions of the matrix equations $X - AXB = C$ and $X - A\overline{X}B = C$, Linear Algebra Appl. 367 (2003) 225-233.

[29] A. Kılıçman, Z.A.A. Al Zhour, Vector least-squares solutions for coupled singular matrix equations, J. Comput. Appl. Math. 206 (2007) 1051-1069.

[30] J.F. Li, X.Y. Hu, L. Zhang, The submatrix constraint problem of matrix equation $AXB + CYD = E$, Appl. Math. Comput. 215 (2009) 2578-2590.

[31] J.F. Li, X.Y. Hu, L. Zhang,, The nearness problems for symmetric centrosymmetric with a special submatrix constraint, Numerical Algorithms, In Press.

[32] F. Piao, Q. Zhang, Z. Wang, The solution to matrix equation $AX + X^T C = B$, J. Franklin Institute 344 (2007) 1056-1062.

[33] M.A. Ramadan, M.A.A. Naby, A.M.E. Bayoumi, On the explicit solutions of forms of the Sylvester and the Yakubovich matrix equations, Math. Comput. Model. 50 (2009) 1400-1408.

[34] J.P. Thiran, M. Matelart, B.L. Bailly, On the generalized ADI method for the matrix equation $X - AXB = C$, J. Comput. Appl. Math. 156 (2003) 285-302.

[35] M.A. Ramadan, E.A. El-Sayed, On the matrix equation $XH = HX$ and the associated controllability problem, Appl. Math. Comput. 186 (2007) 844-859.

[36] Q.W. Wang, The general solution to a system of real quaternion matrix equations, Comput. Math. Appl. 49 (2005) 665-675.

[37] Q. W. Wang, A system of four matrix equations over von Neumann regular rings and Its applications, Acta Mathematica Sinica, English Series, 21 (2005) 323-334.

[38] Q.W. Wang, Bisymmetric and centrosymmetric solutions to systems of real quaternion matrix equations, Comput. Math. Appl. 49 (2005) 641-650.

[39] Q.W. Wang, H.X. Chang, Q. Ning, The common solution to six quaternion matrix equations with applications, Appl. Math. Comput. 198 (2008) 209-226.

[40] Q.W. Wang, C.K. Li, Ranks and the least-norm of the general solution to a system of quaternion matrix equations, Linear Algebra Appl. 430 (2009) 1626-1640.

[41] Q.W Wang, H.S. Zhang, G.J. Song, A new solvable condition for a pair of generalized Sylvester equations, Electron. J. Linear Algebra 18 (2009) 289-301.

[42] Q.W. Wang, J.W. Woude, H.X. Chang, A system of real quaternion matrix equations with applications, Linear Algebra Appl. 431 (2009) 2291-2303.

[43] J. Weaver, Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues, and eigenvectors, Amer. Math. Monthly 92 (1985) 711-717.

[44] L. Xie, J. Ding, F. Ding, Gradient based iterative solutions for general linear matrix equations, Comput. Math. Appl. 58 (2009) 1441-1448.

[45] D.X. Xie, X.Y. Hu, Y.P. Sheng, The solvability conditions for the inverse eigenproblems of symmetric and generalized centro-symmetric matrices and their approximations, Linear Algebra Appl. 418 (2006) 142-152.

[46] F.Z. Zhou, X.Y. Hu, L. Zhang, The solvability conditions for the inverse eigenvalue problem of generalized centro-symmetric matrices, Linear Algebra Appl. 364 (2003) 147-160.

[47] L. Zhao, X. Hu, L. Zhang, Linear restriction problem of Hermitian reflexive matrices and its approximation, Appl. Math. Comput. 200 (2008) 341-351.

[48] K. Ziętak, On approximation problems with zero-trace matrices, Linear Algebra Appl. 247 (1996) 169-183.

[49] B. Zhou, G.R. Duan, A new solution to the generalized Sylvester matrix equation $AV - EVF = BW$, Systems Control Lett. 55 (2006) 193-198.

[50] B. Zhou, G.R. Duan, Solutions to generalized Sylvester matrix equation by Schur decomposition, Internat. J. Systems Sci. 38 (2007) 369-375.

[51] B. Zhou, J. Lam, G.R. Duan, On Smith-type iterative algorithms for the Stein matrix equation, Appl. Math. Lett. 22 (2009) 1038-1044.

Department of Mathematics, Faculty of Mathematical Sciences,
Shahid Beheshti University, General Campus,
Evin, Tehran 19839, Iran
Email: m_hajarian@sbu.ac.ir; mhajarian@aut.ac.ir; masoudhajarian@ gmail.com

Department of Applied Mathematics
Faculty of Mathematics and Computer Science,
Amirkabir University of Technology,
No.424, Hafez Avenue, Tehran 15914, Iran
Email: mdehghan@aut.ac.ir, mdehghan.aut@gmail.com