# Deficiency of E-valued meromorphic functions

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**Abstract**

The purpose of this paper is to discuss the deficiency of an E-valued meromorphic function $f$. Results are obtained to extend the related results for meromorphic vector valued function of Lahiri and Ziegler.

## 1 Introduction of E-valued meromorphic function

In 1982, Ziegler [14] succeeded in extending the Nevanlinna theory of meromorphic function to the vector-valued meromorphic function in finite dimensional spaces. After Ziegler some works related to vector valued meromorphic function were done in 1990s [5]-[7]. Later, Hu and Yang [4] established the Nevanlinna theory of meromorphic mappings with the range in an infinite-dimensional Hilbert spaces. In 2006, Hu and Hu [3] established the Nevanlinna's first and second main theorems of meromorphic mappings with the range in an infinite-dimensional Banach spaces $E$ with a Schauder basis. Recently, Xuan and Wu [11] established the Nevanlinna's first and second main theorems for an $E$-valued meromorphic mapping from a generic domain $D \subseteq \mathbb{C}$ to an to an infinite-dimensional Banach spaces $E$ with a Schauder basis. For a meromorphic scalar valued function $f(z)$. On the deficiency of $f(z)$ has been studied in Hayman [2], Yang [12] and Zheng [13]. For a meromorphic vector valued function $f(z)$. On the deficiency of $f(z)$ has been studied in Lahiri [5]-[7] and Ziegler [14]. Recently, Bhoosnurmath and Pujari [1] studied the E-Valued Borel Exceptional Values of Meromorphic Functions.But [1], [3] or [11] does not contain deficiency. In this paper, we discuss this problem.

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For the sake of convenience, we introduce some fundamental definition and notation of $E$-valued meromorphic function which was introduced by [1]. [3] and [11].

Let $(E, \| \bullet \|)$ be a complex Banach space with Schauder basis $\{e_j\}$ and the norm $\| \bullet \|$. Thus an $E$-valued meromorphic function $f(z)$ defined in $C_R = \{|z| < R\}, 0 < R \leq +\infty, C_{+\infty} = \mathbb{C}$ can be written as $f(z) = (f_1(z), f_2(z), \cdots, f_k(z), \cdots)$. Let $E_n$ be an $n$-dimensional projective space of $E$ with a basis $\{e_j\}_1^n$. The projective operator $P_n : E \to E_n$ is a realization of $E_n$ associated to the basis.

The elements of $E$ are called vectors and are usually denoted by letters from the alphabet: $a, b, c, \cdots$. The symbol 0 denotes the zero vector of $E$. We denote vector infinity, complex number infinity, and the norm infinity by $\widehat{\infty}, \infty$, and $+\infty$, respectively. A vector-valued mappings is called holomorphic (meromorphic) if all $f_j(z)$ are holomorphic (some of $f_j(z)$ are meromorphic). The $j$th derivative $j = 1, 2, \cdots$ of $f(z)$ are defined by

$$f^{(j)}(z) = (f_1^{(j)}(z), f_2^{(j)}(z), \cdots, f_k^{(j)}(z), \cdots).$$

We assume that $f^{(0)}(z) = f(z)$. A point $z_0 \in C_R$ is called a pole of $f(z) = (f_1(z), f_2(z), \cdots, f_k(z), \cdots)$ if $z_0$ is a pole of at least one of the component functions $f_k(z)(k = 1, 2, \cdots)$. We denote $\|f(z)\| = +\infty$ when $z_0$ is a pole. A point $z_0 \in C_R$ is called a "zero" of $f(z) = (f_1(z), f_2(z), \cdots, f_k(z), \cdots)$ if $z_0$ is a zero of all the component functions $f_k(z)(k = 1, 2, \cdots)$.

Let $n(r, f)$ or $n(r, \widehat{\infty})$ denote the number of poles of $f(z)$ in $|z| \leq r$, and $n(r, a)$ denote the number of $a$-points of $f(z)$ in $|z| \leq r$, counting with multiplicities. Define the volume function associated with $E$-valued meromorphic function $f(z)$

$$V(r, \widehat{\infty}, f) = V(r, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi)\| dx \wedge dy, \quad \xi = x + iy;$$

$$V(r, a) = V(r, a, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi) - a\| dx \wedge dy, \quad \xi = x + iy;$$

and the counting function of finite or infinite $a-$points by

$$N(r, f) = n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt,$$

$$N(r, \widehat{\infty}) = n(0, \widehat{\infty}) \log r + \int_0^r \frac{n(t, \widehat{\infty}) - n(0, \widehat{\infty})}{t} dt,$$

and

$$N(r, a) = n(0, a) \log r + \int_0^r \frac{n(t, a) - n(0, a)}{t} dt.$$

respectively. Next, we define

$$m(r, f) = m(r, \widehat{\infty}, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\| f(re^{i\theta}) \right\| d\theta;$$

$$m(r, a) = m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\|f(re^{i\theta}) - a\|} d\theta;$$

$$T(r, f) = m(r, f) + N(r, f).$$

Let $\overline{n}(r, f)$ or $\overline{n}(r, \widehat{\infty})$ denote the number of poles of $f(z)$ in $|z| \leq r$, and $\overline{n}(r, a)$ denote the number of $a$-points of $f(z)$ in $|z| \leq r$, ignoring multiplicities. Similarly, we can define the counting function $\overline{N}(r, f)$, $\overline{N}(r, \widehat{\infty})$ and $\overline{N}(r, a)$ of $\overline{n}(r, f)$, $\overline{n}(r, \widehat{\infty})$ and $\overline{n}(r, a)$.

If $f(z)$ is an $E$-valued meromorphic function in the whole complex plane, then the order and the lower order of $f(z)$ are defined by

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r};$$

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

We call the E-valued meromorphic function $f$ admissible if

$$\limsup_{r\to+\infty} \frac{T(r, f)}{\log r} = +\infty.$$

**Definition 1.1.** *For a meromorphic function $f(z)$ (E-valued or scalar valued) we denote by $S(r, f)$ any quantity such that*

$$S(r, f) = O(\log T(r, f) + \log r), \ \ r \to +\infty$$

*or*

$$S(r, f) = o(T(r, f)), \ \ r \to +\infty$$

*without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of $r$ of finite linear measure.*

**Definition 1.2.** *(see [3]) An E-valued meromorphic function $f(z)$ in $C_R = \{|z| < R\}, 0 < R \leq +\infty$ is of compact projection, if for any given $\varepsilon > 0$, $\|P_n(f(z)) - f(z)\| < \varepsilon$ has sufficiently large $n$ in any fixed compact subset $D \subset C_R$.*

In 2006, C. G. Hu and Qijian Hu[3] proved the following theorems.

THEOREM A (the E-valued Nevanlinna's first fundamental theorem) Let $f(z)$ be a nonconstant $E$-valued meromorphic function in $C_R$. Then for $0 < r < R, a \in E, f(z) \not\equiv a$,

$$T(r, f) = V(r, a) + N(r, a) + m(r, a) + \log^+ \|c_q(a)\| + \varepsilon(r, a).$$

Here $\varepsilon(r, a)$ is a function such that

$$|\varepsilon(r, a)| \leq \log^+ \|a\| + \log 2, \varepsilon(r, 0) \equiv 0,$$

and $c_q(a) \in E$ is the coefficient of the first term in the Laurent series at the point $a$.

THEOREM B (the E-valued Nevanlinna's second fundamental theorem) Let $f(z)$ be a nonconstant $E$-valued meromorphic function of compact projection in $\mathbb{C}$ and $a^{[k]} \in E(k = 1, 2, \cdots, q)$ be $q \geq 3$ distinct points. Then for $0 < r < R$,

$$(q-2)T(r,f) + G(r,f) \leq \sum_{k=1}^{q}[V(r,a^{[k]}) + \overline{N}(r,a^{[k]})] + S(r,f).$$

and

$$(q-1)T(r,f) + G(r,f) \leq \sum_{k=1}^{q}[V(r,a^{[k]}) + \overline{N}(r,a^{[k]})] + \overline{N}(r,\widehat{\infty}) + S(r,f),$$

where

$$G(r,f) = \int_0^r \frac{1}{2\pi}dt \int_{C_r} \Delta \log \|f(\xi)\| dx \wedge dy, \quad \xi = x + iy;$$

THEOREM C Let $f(z)$ be a nonconstant $E$-valued meromorphic function of compact projection in $\mathbb{C}$. Then the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{\|f'(re^{i\theta})\|}{\|f(re^{i\theta}) - a\|} d\theta = S(r,f), a \in E,$$

holds.

## 2 Deficiency of E-valued meromorphic function

Follow Ziegler [14] or Bhoosnurmath, Pujari [1], we define the Nevanlinna deficient value and deficiency for the E-valued meromorphic function. For any vector $a \in E$, we define the number $\delta(a) = \delta(a, f)$ by putting

$$\delta(a) = \delta(a,f) = \liminf_{r \to +\infty} \frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r \to +\infty} \frac{V(r,a) + N(r,a)}{T(r,f)};$$

$$\delta(\widehat{\infty}) = \delta(\widehat{\infty}, f) = \liminf_{r \to +\infty} \frac{m(r,f)}{T(r,f)} = 1 - \limsup_{r \to +\infty} \frac{N(r,f)}{T(r,f)};$$

and $\Theta(a) = \Theta(a, f)$ by putting

$$\Theta(a) = \Theta(a,f) = 1 - \limsup_{r \to +\infty} \frac{V(r,a) + \overline{N}(r,a)}{T(r,f)};$$

$$\Theta(\widehat{\infty}) = \Theta(\widehat{\infty}, f) = 1 - \limsup_{r \to +\infty} \frac{\overline{N}(r,f)}{T(r,f)}.$$

For any $a \in E$, it is easy to derive

$$0 \leq \delta(a) \leq \Theta(a) \leq 1. \tag{2.1}$$

We also define

$$\delta_G = \liminf_{r \to +\infty} \frac{G(r,f)}{T(r,f)}.$$

**Theorem 2.1.** *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$. Then the set $\{a \in E \cup \{\widehat{\infty}\}, \delta(a) > 0\}$ is at most countable and summing over all such points we have*

$$\sum_a \delta(a) + \delta_G \leq \sum_a \Theta(a) + \delta_G \leq 2.$$

Theorem 2.1 extend the relative result of meromorphic vector valued function in Ziegler [14].

*Proof.* Since $f(z)$ is admissible, then there is a sequence $\{r_v\}$ outside the exceptional set of Definition 1.1 such that

$$\lim_{v \to +\infty} \frac{S(r_v, f)}{T(r_v, f)} = 0.$$

Assume $a^{[k]} \in E (k = 1, 2, \cdots, q)$ are $q \geq 2$ distinct points. In view of Theorem B, we get

$$(q-1)T(r_v, f) + G(r_v, f) \leq \sum_{k=1}^{q} [V(r_v, a^{[k]}) + \overline{N}(r_v, a^{[k]})] + \overline{N}(r_v, \widehat{\infty}) + S(r_v, f).$$

Hence, dividing by $T(r_v, f)$, we get

$$q - 1 \leq \sum_{k=1}^{q} \frac{[V(r_v, a^{[k]}) + \overline{N}(r_v, a^{[k]})]}{T(r_v, f)} + \frac{\overline{N}(r_v, \widehat{\infty})}{T(r_v, f)} + \frac{S(r_v, f)}{T(r_v, f)}.$$

Thus

$$q - 1 + \frac{G(r_v, f)}{T(r_v, f)} \leq \sum_{k=1}^{q} \limsup_{r \to +\infty} \frac{[V(r, a^{[k]}) + \overline{N}(r, a^{[k]})]}{T(r, f)} + \limsup_{r \to +\infty} \frac{\overline{N}(r, \widehat{\infty})}{T(r, f)}.$$

So

$$\sum_{k=1}^{q} \Theta(a^{[k]}) + \Theta(\widehat{\infty}) \leq 2 - \delta_G. \tag{2.2}$$

Also

$$\sum_{k=1}^{q} \delta(a^{[k]}) + \delta(\widehat{\infty}) \leq 2 - \delta_G \leq 2. \tag{2.3}$$

Hence the number of vector in the set $\{a \in E \cup \{\widehat{\infty}\}, \delta(a) > \frac{1}{l}\}$ is at most $2l - 1$. So the set $\{a \in E \cup \{\widehat{\infty}\}, \delta(a) > 0\} = \cup_{l=1}^{+\infty} \{a \in E \cup \{\widehat{\infty}\}, \delta(a) > \frac{1}{l}\}$ is at most countable. Since (2.2) and (2.3) hold for all $q \geq 2$, letting $q \to +\infty$ and combining (2.1) we get

$$\sum_a \delta(a) + \delta_G \leq \sum_a \Theta(a) + \delta_G \leq 2. \qquad \blacksquare$$

It is easy to derive the following result from Theorem 2.1.

**Corallary 2.2.** *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$. Then the set $\{a \in E \cup \{\widehat{\infty}\}, \delta(a) > 0\}$ is at most countable and summing over all such points we have*

$$\sum_a \delta(a) \leq \sum_a \Theta(a) \leq 2.$$

For vector valued transcendental integral function $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))$, Lahiri [5] have prove the following

THEOREM D [5]. Let $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))$ be a vector-valued transcendental integral function (see [5]) of finite order. Then

$$\sum_{a \in \mathbb{C}^n} \delta(a) \leq \delta(0, f').$$

Most recently, Wu and Chen [9] extend Theorem D to admissible meromorphic vector valued function and prove

THEOREM E [9]. Let $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\delta(\infty) = 1$. Then

$$\sum_{a \in \mathbb{C}^n} \delta(a) \leq \delta(0, f').$$

It is natural to consider whether there exists a similar results, if meromorphic vector valued function $f(z)$ is replaced by E-valued meromorphic function $f(z)$. In this following, the main contribution is to extend the above theorem to E-valued meromorphic function.

**Theorem 2.3.** *Let $f(z)$ be a finite order admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and assume $\delta(\widehat{\infty}) = 1$. Then*

$$\sum_{a \in E} \delta(a) \leq \delta(0, f').$$

*Proof.* For any $q \geq 2$ vectors $\{a^{[\mu]}\}$ in $E$, put

$$F(z) = \sum_{j=1}^{q} \frac{1}{\|f(z) - a^{[j]}\|}.$$

We can get

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ F(re^{i\theta}) d\theta \leq m\left(r, 0, f'\right) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \{F(re^{i\theta}) \|f'(re^{i\theta})\|\} d\theta; \quad (2.4)$$

Put

$$\delta = \min_{i \neq j} \|a^{[i]} - a^{[j]}\|.$$

Let for the moment $\mu \in \{1, 2, \cdots, q\}$ be fixed. Then we get in every point where

$$\|f(z) - a^{[\mu]}\| < \frac{\delta}{2q} \leq \frac{\delta}{4}, \quad (2.5)$$

the inequality

$$\|f(z) - a^{[\nu]}\| \geq \|a^{[\mu]} - a^{[\nu]}\| - \|f(z) - a^{[\mu]}\| \geq \frac{3\delta}{4},$$

for $\mu \neq \nu$. Therefore the set of points on $\partial\mathbb{C}_r$ which is determined by (2.5) is either empty or any two such sets for different $\mu$ have empty intersection. In any case

$$\begin{aligned}&\tfrac{1}{2\pi}\int_0^{2\pi}\log^+ F(re^{i\theta})d\theta \geq \tfrac{1}{2\pi}\sum_{\mu=1}^{q}\int_{\|f(z)-a^{[\mu]}\|<\frac{\delta}{2q},|z|=r}\log^+ F(re^{i\theta})d\theta\\ &\geq \tfrac{1}{2\pi}\sum_{\mu=1}^{q}\int_{\|f(z)-a^{[\mu]}\|<\frac{\delta}{2q},|z|=r}\log^+\tfrac{1}{\|f(re^{i\theta})-a^{[\mu]}\|}d\theta.\end{aligned}$$

Because of

$$\begin{aligned}&\tfrac{1}{2\pi}\int_{\|f(z)-a^{[\mu]}\|<\frac{\delta}{2q},|z|=r}\log^+\tfrac{1}{\|f(re^{i\theta})-a^{[\mu]}\|}d\theta\\ &= m(r,a^{[\mu]}) - \tfrac{1}{2\pi}\int_{\|f(z)-a^{[\mu]}\|\geq\frac{\delta}{2q},|z|=r}\log^+\tfrac{1}{\|f(re^{i\theta})-a^{[\mu]}\|}d\theta\\ &\geq m(r,a^{[\mu]}) - \log^+\tfrac{2q}{\delta},\end{aligned}$$

it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+ F(re^{i\theta})d\theta \geq \sum_{\mu=1}^{q} m(r,a^{[\mu]}) - \log^+\frac{2q}{\delta}, \tag{2.6}$$

so that by (2.4)

$$\sum_{\mu=1}^{q} m(r,a^{[\mu]}) \leq m\left(r,0,f'\right) + \frac{1}{2\pi}\int_0^{2\pi}\log^+\{F(re^{i\theta})\|f'(re^{i\theta})\|\}d\theta + \log^+\frac{2q}{\delta}. \tag{2.7}$$

Thus by Theorem C, we have (The proof of the following inequality quoted [3])

$$\sum_{\mu=1}^{q} m(r,a^{[\mu]}) \leq m\left(r,0,f'\right) + S(r,f). \tag{2.8}$$

It follows from Theorem A that

$$T(r,f') = m\left(r,0,f'\right) + N\left(r,0,f'\right) + V\left(r,0,f'\right) + O(1). \tag{2.9}$$

Thus from (2.8) and (2.9) we deduce

$$\sum_{\mu=1}^{q} m(r,a^{[\mu]}) + N\left(r,0,f'\right) + V\left(r,0,f'\right) \leq T\left(r,f'\right) + S(r,f).$$

Therefore, we have

$$\frac{N\left(r,0,f'\right)+V\left(r,0,f'\right)}{T(r,f')} + \frac{T(r,f)}{T(r,f')}\left(\frac{\sum\limits_{\mu=1}^{q} m(r,a^{[\mu]})}{T(r,f)} - o(1)\right) \leq 1, \quad r\to+\infty.$$

On the other hand, one has

$$\begin{aligned} T(r,f') &= m(r,f') + N(r,f') \\ &\le m(r,f) + N(r,f') + \tfrac{1}{2\pi}\int_0^{2\pi}\log^+\tfrac{\|f'(re^{i\theta})\|}{\|f(re^{i\theta})\|}d\theta \\ &\le m(r,f) + N(r,f) + \overline{N}(r,f) + \tfrac{1}{2\pi}\int_0^{2\pi}\log^+\tfrac{\|f'(re^{i\theta})\|}{\|f(re^{i\theta})\|}d\theta \\ &\le T(r,f) + N(r,f) + S(r,f). \end{aligned}$$

Hence

$$\limsup_{r\to+\infty}\frac{T(r,f')}{T(r,f)} \le 2-\delta(\widehat{\infty}).$$

So

$$\begin{aligned} 1 &\ge \limsup_{r\to+\infty}\left[\tfrac{N(r,0,f')+V(r,0,f')}{T(r,f')} + \tfrac{T(r,f)}{T(r,f')}\left(\tfrac{\sum\limits_{\mu=1}^{q} m(r,a^{[\mu]})}{T(r,f)} - o(1)\right)\right] \\ &\ge \limsup_{r\to+\infty}\tfrac{N(r,0,f')+V(r,0,f')}{T(r,f')} + \liminf_{r\to+\infty}\tfrac{T(r,f)}{T(r,f')}\left(\tfrac{\sum\limits_{\mu=1}^{q} m(r,a^{[\mu]})}{T(r,f)} - o(1)\right) \\ &\ge \limsup_{r\to+\infty}\tfrac{N(r,0,f')+V(r,0,f')}{T(r,f')} + \liminf_{r\to+\infty}\tfrac{T(r,f)}{T(r,f')}\liminf_{r\to+\infty}\tfrac{\sum\limits_{\mu=1}^{q} m(r,a^{[\mu]})}{T(r,f)} \\ &\ge \limsup_{r\to+\infty}\tfrac{N(r,0,f')+V(r,0,f')}{T(r,f')} + \tfrac{\sum\limits_{j=1}^{q}\delta\left(a^{[j]}\right)}{2-\delta(\widehat{\infty})}. \end{aligned}$$

Since $q > 0$ were arbitrary and $\delta(\widehat{\infty}) = 1$, we have

$$\sum_{a\in E}\delta(a) \le 1 - \limsup_{r\to+\infty}\frac{N(r,0,f') + V(r,0,f')}{T(r,f')} = \delta(0,f'). \qquad \blacksquare$$

**Example 2.4.** *Put $f(z) = (e^z, e^z, \cdots, e^z, \cdots)$. Then*

$$f^{(j)}(z) = (e^z, e^z, \cdots, e^z, \cdots), \quad j = 1, 2, \cdots.$$

*For any non-zero vector $a \in E$, we have $\delta(a) = 0$, $\delta(\widehat{\infty}) = 1, \delta(0) = 1$ and $\delta(0, f') = 1$. Thus*

$$\sum_{a\in E}\delta(a) \le \delta(0,f').$$

**Corallary 2.5.** *Let $f(z)$ be a finite order admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and assume $\delta(\widehat{\infty}) = 1$. If $f(z)$ has at least one deficient vector $a \in E$, then the vector 0 is the deficient vector of $f'$.*

**Corallary 2.6.** *Let $f(z)$ be a finite order admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and assume $\sum\limits_{a\in E}\delta(a) = 1$ and $\delta(\widehat{\infty}) = 1$. Then $\delta(0, f') = 1$.*

## 3 Relative deficiency of E-valued meromorphic function

The concept of the relative Nevanlinna defect of meromorphic function was due to Milloux [8] and Xiong qinglai (also Hiong qinglai [10]). In 1990, Lahiri [5] extended this concept to meromorphic vector valued function and prove

THEOREM F Let $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))$ be a meromorphic vector valued function and let $a^{[\mu]}(\mu = 1, 2, \cdots, p)$ and $b^{[\lambda]}(\lambda = 1, 2, \cdots, q), q \geq 2$, are elements of $\mathbb{C}^n$, distinct within each set. Then for all positive integers $k$,

$$\Theta(\infty, f) + \sum_{\lambda=1}^{q} \Theta^{(k)}(b^{[\lambda]}, f) + \frac{q-2}{n} \sum_{\mu=1}^{p} \delta(a^{[\mu]}, f) \leq q + 1.$$

where $\Theta^{(k)}(a, f)$ is the relative deficiency (see Lahiri [5]) of the vector $a$ in $\mathbb{C}^n$.

It is natural to consider whether there exists a similar results, if meromorphic vector valued function $f(z)$ is replaced by E-valued meromorphic function $f(z)$. In this section, the main contribution is to extend the above theorem to E-valued meromorphic function.

**Definition 3.1.** *If k is a positive integer then the number*

$$\Theta^{(k)}(a, f) = 1 - \limsup_{r \to +\infty} \frac{V(r, a, f^{(k)}) + \overline{N}(r, a, f^{(k)})}{T(r, f)}$$

*is called the relative deficiency of the value $a \in E$ with respect to distinct zeros.*

**Theorem 3.2.** *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and let $a^{[\mu]}(\mu = 1, 2, \cdots, p)$ and $b^{[\lambda]}(\lambda = 1, 2, \cdots, q), q \geq 2$, are elements of E, distinct within each set. Then for all positive integers k,*

$$\sum_{\lambda=1}^{q} \Theta^{(k)}(b^{[\lambda]}, f) + (q-2) \sum_{\mu=1}^{p} \delta(a^{[\mu]}, f) \leq q.$$

In order to prove Theorem 3.2, we need the following Lemmas.

**Lemma 3.3.** *[3]. If an E-valued meromorphic function $f(z)$ in $\mathbb{C}$ is of compact projection, then $f^{(k)}(z)$ is also of compact projection in $\mathbb{C}$ for all positive integers k.*

**Lemma 3.4.** *[3]. Let $f(z)$ be of compact projection in $\mathbb{C}$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{\|f'(re^{i\theta})\|}{\|f(re^{i\theta})\|} d\theta = O(\log T(r, f) + \log r)$$

*without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of r of finite linear measure.*

**Lemma 3.5.** *[11]. If an E-valued meromorphic function $f(z)$ in $\mathbb{C}$ is of compact projection, then*

$$T(r, f') \leq 2T(r, f) + O(\log T(r, f) + \log r)$$

*without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of r of finite linear measure.*

From Lemma 3.3-3.5, we can get

**Lemma 3.6.** *If an E-valued meromorphic function $f(z)$ in $\mathbb{C}$ is of compact projection, then*

$$S(r, f^{(k)}) = S(r, f),$$

*and*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{\|f^{(k)}(re^{i\theta})\|}{\|f^{(k-1)}(re^{i\theta})\|} d\theta = S(r, f),$$

*holds for all positive integers k.*

**Lemma 3.7.** *Let $f(z)$ be of compact projection in $\mathbb{C}$, then for an positive integers k, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{\|f^{(k)}(re^{i\theta})\|}{\|f(re^{i\theta})\|} d\theta = S(r, f).$$

*Proof.*

$$\begin{aligned}
&\tfrac{1}{2\pi}\int_0^{2\pi} \log^+ \tfrac{\|f^{(k)}(re^{i\theta})\|}{\|f(re^{i\theta})\|} d\theta \\
&= \tfrac{1}{2\pi}\int_0^{2\pi} \log^+ \prod_{\lambda=1}^{k} \tfrac{\|f^{(\lambda)}(re^{i\theta})\|}{\|f^{(\lambda-1)}(re^{i\theta})\|} d\theta \\
&\le \sum_{\lambda=1}^{k} \tfrac{1}{2\pi}\int_0^{2\pi} \log^+ \tfrac{\|f^{(\lambda)}(re^{i\theta})\|}{\|f^{(\lambda-1)}(re^{i\theta})\|} d\theta
\end{aligned}$$

Combining the above inequality and Lemma 3.6, Lemma 3.7 follows. ∎

Now, we are in the position to prove Theorem 3.2.

*Proof.* We set

$$F(z) = \sum_{j=1}^{p} \frac{1}{\|f(z) - a^{[j]}\|}.$$

then

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ F(re^{i\theta}) d\theta \le m\left(r, 0, f^{(k)}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log^+\{F(re^{i\theta})\|f^{(k)}(re^{i\theta})\|\} d\theta.$$

From this and (2.6), we have

$$\sum_{\mu=1}^{p} m(r, a^{[\mu]}) \le m\left(r, 0, f^{(k)}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log^+\{F(re^{i\theta})\|f^{(k)}(re^{i\theta})\|\} d\theta + \log^+ \frac{2q}{\delta}.$$

Hence, we can get from the above and Lemma 3.7 that

$$\sum_{\mu=1}^{p} m(r, a^{[\mu]}) \le m\left(r, 0, f^{(k)}\right) + S(r, f). \tag{3.1}$$

It follows from Theorem A that

$$T(r, f^{(k)}) = m\left(r, 0, f^{(k)}\right) + N\left(r, 0, f^{(k)}\right) + V\left(r, 0, f^{(k)}\right) + O(1). \tag{3.2}$$

Thus from (3.1) and (3.2) we deduce

$$\sum_{\mu=1}^{p} m(r, a^{[\mu]}) \le T\left(r, f^{(k)}\right) + S(r, f).$$

By Theorem A,

$$pT(r,f) \le T\left(r, f^{(k)}\right) + \sum_{\mu=1}^{p}[N(r,a^{[\mu]}) + V(r,a^{[\mu]})] + S(r,f). \tag{3.3}$$

Now it follows from Theorem B and Lemma 3.7 that

$$(q-2)T(r, f^{(k)}) \le \sum_{\lambda=1}^{q}[V(r, b^{[\lambda]}, f^{(k)}) + \overline{N}(r, b^{[\lambda]}, f^{(k)})] + S(r,f). \tag{3.4}$$

Therefore from (3.3) and (3.4) we get

$$\begin{aligned} p(q-2)T(r,f) &\le \sum_{\lambda=1}^{q}[V(r,b^{[\lambda]},f^{(k)}) + \overline{N}(r,b^{[\lambda]},f^{(k)})] \\ &+(q-2)\sum_{\mu=1}^{p}[N(r,a^{[\mu]}) + V(r,a^{[\mu]})] + S(r,f). \end{aligned}$$

Hence

$$\begin{aligned} p(q-2) &\le \sum_{\lambda=1}^{q} \limsup_{r\to+\infty} \frac{[V(r,b^{[\lambda]},f^{(k)})+\overline{N}(r,b^{[\lambda]},f^{(k)})]}{T(r,f)} \\ &+(q-2)\sum_{\mu=1}^{p} \limsup_{r\to+\infty} \frac{[N(r,a^{[\mu]})+V(r,a^{[\mu]})]}{T(r,f)} \\ &= \sum_{\lambda=1}^{q}(1-\Theta^{(k)}(b^{[\lambda]},f)) + (q-2)\sum_{\mu=1}^{p}(1-\delta(a^{[\mu]},f)). \end{aligned}$$

Hence

$$\sum_{\lambda=1}^{q}\Theta^{(k)}(b^{[\lambda]},f) + (q-2)\sum_{\mu=1}^{p}\delta(a^{[\mu]},f) \le q. \qquad \blacksquare$$

**Example 3.8.** *Put $f(z) = (e^z, e^z, \cdots, e^z, \cdots)$. Then*

$$f^{(k)}(z) = (e^z, e^z, \cdots, e^z, \cdots), \quad k = 1, 2, \cdots.$$

*For any non-zero vector $a \in E$, we have $\delta(a,f) = 0$, $\delta(\widehat{\infty}, f) = 1, \delta(0,f) = 1$ and $\Theta^{(k)}(a,f) = \delta(a,f) = 0$, $\Theta^{(k)}(\widehat{\infty},f) = \delta(\widehat{\infty},f) = 1, \Theta^{(k)}(0,f) = \delta(0,f) = 1$. Let $a^{[\mu]}(\mu = 1,2,\cdots,p)$ and $b^{[\lambda]}(\lambda = 1,2,\cdots,q; q \ge 2)$ be elements of E, distinct within each set. Then for all positive integers k, we have*

$$\sum_{\lambda=1}^{q}\Theta^{(k)}(b^{[\lambda]},f) = \sum_{\lambda=1}^{q}\delta(b^{[\lambda]},f) \le 2 - \delta(\widehat{\infty},f) = 1,$$

$$\sum_{\mu=1}^{p}\delta(a^{[\mu]},f) \le 2 - \delta(\widehat{\infty},f) = 1.$$

*Therefore,*

$$\sum_{\lambda=1}^{q} \Theta^{(k)}(b^{[\lambda]}, f) + (q-2) \sum_{\mu=1}^{p} \delta(a^{[\mu]}, f) \leq 1 + (q-2) = q - 1 < q.$$

**Corallary 3.9.** *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and let $a^{[\mu]}(\mu = 1, 2, \cdots, p)$ and $b^{[\lambda]}(\lambda = 1, 2, \cdots, q), q \geq 2$, are elements of E, distinct within each set. Then for all positive integers k,*

$$\Theta(\infty, f) + \sum_{\lambda=1}^{q} \Theta^{(k)}(b^{[\lambda]}, f) + (q-2) \sum_{\mu=1}^{p} \delta(a^{[\mu]}, f) \leq q + 1.$$

**Corallary 3.10.** *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and let $a^{[\mu]}(\mu = 1, 2, \cdots, p)$ and $b^{[\lambda]}(\lambda = 1, 2, \cdots, q), q \geq 2$, are elements of E, distinct within each set. Then for all positive integers k and n,*

$$\Theta(\infty, f) + \sum_{\lambda=1}^{q} \Theta^{(k)}(b^{[\lambda]}, f) + \frac{q-2}{n} \sum_{\mu=1}^{p} \delta(a^{[\mu]}, f) \leq q + 1.$$

**Corallary 3.11.** *Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$,*

$$\frac{1}{q} \sum_{\lambda=1}^{q} \Theta^{(k)}(b^{[\lambda]}, f) \leq 1.$$

*i.e., the mean value of $\Theta^{(k)}(b^{[\lambda]}, f)$'s does not exceed 1.*

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