A Phragmén - Lindelöf principle for slice regular functions

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1   Introduction

The celebrated 100-year old Phragmén-Lindelöf theorem, [19, 20], is a far reaching extension of the maximum modulus theorem for holomorphic functions. In its simplest form, it can be stated as follows:

Theorem 1.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain whose boundary contains the point at infinity. If f is a bounded holomorphic function on Ω and $\limsup_{z \to z_0} |f(z)| \leq M$ at each finite boundary point z_0, then $|f(z)| \leq M$ for all $z \in \Omega$.*

The term Phragmén-Lindelöf also applies to a number of variations of this result, which guarantee a bound for holomorphic functions, when conditions are known on their growth. The two most famous variations deal with functions which are holomorphic in an angle or in a strip, and they can be stated as follows (see, for instance, [4, 17] as well as [1, 12]).

Theorem 1.2. *Let f be a holomorphic function on an angle Ω of opening $\frac{\pi}{\alpha}$. Suppose f is continuous up to the boundary and such that, for some $\rho < \alpha$, $|f(z)| \leq \exp(|z|^\rho)$ asymptotically. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial\Omega$ then $|f| \leq M$ in Ω.*

Theorem 1.3. *Let f be a holomorphic function on a strip Ω of width 2γ, continuous up to the boundary. Suppose that $|f(z)| \leq N \exp(e^{k|z|})$ in Ω for some positive constants N and $k < \frac{\pi}{2\gamma}$. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial\Omega$ then $|f| \leq M$ in Ω.*

In some recent papers [13, 14, 15, 16] there has been a resurgence of interest in Phragmén-Lindelöf type theorems. Specifically, [13, 14] consider solutions of suitable partial differential equations while [15, 16] deal with the case of functions of a hypercomplex variable. In the present article we obtain the analogs of theorems 1.1, 1.2 and 1.3 for slice regular functions, a class of functions of a quaternionic variable introduced in [9, 10] and studied in subsequent papers (e.g. [2, 3, 7, 8, 21] and references therein).

In section 2 we will provide the necessary background about the theory of slice regular quaternionic functions. In section 3 we give direct proofs of quaternionic analogs of theorems 1.1 and 1.2. Finally, in section 4 we use a different approach, which exploits the intrinsic nature of slice regular functions, to extend our results.

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2 Preliminaries

Let $\mathbb{H}$ denote the skew field of quaternions. Its elements are of the form $q = x_0 + ix_1 + jx_2 + kx_3$ where the x_l are real, and i, j, k are such that

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

We set

$$Re(q) = x_0, \qquad Im(q) = ix_1 + jx_2 + kx_3, \qquad |q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

$Re(q)$, $Im(q)$ and $|q|$ are called the *real part*, the *imaginary part* and the *module* of q, respectively. The quaternion

$$\bar{q} = Re(q) - Im(q) = x_0 - ix_1 - jx_2 - kx_3$$

is called the *conjugate* of q and it satisfies

$$|q| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q}.$$

The *inverse* of any element $q \neq 0$ is given by

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

We denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

so that every quaternion q which is not real (i.e. with $Im(q) \neq 0$) can be written as $q = x + Iy$ for $x = Re(q), y = |Im(q)|$ and $I = \frac{Im(q)}{|Im(q)|} \in \mathbb{S}$. Also, if we set $r = |q|$ then $q = re^{I\vartheta}$ for some $I \in \mathbb{S}$ and $\vartheta \in \mathbb{R}$.

Beginning with the seminal papers of Fueter [5, 6], many mathematicians have developed theories of holomorphicity in the quaternionic setting (for an overview, see the introduction of [8]). More recently, beginning with [9, 10], the authors proposed a new notion of holomorphicity (called *slice regularity*) for quaternion-valued functions of a quaternionic variable. Unlike Fueter's, this theory includes the polynomials and the power series of the quaternionic variable q of the type $\sum_{n \geq 0} q^n a_n$, with coefficients $a_n \in \mathbb{H}$. Furthermore, analogs of most of the fundamental properties of holomorphic functions of one complex variable were proven in this new setting (e.g. [2, 3, 7, 8, 21] and references therein).

Definition 2.1. *Let Ω be an open set in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is said to be* slice regular *if, for every $I \in \mathbb{S}$, its restriction f_I to the complex line $L_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and I is C^1 and it satisfies*

$$\overline{\partial}_I f(x + yI) := \frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0,$$

in $\Omega_I = \Omega \cap L_I$.

The following result of [10], which appears in its most general form in [8], is a key tool in the study of slice regular functions, and it will be used extensively in section 4.

Lemma 2.2 (Splitting Lemma)**.** *If f is a slice regular function on an open set Ω then, for every $I \in \mathbb{S}$ and every $J \perp I$ in $\mathbb{S}$, there exist two holomorphic functions $F, G : \Omega_I \to L_I$ such that*

$$f_I(z) = F(z) + G(z)J$$

for all $z \in \Omega_I$.

The following type of domains, introduced in [2, 3, 21], naturally qualify as domains of definition of regular functions.

Definition 2.3. *Let Ω be a domain in $\mathbb{H}$. We say that Ω is a* slice domain *if $\Omega \cap \mathbb{R}$ is non empty and if $\Omega_I = \Omega \cap L_I$ is a domain in L_I for all $I \in \mathbb{S}$.*

Indeed, an analog of the identity principle holds for slice regular functions on slice domains. It was proven in [10] and, in its most general form, in [21].

Theorem 2.4. *Let $f, g : \Omega \to \mathbb{H}$ be slice regular functions on a slice domain Ω. If f and g coincide in $T \subseteq \Omega$ and if there exists $I \in \mathbb{S}$ such that $T_I = T \cap L_I$ has an accumulation point in Ω_I, then f and g coincide in Ω.*

Furthermore, an analog of the maximum modulus principle holds (see [7]).

Theorem 2.5. *Let $f : \Omega \to \mathbb{H}$ be a slice regular function on a slice domain Ω. If $|f|$ has a relative maximum point in Ω, then f is constant in Ω.*

Theorem 2.5 immediately implies the next corollary.

Corollary 2.6. *Let $f : \Omega \to \mathbb{H}$ be a slice regular function on a bounded slice domain Ω. If $\limsup_{q \to q_0} |f(q)| \leq M$ for all $q_0 \in \partial\Omega$ then $|f| \leq M$ in Ω.*

Due to the non-commutativity of $\mathbb{H}$, pointwise multiplication and composition do not preserve slice regularity in general. Nevertheless, slice regularity is preserved for the following class of functions.

Definition 2.7. *Let $f : \Omega \to \mathbb{H}$ be a slice regular function. We say that f is a* slice preserving *function if $f(\Omega_I) \subseteq L_I$ for all $I \in \mathbb{S}$.*

Indeed, the following propositions can be proven by direct computation.

Proposition 2.8. *Let $f, g : \Omega \to \mathbb{H}$ be slice regular functions. If f is a slice preserving function then the product $f \cdot g$ is slice regular.*

Proposition 2.9. *Let $f : \Omega \to \Omega' \subseteq \mathbb{H}$ and $g : \Omega' \to \mathbb{H}$ be slice regular functions. If f is a slice preserving function then the composition $g \circ f$ is slice regular.*

3   The Phragmén - Lindelöf principle

In this section we will give a direct proof of the Phragmén-Lindelöf principle for slice regular functions defined on suitable domains Ω in the quaternionic space $\mathbb{H}$. We will also study the special case in which Ω is a cone.

As in the complex case, the quaternionic Phragmén-Lindelöf principle generalizes the maximum modulus principle 2.6 to unbounded domains. Let $\widehat{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ denote the Alexandroff compactification of $\mathbb{H}$. We define the *extended boundary* $\partial_\infty \Omega$ of any $\Omega \subseteq \widehat{\mathbb{H}}$ to be the boundary of the closure of Ω in $\widehat{\mathbb{H}}$. As customary, we will denote by $\partial \Omega = \partial_\infty \Omega \setminus \{\infty\}$ the finite boundary of Ω.

Theorem 3.1 (Phragmén-Lindelöf principle). *Let $\Omega \subset \mathbb{H}$ be a domain whose extended boundary contains the point at infinity and suppose that there exists a real point $t \in \mathbb{R} \cap \Omega$ such that $\Omega \setminus (-\infty, t]$ (or $\Omega \setminus [t, +\infty)$) is a slice domain. If f is a bounded slice regular function on Ω and $\limsup_{q \to q_0} |f(q)| \leq M$ for all $q_0 \in \partial \Omega$, then $|f(q)| \leq M$ for all $q \in \Omega$.*

Proof. Since $q \mapsto q + t$ and $q \mapsto -q$ are slice preserving functions, by proposition 2.9 we can assume that $t = 0$ and $\Omega \setminus (-\infty, 0]$ is a slice domain. Choose $r > 0$ such that the closure of $B_r = B(0, r)$ is contained in Ω and let $\omega_r(q) = q^{-1}r$ for $q \neq 0$. Notice that $|\omega_r| < 1$ in $\mathbb{H} \setminus \overline{B_r}$, that $|\omega_r| = 1$ on ∂B_r, and that ω_r is a slice preserving regular function (see [11] and references therein).

For all $q \in \mathbb{H} \setminus (-\infty, 0]$, define the principal logarithm of q as

$$Log(q) = \ln |q| + \arccos \left(\frac{Re(q)}{|q|} \right) \frac{Im(q)}{|Im(q)|}.$$

Notice that the principal logarithm is a slice regular function and it is slice preserving. By proposition 2.9, setting $\omega_r^\delta(q) := e^{\delta Log(\omega_r(q))}$ for all $q \in \mathbb{H} \setminus (-\infty, 0]$ defines a slice regular function. Finally, by proposition 2.8 the product $\omega_r^\delta f$ is a slice regular function on $\Omega' = \Omega \setminus (\overline{B_r} \cup (-\infty, r])$, which by hypothesis is a slice domain when r is sufficiently small. The behavior of $|\omega_r^\delta f|$ on the extended boundary $\partial_\infty \Omega' = \{\infty\} \cup \partial \Omega \cup \partial B_r \cup (\Omega \cap (-\infty, r])$ is the following:

1. $\limsup_{q\to\infty} |\omega_r^\delta f(q)| = \limsup_{q\to\infty} |f(q)| \frac{r^\delta}{|q|^\delta} = 0,$

2. $\limsup_{q\to q_0} |\omega_r^\delta f(q)| < \limsup_{q\to q_0} |f(q)| \leq M$ for all $q_0 \in \partial\Omega,$

3. $\limsup_{q\to q_0} |\omega_r^\delta f(q)| = |f(q_0)| \leq \max_{\partial B_r} |f| =: M_r$ for all $q_0 \in \partial B_r,$

4. $\limsup_{q\to q_0} |\omega_r^\delta f(q)| \leq \sup_{\Omega\cap(-\infty,r]} \left(\frac{r^\delta}{|q|^\delta} |f(q)| \right) =: N$ for all $q_0 \in \Omega \cap (-\infty, r].$

Let us prove that N is finite. Choose $\{a_n\}_{n\in\mathbb{N}} \subset \overline{\Omega} \cap (-\infty, r]$ such that $\lim_{n\to\infty} \frac{r^\delta}{|a_n|^\delta} |f(a_n)| = N$. If $N = 0$, there is nothing to prove. Otherwise, by point 1, $\{a_n\}_{n\in\mathbb{N}}$ must be bounded. By possibly extracting a subsequence, we may suppose $\{a_n\}_{n\in\mathbb{N}}$ to converge to some $q_0 \in \overline{\Omega} \cap (-\infty, r]$. If $q_0 \in \partial\Omega$ then $N \leq M$ by hypothesis. Else $q_0 \in \Omega$ and $N = \frac{r^\delta}{|q_0|^\delta} |f(q_0)|$.

As a consequence of points 1-4, $\limsup_{q\to q_0} |\omega_r^\delta f| \leq \max\{M, M_r, N\}$ for all $q_0 \in \partial_\infty \Omega'$ and, by an easy application of the maximum modulus principle 2.6, $|\omega_r^\delta f| \leq \max\{M, M_r, N\}$ in Ω'.

Now let us prove that $N \leq \max\{M, M_r\}$. Suppose by contradiction that the opposite inequality holds. In particular $N > M$ and (as we explained above) there exists $q_0 \in \Omega \cap (-\infty, r])$ such that $N = \frac{r^\delta}{|q_0|^\delta} |f(q_0)|$. In a ball $B(q_0, \varepsilon)$ contained in $\Omega \setminus \overline{B_r}$, we define a new branch of logarithm log by letting

$$log(q) = \ln|q| + \left[\arccos\left(\frac{Re(q)}{|q|} \right) - \pi \right] \frac{Im(q)}{|Im(q)|}.$$

As before, the function $g = e^{\delta log \omega_r} f$ is slice regular in $B(q_0, \varepsilon)$ and $|g(q)| = \frac{r^\delta}{|q|^\delta} |f(q)|$ for all $q \in B(q_0, \varepsilon)$. As a consequence, $|g(q_0)| \geq |g(q)|$ for all $q \in B(q_0, \varepsilon)$. Indeed:

1. for all $q \in (q_0 - \varepsilon, q_0 + \varepsilon)$, $|g(q)| \leq \sup_{\Omega\cap(-\infty,r]} \left(\frac{r^\delta}{|q|^\delta} |f(q)| \right) = N = |g(q_0)|$;

2. for all $q \in B(q_0, \varepsilon) \setminus (q_0 - \varepsilon, q_0 + \varepsilon)$, we proved $|g(q)| = |\omega_r^\delta(q) f(q)| = \frac{r^\delta}{|q|^\delta} |f(q)| \leq \max\{M, M_r, N\} = N = |g(q_0)|$.

Hence $|g|$ has a maximum at q_0 and g must be constant. Therefore $|\omega_r^\delta f| = |g| \equiv N$ in $B(q_0, \varepsilon) \setminus (q_0 - \varepsilon, q_0 + \varepsilon)$. In particular $\omega_r^\delta f$, which is a slice regular function on the slice domain Ω', has an interior maximum point. As before, the maximum modulus principle 2.6 yields that $\omega_r^\delta f$ must be constant. As a consequence, there exists a constant c such that $f(q) = q^\delta c$ in Ω', a contradiction with the hypothesis that f is bounded.

So far, we proved that $|\omega_r|^\delta |f| \leq \max\{M, M_r\}$ in $\Omega \setminus \overline{B_r}$. We deduce that

$$|f| \leq \frac{\max\{M, M_r\}}{|\omega_r|^\delta}$$

in $\Omega \setminus \overline{B_r}$ and letting $\delta \to 0^+$ we conclude that $|f| \leq \max\{M, M_r\}$ in $\Omega \setminus \overline{B_r}$.

If we let $r \to 0^+$ we obtain $|f(q)| \leq \max\{M, |f(0)|\}$ for all $q \in \Omega \setminus \{0\}$, hence for all $q \in \Omega$. Finally, we prove that $|f(0)| \leq M$: if it were not so, then $|f|$ would have a maximum at 0, a contradiction by the maximum modulus principle 2.5. ∎

We now tackle the case in which Ω is the *circular cone*

$$C(\varphi) = \{re^{I\vartheta} : r > 0, |\vartheta| < \varphi/2, I \in \mathbb{S}\}$$

for some $\varphi < 2\pi$. Such cones certainly satisfy the hypotheses of theorem 3.1. Moreover, we can prove that it is not necessary to suppose that f is bounded as long as the *opening* φ of the cone is suitably related to the growth order of f, defined as follows. If f is a slice regular function on $\Omega = C(\varphi)$, continuous up to the boundary, we set

$$M_f(r, \Omega) = \max\{|f(q)| : q \in \overline{\Omega}, |q| = r\} \tag{1}$$

and define the *order* ρ of f as

$$\rho = \limsup_{r \to +\infty} \frac{\ln^+ \ln^+ M_f(r, \Omega)}{\ln r}. \tag{2}$$

Theorem 3.2 (Phragmén-Lindelöf principle for circular cones). *Let f be a slice regular function in $C\left(\frac{\pi}{\alpha}\right)$, continuous up to the boundary. Suppose the order ρ of f to be strictly less than α. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial C\left(\frac{\pi}{\alpha}\right)$ then $|f| \leq M$ in $C\left(\frac{\pi}{\alpha}\right)$.*

Proof. Choose γ such that $\rho < \gamma < \alpha$. For $q \in C\left(\frac{\pi}{\alpha}\right)$ we define $\omega(q) = e^{-q^\gamma}$ with $q^\gamma = e^{\gamma Log(q)}$ (where $Log(q)$ is the principal logarithm of q). For all $\delta > 0$ we set $\omega^\delta(q) = e^{-\delta q^\gamma}$ and we have that

$$|\omega^\delta(re^{I\vartheta})| = e^{-\delta r^\gamma \cos(\gamma\vartheta)}.$$

For $-\frac{\pi}{2\alpha} < \vartheta < \frac{\pi}{2\alpha}$ and $\rho < \rho_1 < \gamma$ the following holds asymptotically:

$$|\omega^\delta(re^{I\vartheta})f(re^{I\vartheta})| < e^{r^{\rho_1} - \delta r^\gamma \cos(\gamma\vartheta)}.$$

Since $\gamma < \alpha$, we have $-\frac{\pi}{2} < \gamma\vartheta < \frac{\pi}{2}$ so that $\cos(\gamma\vartheta) > 0$. Since $\rho_1 < \gamma$ we conclude that in $C\left(\frac{\pi}{\alpha}\right)$

$$\lim_{q \to \infty} |\omega^\delta(q)f(q)| = 0.$$

Since for all $q \in \partial C\left(\frac{\pi}{\alpha}\right)$ we have $|\omega^\delta(q)f(q)| < |f(q)| \leq M$, we conclude that $\lim_{q \to q_0} |\omega^\delta(q)f(q)| \leq M$ for all $q_0 \in \partial_\infty C\left(\frac{\pi}{\alpha}\right)$. Applying the maximum modulus principle 2.6, we get $|\omega^\delta f| \leq M$ in $C\left(\frac{\pi}{\alpha}\right)$. The inequality $|f| \leq \frac{M}{|\omega|^\delta}$, which holds for all $\delta > 0$, yields that $|f| \leq M$ in $C\left(\frac{\pi}{\alpha}\right)$. ∎

In the next section we will offer an alternative proof extending theorem 3.2 to a larger class of domains.

4 A slicewise approach

The proofs we gave in the previous section are intrinsic to the quaternionic setting. However, in the theory of slice regular functions it is often possible to use a different technique. Specifically, one can apply the splitting lemma 2.2 to reduce to the case of holomorphic functions of one complex variable. In this section, we employ this technique to give different proofs of theorems 3.1 and 3.2. In fact, this allows slight variations of the hypotheses. We also prove two other results, which we were unable to obtain using a direct approach.

Theorem 4.1. *Let $\Omega \subset \mathbb{H}$ be a domain whose extended boundary contains the point at infinity and such that, for all $I \in \mathbb{S}$, $\Omega_I = \Omega \cap L_I$ is simply connected. If f is a bounded slice regular function on Ω and $\limsup_{q \to q_0} |f(q)| \leq M$ for all $q_0 \in \partial\Omega$, then $|f(q)| \leq M$ for all $q \in \Omega$.*

Proof. Suppose that there exists $p \in \Omega$ such that $|f(p)| > M$. Possibly multiplying f by the constant $\frac{\overline{f(p)}}{|f(p)|}$, we may suppose $f(p) > 0$. Let $I \in \mathbb{S}$ be such that $p \in L_I$, choose $J \in \mathbb{S}$ such that $J \perp I$ and let $F, G : \Omega_I = \Omega \cap L_I \to L_I$ be holomorphic functions such that $f_I = F + GJ$ (see lemma 2.2). Then $f_I(p) = F(p)$. On the other hand, since $|F| \leq |f_I| \leq M$ in $\partial\Omega_I \subseteq \partial\Omega$ and $|F| \leq |f_I|$ is bounded in Ω_I, we must have $|F| \leq M$ in Ω_I by the complex Phragmén-Lindelöf principle 1.1. ∎

Remark 4.2. *We required Ω_I to be simply connected, as in the classic (complex) Phragmén-Lindelöf principle 1.1. Notice however that this hypothesis can be weakened (see exercise 1 in [4]).*

A similar proof allows us to extend theorem 3.2 to a larger class of domains.

Definition 4.3. *We call a slice domain $\Omega \subset \mathbb{H}$ an* angular domain *if, for all $I \in \mathbb{S}$, Ω_I is an angle $\{re^{I(\zeta_I + \vartheta)} : r > 0, |\vartheta| < \varphi_I/2\}$ for some ζ_I, φ_I with $\zeta_I \in \mathbb{R}, 0 < \varphi_I < 2\pi$. The* opening *of Ω is defined to be $\sup_{I \in \mathbb{S}} |\varphi_I|$.*

The following proposition shows, once again, the surprising geometrical properties of the quaternions.

Proposition 4.4. *Let Ω be an open subset of $\mathbb{H}$ such that, for all $I \in \mathbb{S}$, Ω_I is an angle $\{re^{I(\zeta_I + \vartheta)} : r > 0, |\vartheta| < \varphi_I/2\}$. If $I \mapsto \zeta_I$ and $I \mapsto \varphi_I$ are continuous in $\mathbb{S}$ then Ω is automatically a slice domain.*

Proof. In order to prove our assertion it suffices to show that at least one slice Ω_I contains a real half line. Notice that, for all $I \in \mathbb{S}$, $\Omega_I = \Omega_{-I}$. Hence $\varphi_I = \varphi_{-I}$ and $\zeta_I = 2k\pi - \zeta_{-I}$ for some $k \in \mathbb{Z}$. Since

$$I \mapsto (\zeta_I, \varphi_I)$$

is a continuous function from $\mathbb{S} \simeq S^2$ to $\mathbb{R}^2$, by the Borsuk-Ulam theorem (see Corollary 9.3 in [18]) there exist two antipodal points of $\mathbb{S}$ having the same image. In particular, there exists a $J \in \mathbb{S}$ such that $\zeta_J = \zeta_{-J}$ and we conclude that $\zeta_J = k\pi$ for some $k \in \mathbb{Z}$. ∎

The *order* of a slice regular function f on an angular domain Ω is defined by equations (1) and (2) as in the case of circular cones.

Theorem 4.5. *Let f be a slice regular function on an angular domain Ω of opening $\frac{\pi}{\alpha}$. Suppose f is continuous up to the boundary and has order $\rho < \alpha$. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial\Omega$ then $|f| \leq M$ in Ω.*

The proof of theorem 4.5 is completely analogous to that of theorem 4.1 and it makes use of the Phragmén-Lindelöf principle for complex angles 1.2. As in the complex case, the hypothesis that $\rho < \alpha$ cannot be weakened. Indeed, we have the following fact.

Example 4.6. *We can define a slice regular function f of order $\rho > 0$ on $C\left(\frac{\pi}{\rho}\right)$ by setting $f(q) = e^{q^\rho}$ where $q^\rho = e^{\rho Log(q)}$. We notice that, for all $q = re^{\pm I\frac{\pi}{2\rho}} \in \partial C\left(\frac{\pi}{\rho}\right)$, $|f(q)| = |\exp(r^\rho e^{\pm I\frac{\pi}{2}})| = |\exp(\pm Ir^\rho)| = 1$, while the function f is unbounded in $C\left(\frac{\pi}{\rho}\right)$.*

Nevertheless, as in the complex case, when a function f has order ρ in angular domain of opening $\frac{\pi}{\rho}$ we can control the growth of f in terms of its *type* σ, defined as

$$\sigma = \limsup_{r \to +\infty} \frac{\ln^+ M_f(r, \Omega)}{r^\rho}. \tag{3}$$

Theorem 4.7. *Let Ω be an angular domain with $\Omega_I = \{re^{I(\zeta_I + \vartheta)} : r > 0, |\vartheta| < \varphi_I/2\}$ for all $I \in \mathbb{S}$. Let f be a slice regular function of order ρ and type σ on Ω, continuous up to the boundary. If the opening of Ω is not greater than $\frac{\pi}{\rho}$ and $|f|$ is bounded by M in $\partial\Omega$, then for all $I \in \mathbb{S}$*

$$|f(re^{I(\zeta_I + \vartheta)})| \leq Me^{\sigma r^\rho \cos(\rho\vartheta)} \tag{4}$$

for $r > 0$ and $|\vartheta| < \varphi_I/2$.

Proof. Suppose that there exists $p = re^{I(\zeta_I + \vartheta)} \in \Omega_I$ such that

$$|f(p)| > Me^{\sigma r^\rho \cos(\rho\vartheta)}.$$

As in the proof of theorem 4.1, we may suppose $f(p) > 0$. Choosing $J \in \mathbb{S}$ with $J \perp I$ and holomorphic functions $F, G : \Omega_I \to L_I$ such that $f_I = F + GJ$, we have $f_I(p) = F(p)$. Now, $|F| \leq |f_I| \leq M$ in $\partial\Omega_I$ and F has order less than or equal to ρ and type less than or equal to σ in Ω_I. By theorem 22 in [17], we conclude $|F(p)| \leq Me^{\sigma r^\rho \cos(\rho\vartheta)}$, a contradiction. ∎

An analogous proof allows us to derive the quaternionic version of theorem 1.3.

Definition 4.8. *We call a slice domain $\Omega \subset \mathbb{H}$ a strip domain if, for all $I \in \mathbb{S}$, there exist a line ℓ_I in L_I and a positive real number γ_I such that Ω_I is the strip $\{z \in L_I : |z - \ell_I| < \gamma_I/2\}$. The width of Ω is defined to be $\sup_{I \in \mathbb{S}} |\gamma_I|$.*

Theorem 4.9. *Let f be a slice regular function on a strip domain Ω of width γ, continuous up to the boundary. Suppose that $|f(q)| \leq N \exp(e^{k|q|})$ in Ω for some positive constants N and $k < \frac{\pi}{\gamma}$. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial\Omega$ then $|f| \leq M$ in Ω.*

The slicewise approach adopted in this section has an important bearing for entire functions, i.e. for slice regular functions on $\Omega = \mathbb{H}$. Indeed, if we define the order and the type of the entire function f by equations (1), (2) and (3), then the quaternionic Liouville theorem proven in [10] generalizes as follows.

Theorem 4.10. *Let f be a quaternionic entire function of first order at most, having type 0. In other words, for all $\varepsilon > 0$ we suppose $|f(q)| < e^{\varepsilon|q|}$ when $|q|$ is large enough. If, for some $I \in \mathbb{S}$, the plane L_I contains a line on which $|f|$ is bounded then f is constant.*

Proof. Let $J \in \mathbb{S}$ be orthogonal to I and let $F, G : L_I \to L_I$ be holomorphic functions such that $f_I = F + GJ$. By the corollary to theorem 22 in [17], F and G are constant. Hence f_I is constant and, by the identity principle 2.4, f must be constant too. ∎

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