Secondary Cohomology and k-invariants*

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Abstract

We give a construction that associates to a pointed topological space (X, x_0) a homotopy invariant ${}_2\kappa^4$ which we call the secondary invariant. This construction can be seen a "3-type" generalization of the classical k-invariant.

Introduction

To a pointed topological topological space (X, x_0) one can associate the n-th homotopy group $\pi_n(X)$. It is a well known result that $\pi_n(X)$ is a homotopy invariant for the space X. Moreover one can show that if $f : X \to Y$ induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all $n > 0$, then the map f is a homotopy equivalence. The naive converse of this result is not true; namely there are spaces X and Y that are not homotopy equivalent but have isomorphic homotopy groups. A fundamental problem in homotopy theory is to find under what conditions a family of isomorphism $\phi_i : \pi_i(X) \to \pi_i(Y)$ is induced by a morphism $f : X \to Y$. This is called the realization problem and it was formulated by Whitehead. In general the problem is very difficult and highly unsolved.

The only case where a nice solution exists is for spaces X with $\pi_i(X) = 0$ for all $i > 2$ (these are called spaces of 2-type). The solution is in terms of the first k-invariant introduced by Eilenberg and MacLane in [EM]. The first k-invariant is a homotopy invariant that belongs to $H^3(\pi_1(X), \pi_2(X))$. MacLane and Whitehead have proved in [MW] that equivalence classes of so called crossed modules are in bijection with the elements of the third cohomology group $H^3(G, A)$. They used

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this description to show that the 2-type of a topological space is determined by triples $(\pi_1(X), \pi_2(X), \kappa^3)$, where κ^3 is the k-invariant mentioned above.

A space X with the property that $\pi_i(X) = 0$ for all $i > n$ is called of n-type. After the result of MacLane and Whitehead, there was a shift in the approach for classification of n-types. Postnikov introduced a construction that is now known as Postnikov invariant. Although it provides a complete classification for all n-types, this construction is not very satisfying since it is not intrinsic to the space X and does not have the same algebraic elegance as the first k-invariant.

Spaces of 3-type were classified by Baues [B] in terms of quadratic modules. There are also classifications of spaces of 3-type in terms of 2-crossed modules due to Conduche [C], and crossed squares due to Loday [L]. To our best knowledge there is no description for the 3-type in terms of some cohomology class.

In this paper we propose a construction that associates to a pointed topological space (X, x_0) an invariant ${}_2\kappa^4$ that is an element in a certain cohomology group we introduce. The construction is similar with that of the first k-invariant but also has a Postnikov-invariant flavor. We believe that ${}_2\kappa^4$ is a natural candidate to classify the 3-type of a space.

Here is how the paper is organized. In the first section we recall general facts about the first k-invariant. In the second section, as a warm up, we treat the case of simply connected spaces. More precisely for two commutative groups A and B we introduce the secondary cohomology group ${}_2H^n(A, B)$. Then to a simply connected topological space X we associate a topological invariant ${}_2\kappa^4 \in {}_2H^4(\pi_2(X), \pi_3(X))$. This construction is very similar with the construction of the k-invariant, one just has to go up one dimension. The key result is the definition of the secondary cohomology groups. Having the right cohomology theory, the proof that ${}_2\kappa^4$ is an invariant is almost cut and paste from [EM].

In the third section we give the result for general topological spaces. We start with a group G (possibly noncommutative), two G-modules A and B, a 3-cocycle $\kappa \in H^3(G, A)$ and we define ${}_2H^4(G, A, \kappa; B)$ the secondary cohomology of the triple (G, A, κ) with coefficients in B. One can then associate to any pointed space (X, x_0) a topological invariant ${}_2\kappa^4 \in {}_2H^4(\pi_1(X), \pi_2(X), \kappa^3; \pi_3(X))$. Obviously if X is simply connected we get the invariant ${}_2\kappa^4$ from section two. Also if $\pi_2(X) = 0$ we get the second k-invariant $\kappa^4 \in H^4(\pi_1(X), \pi_3(X))$ introduced in [EM]. As an application we give an algebraic description of the cohomology group $H^3(X, K^*)$. We conclude the paper with remarks on possible generalizations and research problems in this direction.

1 Preliminaries

We recall from [EM] and [EM1] some notations about group cohomology and the construction of the first k-invariant.

Let G be a group and A a G-module. We set $C^n(G, A) = \{\sigma : G^n \to A\}$ and define $\partial_n : C^n(G, A) \to C^{n+1}(G, A)$ by

$$\partial_n(\sigma)(g_1, ..., g_{n+1}) = g_1\sigma(g_2, ..., g_{n+1}) - \sigma(g_1g_2, g_3, ..., g_{n+1}) + ... + \\ +(-1)^n\sigma(g_1, ..., g_ng_{n+1}) + (-1)^{n+1}\sigma(g_1, ..., g_n).$$

In this way we obtain a chain complex. Its homology groups are denoted by $H^n(G, A)$ and are called the cohomology of G with coefficients in A.

Let X be an arc connected topological space with base point x_0. For each element $\alpha \in \pi_1(X)$ we fix a representative $r(\alpha)$. For each pair of elements $(\alpha, \beta) \in \pi_1(X) \times \pi_1(X)$ we consider a singular 2-simplex $r(\alpha, \beta) : \Delta_2 \to X$ such that the edges $[0,1]$, $[1,2]$ and $[0,2]$ map according to $r(\alpha)$, $r(\beta)$ and $r(\alpha\beta)$. For α, β, $\gamma \in \pi_1(X)$ we define a map $R(\alpha, \beta, \gamma) : \partial(\Delta_3) \to X$ such that $R_{|[0,1,2]} = r(\alpha, \beta)$, $R_{|[1,2,3]} = r(\beta, \gamma)$, $R_{|[0,2,3]} = r(\alpha\beta, \gamma)$, and $R_{|[0,1,3]} = r(\alpha, \beta\gamma)$. In this way we get an element of $\kappa(\alpha, \beta, \gamma) \in \pi_2(X)$.

Theorem 1.1. *[EM] The cochain $(\alpha, \beta, \gamma) \to \kappa(\alpha, \beta, \gamma)$ is a cocycle. A change of the representatives $r(\alpha)$ and $r(\alpha, \beta)$ alters κ by a coboundary. Thus κ determines a unique cohomology class $\kappa^3 \in H^3(\pi_1(X), \pi_2(X))$ which is a topological invariant of (X, x_0).*

If X is a space with the property that $\pi_i(X) = 0$ for $1 < i < n$, then the above construction can generalized to obtain an invariant $\kappa^{n+1} \in H^{n+1}(\pi_1(X), \pi_n(X))$. The element κ^{n+1} is called the $(n-1)$-th k-invariant.

2 The simply connected case

2.1 Secondary cohomology for commutative groups

In this section A and B are commutative groups. For A we use multiplicative notation while for B we use the additive notation. Define ${}_2C^n(A, B) = Map(A^{\frac{n(n-1)}{2}}, B)$. The elements of $A^{\frac{n(n-1)}{2}}$ are $\frac{n(n-1)}{2}$-tuples $(a_{i,j})_{(0 \le i < j \le n-1)}$ with the index in the lexicographic order:

$$(a_{0,1}, a_{0,2}, ..., a_{0,n-1}, a_{1,2}, a_{1,3}, ..., a_{1,n-1}, ..., a_{n-2,n-1})$$

For every $0 \le k \le n+1$ we define $d_n^k : A^{\frac{(n+1)n}{2}} \to A^{\frac{n(n-1)}{2}}$, $d_n^k((a_{i,j})_{(0 \le i < j \le n)}) = (b_{i,j})_{(0 \le i < j \le n-1)}$ where

$$b_{i,j} = \begin{cases} a_{i,j} \text{ if } 0 \le i < j < k-1 \\ a_{i,k-1} a_{i,k} a_{k-1,k}^{-1} \text{ if } 0 \le i < j = k-1 \\ a_{i,j+1} \text{ if } 0 \le i \le k-1 < j \\ a_{i+1,j+1} \text{ if } k-1 < i < j \end{cases}$$

One can check that

$$d_{n-1}^k d_n^l = d_{n-1}^{l-1} d_n^k \text{ if } k < l$$

Let $\delta_n : {}_2C^n(A, B) \to {}_2C^{n+1}(A, B)$ defined by:

$$\delta_n(f) = f d_n^0 - f d_n^1 + f d_n^2 - ... + (-1)^{n+1} f d_n^{n+1} \tag{2.1}$$

Example 2.1. When $n = 2$, $n = 3$ or $n = 4$, and $f \in {}_2C^n(A, B)$ we have

$$\delta_2(f)(a_{01}, a_{02}, a_{12}) = f(a_{12}) - f(a_{02}) + f(a_{01}a_{02}a_{12}^{-1}) - f(a_{01})$$

$$\begin{aligned}&\delta_3(f)(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}) = f(a_{12}, a_{13}, a_{23}) - f(a_{02}, a_{03}, a_{23})\\&+f(a_{01}a_{02}a_{12}^{-1}, a_{03}, a_{13}) - f(a_{01}, a_{02}a_{03}a_{23}^{-1}, a_{12}a_{13}a_{23}^{-1}) + f(a_{01}, a_{02}, a_{12})\end{aligned}$$

$$\begin{aligned}&\delta_4(f)(a_{01}, a_{02}, a_{03}, a_{04}, a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}) =\\&f(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}) - f(a_{02}, a_{03}, a_{04}, a_{23}, a_{24}, a_{34})\\&+f(a_{01}a_{02}a_{12}^{-1}, a_{03}, a_{04}, a_{13}, a_{14}, a_{34})\\&-f(a_{01}, a_{02}a_{03}a_{23}^{-1}, a_{04}, a_{12}a_{13}a_{23}^{-1}, a_{14}, a_{24})\\&+f(a_{01}, a_{02}, a_{03}a_{04}a_{34}^{-1}, a_{12}, a_{13}a_{14}a_{34}^{-1}, a_{23}a_{24}a_{34}^{-1})\\&-f(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})\end{aligned}$$

It is obvious that $\delta_{n+1}\delta_n(f) = 0$ for all $n \geq 1$ and all $f \in {}_2C^n(A, B)$, which means that we have a complex $({}_2C^*(A, B), \delta_*)$.

Definition 2.2. We denote the homology of $({}_2C^*(A, B), \delta_*)$ by ${}_2H^*(A, B)$ and we call it the **secondary cohomology** of the group A with coefficients in B.

2.2 The secondary k-invariant for simply connected spaces

Let X be a simply connected topological space. For any element $a \in \pi_2(X)$ we fix a map $r(a) : \Delta_2 \to X$ that represents a (notice that $r(a)_{|\partial(\Delta_2)} = x_0$). For each a_{01}, a_{02} and $a_{12} \in \pi_2(X)$ we fix a singular 3-simplex $r(a_{01}, a_{02}, a_{12}) : \Delta_3 \to X$ such that $r(a_{01}, a_{02}, a_{12})_{|[0,1,2]} = r(a_{01})$, $r(a_{01}, a_{02}, a_{12})_{|[0,2,3]} = r(a_{02})$, $r(a_{01}, a_{02}, a_{12})_{|[1,2,3]} = r(a_{12})$ and $r(a_{01}, a_{02}, a_{12})_{|[0,1,3]} = r(a_{01}a_{02}a_{12}^{-1})$. For each a_{01}, a_{02}, a_{03}, a_{12}, a_{13} and $a_{23} \in \pi_2(X)$ we define:

$$R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}) : \partial(\Delta_4) \to X \tag{2.2}$$

such that the restriction of $R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})$ on each of the 3-simplices that make the boundary of Δ_4 is given by:

$$\begin{aligned}R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})_{|[0,1,2,3]} &= r(a_{01}, a_{02}, a_{12})\\R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})_{|[0,1,2,4]} &= r(a_{01}, a_{02}a_{03}a_{23}^{-1}, a_{12}a_{13}a_{23}^{-1})\\R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})_{|[0,1,3,4]} &= r(a_{01}a_{02}a_{12}^{-1}, a_{03}, a_{13})\\R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})_{|[0,2,3,4]} &= r(a_{02}, a_{03}, a_{23})\\R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})_{|[1,2,3,4]} &= r(a_{12}, a_{13}, a_{23})\end{aligned}$$

It is obvious that $R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})$ determines a unique element of $\pi_3(X)$, and so $R \in {}_2C^4(\pi_2(X), \pi_3(X))$.

Let's see that R is a cocycle. Take a_{01}, a_{02}, a_{03}, a_{04}, a_{12}, a_{13}, a_{14}, a_{23} a_{24} and $a_{34} \in \pi_2(X)$. We notice that there exists a map F from the 3-dimensional skeleton of Δ_5 to X such that:

$$\begin{aligned}
F_{|\partial([0,1,2,3,4])} &= R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}) \\
F_{|\partial([0,1,2,3,5])} &= R(a_{01}, a_{02}, a_{03}a_{04}a_{34}^{-1}, a_{12}, a_{13}a_{14}a_{34}^{-1}, a_{23}a_{24}a_{34}^{-1}) \\
F_{|\partial([0,1,2,4,5])} &= R(a_{01}, a_{02}a_{03}a_{23}^{-1}, a_{04}, a_{12}a_{13}a_{23}^{-1}, a_{14}, a_{24}) \\
F_{|\partial([0,1,3,4,5])} &= R(a_{01}a_{02}a_{12}^{-1}, a_{03}, a_{04}, a_{13}, a_{14}, a_{34}) \\
F_{|\partial([0,2,3,4,5])} &= R(a_{02}, a_{03}, a_{04}, a_{23}, a_{24}, a_{34}) \\
F_{|\partial([1,2,3,4,5])} &= R(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})
\end{aligned}$$

Moreover each 3-simplex of Δ_5 appears exactly twice (once for each orientation) in the following element of $\pi_3(X)$.

$$\begin{aligned}
&R(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}) - R(a_{02}, a_{03}, a_{04}, a_{23}, a_{24}, a_{34}) \\
&+R(a_{01}a_{02}a_{12}^{-1}, a_{03}, a_{04}, a_{13}, a_{14}, a_{34}) \\
&-R(a_{01}, a_{02}a_{03}a_{23}^{-1}, a_{04}, a_{12}a_{13}a_{23}^{-1}, a_{14}, a_{24}) \\
&+R(a_{01}, a_{02}, a_{03}a_{04}a_{34}^{-1}, a_{12}, a_{13}a_{14}a_{34}^{-1}, a_{23}a_{24}a_{34}^{-1}) \\
&-R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})
\end{aligned}$$

This means that $\delta_4(R) = 0$ and so $R \in {}_2C^4(\pi_2(X), \pi_3(X))$ is a 4-cocycle.

If we keep fixed $r(a)$ and we change $r(a, b, c)$ with another map $r'(a, b, c)$ we get a map $h : \pi_2(X) \times \pi_2(X) \times \pi_2(X) \to \pi_3(X)$ (by gluing r and r' along the boundary). One can see that:

$$\begin{aligned}
&R(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}) - R'(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}) \\
&= h(a_{12}, a_{13}, a_{23}) - h(a_{02}, a_{03}, a_{23}) + h(a_{01}a_{02}a_{12}^{-1}, a_{03}, a_{13}) \\
&-h(a_{01}, a_{02}a_{03}a_{23}^{-1}, a_{12}a_{13}a_{23}^{-1}) + h(a_{01}, a_{02}, a_{12})
\end{aligned}$$

And so R and R' are cohomologous equivalent. If we change $r(a)$ with $r'(a)$ then we can chose $r'(a, b, c)$ such that the two maps $R, R' : \pi_2(X)^6 \to \pi_3(X)$ are equal. This prove that R defines an unique element ${}_2\kappa^4 \in {}_2H^4(\pi_2(X), \pi_3(X))$ that is a topological invariant of (X, x_0).

3 The general case

3.1 Secondary cohomology of (G, A, κ) with coefficients in B

We want to generalize the above results to topological spaces with $\pi_1(X)$ non-trivial. First we need to construct an analog for ${}_2H^4(A, B)$. We start with a group G (possibly noncommutative), two G-modules A and B and a 3-cocycle $\kappa \in Z^3(G, A)$. Define ${}_2C^n(G, A, \kappa; B) = Map(G^n \times A^{\frac{n(n-1)}{2}}, B)$. The elements of G^n are n-tuples

$$(g) = (g_i)_{(1 \leq i \leq n)}$$

The elements of $A^{\frac{n(n-1)}{2}}$ are $\frac{n(n-1)}{2}$-tuples $(a) = (a_{i,j})_{(0\leq i<j\leq n-1)}$ with the index in the lexicographic order:

$$(a) = (a_{0,1}, a_{0,2}, ..., a_{0,n-1}, a_{1,2}, a_{1,3}, ..., a_{1,n-1}, ..., a_{n-2,n-1})$$

For every $0 \leq k \leq n+1$ we define $d_n^k : G^{n+1} \times A^{\frac{n(n+1)}{2}} \to G^n \times A^{\frac{(n-1)n}{2}}$, $d_n^k((g_i)_{(1\leq i\leq n+1)}, (a_{i,j})_{(0\leq i<j\leq n)}) = ((h_i)_{(1\leq i\leq n)}, (b_{i,j})_{(0\leq i<j\leq n-1)})$ where

$$h_i = \begin{cases} g_i \text{ if } i<k \\ g_i g_{i+1} \text{ if } i=k \\ g_{i+1} \text{ if } k<i \end{cases}$$

$$b_{i,j} = \begin{cases} a_{i,j} \text{ if } 0\leq i<j<k-1 \\ a_{i,k-1} a_{i,k}{}^{g_{i+1}...g_{k-1}}(a_{k-1,k}^{-1})\kappa(g_{i+1}...g_{k-1}, g_k, g_{k+1}) \text{ if } 0\leq i<j=k-1 \\ a_{i,j+1} \text{ if } 0\leq i\leq k-1<j \\ a_{i+1,j+1} \text{ if } k-1<i<j \end{cases}$$

Let $\delta_n^\kappa : {}_2C^n(G,A,\kappa;B) \to {}_2C^{n+1}(G,A,\kappa;B)$ defined by:

$$\begin{aligned}\delta_n^\kappa(f)((g);(a)) &= g_1 f d_n^0((g);(a)) - f d_n^1((g);(a)) \\ &+ f d_n^2((g);(a)) - ... + (-1)^{n+1} f d_n^{n+1}((g);(a))\end{aligned}$$

Example 3.1. For $n=2$ or $n=3$ and $f \in {}_2C^n(G,A,\kappa;B)$ we have:

$$\begin{aligned}&\delta_2^\kappa(f)(g_1,g_2,g_3;a_{01},a_{02},a_{12}) = g_1 f(g_2,g_3;a_{12}) - f(g_1g_2,g_3;a_{02}) \\ &+ f(g_1,g_2g_3;a_{01}a_{02}{}^{g_1}(a_{12}^{-1})\kappa(g_1,g_2,g_3)) - f(g_1,g_2;a_{01})\end{aligned}$$

$$\begin{aligned}&\delta_3^\kappa(f)(g_1,g_2,g_3,g_4;a_{01},a_{02},a_{03},a_{12},a_{13},a_{23}) \\ &= g_1 f(g_2,g_3,g_4;a_{12},a_{13},a_{23}) - f(g_1g_2,g_3,g_4;a_{02},a_{03},a_{23}) \\ &+ f(g_1,g_2g_3,g_4;a_{01}a_{02}{}^{g_1}(a_{12}^{-1})\kappa(g_1,g_2,g_3),a_{03},a_{13}) \\ &- f(g_1,g_2,g_3g_4;a_{01},a_{02}a_{03}{}^{g_1g_2}(a_{23}^{-1})\kappa(g_1g_2,g_3,g_4),a_{12}a_{13}{}^{g_2}(a_{23}^{-1})\kappa(g_2,g_3,g_4)) \\ &+ f(g_1,g_2,g_3;a_{01},a_{02},a_{12})\end{aligned}$$

One can check that $\delta_{n+1}^\kappa \delta_n^\kappa(f) = 0$ for all $f \in {}_2C^n(G,A,\kappa;B)$, and so we have a complex $({}_2C^*(G,A,\kappa;B), \delta_*^\kappa)$.

Definition 3.2. We denote the homology of the complex $({}_2C^*(G,A,\kappa;B), \delta_*^\kappa)$ by ${}_2H^*(G,A,\kappa;B)$ and we call it the **secondary cohomology** of (G,A,κ) with coefficients in B.

Remark 3.3. Let's notice that the above construction depends only on the class of $\kappa \in H^3(G,A)$. Indeed if $\kappa = \kappa' + \delta_2(u)$ then there is an isomorphism of complexes $\Phi_u : {}_2C^*(G,A,\kappa;B) \to {}_2C^*(G,A,\kappa';B)$ defined by:

$$\Phi_u(f)((g);(a)) = f((g);(c)) \tag{3.1}$$

where $c_{i,j} = a_{i,j} u(g_{i+1}...g_{j-1}, g_j)$. One can see that

$$\delta^{\kappa'}\Phi_u = \Phi_u \delta^\kappa \tag{3.2}$$

$$\Phi_u \Phi_v = \Phi_{u+v} \tag{3.3}$$

And so Φ is a natural transformation that allows us to identify ${}_2H^*(G, A, \kappa; B)$ with ${}_2H^*(G, A, \kappa'; B)$.

Example 3.4. If A is trivial then ${}_2H^n(G, 1, \kappa; B)$ is the usual cohomology $H^n(G, B)$. If G is trivial then ${}_2H^*(1, A, \kappa; B)$ is the secondary cohomology group ${}_2H^n(A, B)$ defined in the previous section. Also it is easy to show that ${}_2H^n(1; B) = 0$ and ${}_2H^n(A; 0) = 0$.

Example 3.5. Simple computations show that

$${}_2H^2(\mathbb{Z}_2, B) = \{(b_1, b_2) \in B \times B |\ 2b_1 = 2b_2\} / \{(b, b) | b \in B\}$$

$${}_2H^3(\mathbb{Z}_2, B) = B/2B$$

For example one has ${}_2H^2(\mathbb{Z}_2, \mathbb{Z}) = 0$, ${}_2H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, ${}_2H^3(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2$ and ${}_2H^3(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$.

Next we will mention some functorial properties of the secondary cohomology. First we need to introduce the category of algebraic 2-types. An object in this category is a triple (G, A, κ) where G is a group, A is a G-module and $\kappa \in Z^3(G, A)$ is a 3-cocycle. A morphism between (G_1, A_1, κ) and (G_2, A_2, λ) is a pair (u, v) where $u : G_1 \to G_2$ and $v : A \to B$ are morphisms of groups with the property that $v(ga) = u(g)v(a)$ and the 3-cocycles $v(\kappa)$ and $\lambda(u, u, u)$ are equivalent as elements in $Z^3(G_1, A_2)$. On can show that in this way we obtain a category and that a morphism (u, v) is an isomorphism if and only if u and v are isomorphisms. An object in this category was called an "algebraic 3-type" in [MW], but considering the change (over the years) in terminology for the n-type of a topological space our notation seams appropriate. With this notations one can easily check the following result:

Proposition 3.1. *i) If B is an abelian group, then ${}_2H^n(*; B)$ is a contravariant functor from the category of algebraic 2-types to the category of abelian groups.*
ii) If (G, A, κ) is an algebraic 3-type, then ${}_2H^n(G, A, \kappa; *)$ is a covariant functor from the category of abelian groups to the category of abelian groups.
iii) For every morphism (u, v) from (G_1, A_1, κ) to (G_2, A_2, λ) and any G_2-module B we have a morphism $(u, v)^* : {}_2H^n(G_2, A_2, \lambda; B) \to {}_2H^n(G_1, A_1, \kappa; B)$ where the G_1-module structure on B is induced by u.

In [CCG], the authors introduced a cohomology theory for crossed modules (and implicitly for algebraic 3-types). Since their coefficients are abelian crossed modules, it reasonable to believe that an appropriate specialization of that theory will be equivalent with the secondary cohomology described in this paper. However the construction in [CCG] is not very explicit and we had some trouble to pinpoint the precise connection with our theory.

3.2 Secondary k-invariant

Let (X, x_0) be a pointed topological space. For each $\alpha \in \pi_1(X)$ we fix a representative $r(\alpha) : [0,1] \to X$. For each pair of elements α, $\beta \in \pi_1(X)$ we fix a singular 2-simplex $r(\alpha, \beta) : \Delta_2 \to X$ such that $[0,1]$, $[1,2]$ and $[0,2]$ map according to $r(\alpha)$, $r(\beta)$ and $r(\alpha\beta)$. Just like in the construction of the k-invariant define a map $R(\alpha, \beta, \gamma) : \partial(\Delta_3) \to X$ such that $R_{|[0,1,2]} = r(\alpha, \beta)$, $R_{|[1,2,3]} = r(\beta, \gamma)$, $R_{|[0,2,3]} = r(\alpha\beta, \gamma)$, and $R_{|[0,1,3]} = r(\alpha, \beta\gamma)$. This gives us the classical k-invariant.

For each triple $(\alpha, \beta; a) \in \pi_1(X) \times \pi_1(X) \times \pi_2(X)$ we consider a singular 2-simplex $r(\alpha, \beta; a) : \Delta_2 \to X$ such that $[0,1]$, $[1,2]$ and $[0,2]$ map according to $r(\alpha)$, $r(\beta)$ and $r(\alpha\beta)$ and when we glue $r(\alpha, \beta; a)$ with $r(\alpha, \beta)$ along the boundary we get $a \in \pi_2(X)$. For each $(g) = (g_1, g_2, g_3) \in \pi_1(X)^3$ and $(a) = (a_{01}, a_{02}, a_{12}) \in \pi_2(X)^3$ we fix a singular 3-simplex $r((g); (a)) = r(g_1, g_2, g_3; a_{01}, a_{02}, a_{12}) : \Delta_3 \to X$ such that:

$$\begin{aligned}
r((g);(a))_{|[0,1,2]} &= r(g_1, g_2; a_{01}) \\
r((g);(a))_{|[0,2,3]} &= r(g_1g_2, g_3; a_{02}) \\
r((g);(a))_{|[1,2,3]} &= r(g_2, g_3; a_{12}) \\
r((g);(a))_{|[0,1,3]} &= r(g_1, g_2g_3; a_{01}a_{02}\,{}^{g_1}(a_{12}^{-1})\kappa(g_1, g_2, g_3))
\end{aligned}$$

For each $(g) = (g_1, g_2, g_3, g_4) \in \pi_1(X)^4$ and $(a) = (a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}) \in \pi_2(X)^6$ we define:

$$R((g);(a)) = R(g_1, g_2, g_3, g_4; a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}) : \partial(\Delta_4) \to X \tag{3.4}$$

such that the restriction of $R((g); (a))$ on each of the five 3-simplices that make the boundary of Δ_4 is given by:

$$\begin{aligned}
&R((g);(a))_{|[0,1,2,3]} = r(g_1, g_2, g_3; a_{01}, a_{02}, a_{12}) \\
&R((g);(a))_{|[0,1,2,4]} = r(g_1, g_2, g_3g_4; a_{01}, a_{02}a_{03}\,{}^{g_1g_2}(a_{23}^{-1})\kappa(g_1g_2, g_3, g_4), \\
&a_{12}a_{13}\,{}^{g_2}(a_{23}^{-1})\kappa(g_2, g_3, g_4)) \\
&R((g);(a))_{|[0,1,3,4]} = r(g_1, g_2g_3, g_4; a_{01}a_{02}\,{}^{g_1}(a_{12}^{-1})\kappa(g_1, g_2, g_3), a_{03}, a_{13}) \\
&R((g);(a))_{|[0,2,3,4]} = r(g_1g_2, g_3, g_4; a_{02}, a_{03}, a_{23}) \\
&R((g);(a))_{|[1,2,3,4]} = r(g_2, g_3, g_4; a_{12}, a_{13}, a_{23})
\end{aligned}$$

It is obvious that $R(g_1, g_2, g_3, g_4; a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})$ determines a unique element of $\pi_3(X)$, and so $R \in {}_2C^4(\pi_1(X), \pi_2(X), \kappa; \pi_3(X))$.

Just like in the case X simply connected one can show that R is a 4-cocycle, i.e. $R \in {}_2Z^4(\pi_1(X), \pi_2(X), \kappa; \pi_3(X))$.

We want to show that the class of $R \in {}_2H^4(\pi_1(X), \pi_2(X), \kappa; \pi_3(X))$ does not depend on the choices we made. First we keep fixed $r(\alpha)$ and $r(\alpha, \beta)$ and $r(\alpha, \beta; a)$ and we change $r(\alpha, \beta, \gamma; a, b, c)$ with another map $r'(\alpha, \beta, \gamma; a, b, c)$ we get a map $h : \pi_1(X)^3 \times \pi_2(X)^3 \to \pi_3(X)$ (by gluing r and r' along the boundary). One can see that:

$$R - R' = \delta_3^{\kappa}(h)$$

and so R and R' are cohomologous equivalent. If we change $r(\alpha,\beta;a)$ with $r'(\alpha,\beta;a)$ then we can chose $r'(\alpha,\beta,\gamma;a,b,c)$ such that the two maps $R, R' : \pi_2(X)^6 \to \pi_3(X)$ are equal. If we change $r(\alpha,\beta)$ with $r'(\alpha,\beta))$ we replace κ with some $\kappa' = \kappa - \delta_2(u)$ which gives an isomorphism like in (3.1). If we fix $r(\alpha,\beta;a)$ and $r(\alpha,\beta,\gamma;a,b,c)$ then the 4-cocycle $R' \in {}_2Z^4(\pi_1(X),\pi_2(X),\kappa';\pi_3(X))$ becomes $\Phi_u(R)$. Finally if we change $r(\alpha)$ with $r'(\alpha)$ we can chose all the other r' such that R does not change. We have the following result.

Theorem 3.6. *If (X, x_0) is a topological space then the above construction defines a topological invariant ${}_2\kappa^4 = R \in {}_2H^4(\pi_1(X),\pi_2(X),\kappa;\pi_3(X))$.*

Remark 3.7. If $\pi_2(X) = 0$ then ${}_2\kappa^4$ is the element κ^4 described in the first remark from section 5 in [EM] (see also [EM2]).

Remark 3.8. Notice that the definition of the secondary cohomology is induced by how we add elements in $\pi_2(\Delta^3_2, \Delta^3_1, x_0)$ (where Δ^3_r is the r-skeleton of Δ^3). On the other hand this is essentially equivalent with the homotopy addition lemma (see for example [B] or [BR]). This connection (and its higher dimensional analog) was unknowingly used in [S1] to give a new explicit description for the simplicial group $K(A,2)$ (respectively for $K(A,n)$).

3.3 Description of the third cohomology group of a space X

We recall from [EM] the description of $H^2(X,K^*)$ in terms of $\pi_1(X)$, $\pi_2(X)$ and κ^3. Let $u \in C^2(\pi_1(X),K^*)$ and $v \in Hom(\pi_2(X),K^*)$ such that

$$(\delta_2(u))(g,h,k) = v(\kappa(g,h,k))$$

$$v({}^g a) = v(a)$$

Let $H^2(\pi_1,\pi_2,\kappa;K^*)$ be the quotient group of all pairs (u,v) that satisfy the above relations by the subgroup of all pairs $(\delta_1(p),1)$ where $p \in C^1(\pi_1(X),K^*)$. It was proved in [EM] that for a space X the second cohomology group $H^2(X,K^*)$ is isomorphic with $H^2(\pi_1(X),\pi_2(X),\kappa;K^*)$.

We will give a similar description of $H^3(X,K^*)$ in terms of $\pi_1(X)$, $\pi_2(X)$, $\pi_3(X)$, κ^3 and ${}_2\kappa^4$.

For $\zeta \in H^3(X,K^*)$, the natural morphism $\pi_3(X) \to H_3(X)$ induces a group morphism $v : \pi_3(X) \to K^*$. One can notice that $v({}^\alpha m) = v(m)$ for all $\alpha \in \pi_1(X)$ and $m \in \pi_3(X)$. Also for α, β and $\gamma \in \pi_1(X)$ and a, b and $c \in \pi_2(X)$ we define $u : \pi_1(X)^3 \times \pi_2(X)^3 \to K^*$ by

$$u(\alpha,\beta,\gamma,a,b,c) = \zeta(r(\alpha,\beta,\gamma,a,b,c))$$

Let $(g) = (g_1,g_2,g_3,g_4) \in \pi_1(X)^4$ and $(a) = (a_{01},a_{02},a_{03},a_{12},a_{13},a_{23}) \in \pi_2(X)^6$ we have

$$\delta_3^\kappa(u)((g),(a)) = v({}_2\kappa^4((g),(a)))$$

We define $H^3(\pi_1(X),\pi_2(X),\kappa^3,\pi_3(X),{}_2\kappa^4;K^*)$ the quotient group of all pairs (u,v) that satisfy the above relations by the subgroup of all pairs $(\delta_2^\kappa(p),1)$ where $p \in {}_2C^2(\pi_1(X),\pi_2(X),\kappa;K^*)$. One can show that we have:

Theorem 3.9. $H^3(X,K^*) \simeq H^3(\pi_1(X),\pi_2(X),\kappa^3,\pi_3(X),\,{}_2\kappa^4;K^*)$

Remark 3.10. If $\pi_3(X) = 0$ we get $H^3(X,K^*) \simeq {}_2H^3(\pi_1(X),\pi_2(X),\kappa;K^*)$ and so $H^3(X,K^*)$ can be explicitly described in terms of $\pi_1(X)$, $\pi_2(X)$ and κ^3. This was also noticed in [EM] with the comment "The algebraic constructions involved are quite cumbersome". We hope that this paper makes the construction look more natural.

Example 3.11. When $X = S^2$ one has $\pi_1(S^2) = 1$, $\pi_2(S^2) = \pi_3(S^2) = \mathbb{Z}$. If we assume that ${}_2\kappa^4$ is trivial then we get that $H^3(S^2) \neq 0$ which is not true. This implies that the secondary invariant ${}_2\kappa^4$ associated to S^2 is not trivial.

3.4 Conclusions and Remarks

Having a cohomological invariant that gives information about the n-type of a space can be very useful. For example, in the case of 2-types, the existence of the the first k-invariant κ^3 allows us to avoid the formalism from the definition of crossed modules, but still work in a purely algebraic framework. This was used in [ST] to study 2-dimensional HQFT's, see also [PT] for the approach with crossed modules. We hope that the construction described in this paper will have similar applications for problems involving 3-types.

A natural question is whether the invariant ${}_2\kappa^4$ classify the 3-type of a space. This problem turned out to be much more difficult then one expects (however we are still optimistic about it). A possible approach is to show that equivalences classes of quadratic modules are in bijection with elements of the secondary cohomology group ${}_2H^4(G,A,\kappa;B)$ for the appropriate G, A, κ and C.

If the the above question has a positive answer one could try to define a ternary cohomology group ${}_3H^n(G,A,\kappa,B,\,{}_2\kappa;C)$. Then find a cohomology class ${}_3\kappa^5 \in {}_3H^5(\pi_1(X),\pi_2(X),\kappa^3,\pi_3(X),\,{}_2\kappa^4;\pi_4(X))$ that classify the 4-type of a space, and so on. We can notice that we have a short exact sequence of complexes:

$$0 \to C^*(G,B) \to {}_2C^*(G,A,\kappa;B) \to {}_2C^*(A,B) \to 0$$

This suggest that at the next level we should have:

$$0 \to {}_2C^*(G,A,\kappa;C) \to {}_3C^*(G,A,\kappa,B,\,{}_2\kappa;C) \to {}_3C^*(B,C) \to 0$$

In general we expect that the cohomology theory at step n is a twist between the cohomology from step $n-1$ with an appropriate cohomology theory that depends only on two groups. A first step in this direction was made in [S1] where it was proved that the secondary cohomology ${}_2H^n(A,*)$ corresponds to the simplicial group $K(A,2)$ the same way the usual cohomology $H^n(G,*)$ corresponds to the simplicial group $K(G,1)$. The general case of the secondary cohomology ${}_2H^n(G,A,\kappa;*)$ is obtained using a κ^3-twist between $K(G,1)$ and $K(A,2)$. This is similar with the results from [M] and is also the reason way we said in introduction that our construction has a Postnikov-invariant flavor.

Finally, notice that when we prove $\delta_4\delta_3 = 0$ we use an equality of the type

$$f(f(a_{01},a_{0,2},a_{12}),a_{03},a_{13}) = f(a_{01},f(a_{02},a_{03},a_{23}),f(a_{12},a_{13},a_{23})) \tag{3.5}$$

where $f : A \times A \times A \to A$, $f(a,b,c) = abc^{-1}$. The identity (3.5) is almost the same as the ternary associativity condition discussed in [S]. With the notations from that paper one can take $f(a,b,c) = m(a,Q(c),b)$ and check that f satisfy condition (3.5).

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