Sharp Hardy-type inequalities with Lamb's constants

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Abstract

Let Ω be an n-dimensional convex domain with finite inradius $\delta_0 = \sup_{x\in\Omega} \delta$, where $\delta = dist(x, \partial\Omega)$, and let (p, q) be a pair of positive numbers. For functions vanishing at the boundary of the domain and any $\nu \in [0, p/q]$ we prove the following Hardy-type inequality

$$\int_\Omega \frac{|\nabla f|^2}{\delta^{p-1}}\, dx \;\geq\; h\int_\Omega \frac{|f|^2}{\delta^{p+1}}\, dx + \frac{\lambda^2}{\delta_0^q}\int_\Omega \frac{|f|^2}{\delta^{p-q+1}}\, dx$$

with two sharp constants

$$h = \frac{p^2 - \nu^2 q^2}{4} \geq 0 \qquad \text{and} \qquad \lambda = \frac{q}{2}\lambda_\nu(2p/q) > 0,$$

where $z = \lambda_\nu(p)$ is the Lamb constant defined as the first positive root of the equation $pJ_\nu(z) + 2zJ'_\nu(z) = 0$ for the Bessel function J_ν. We prove that $z = \lambda_\nu(p)$ as a function in p can be found as the solution of an initial value problem for the differential equation

$$\frac{dz}{dp} = \frac{2z}{p^2 - 4\nu^2 + 4z^2}.$$

For $n = 1$ our inequality is an improvement of the original Hardy inequality for finite intervals. For $n \geq 1$ and $p = q/2 = 1$ it gives a new sharp

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form of the Hardy-type inequality due to H. Brezis and M. Marcus. The case $h = 0$, $\nu = 1/2$, $p = 1$ and $q = 2$ coincides with sharp eigenvalue estimates due to J. Hersch for $n = 2$, and L. E. Payne and I. Stakgold for $n \geq 3$.

1 Introduction

The aim of this paper is to obtain a new sharp Hardy-type inequality which constructs a bridge between Hardy-type inequalities of the classical form and sharp estimates of the first eigenvalue $\lambda_1(\Omega)$ of the Laplacian under the Dirichlet boundary condition for n-dimensional convex domains Ω.

Let Ω be an open set in the Euclidean space $\mathbb{R}^n$. There are two famous results on sharp estimates of the first eigenvalue. The first one is the Rayleigh-Faber-Krahn isoperimetric inequality (see, for instance, C. Bandle [5])

$$\lambda_1(\Omega) \ \geq \ \frac{\omega_n^{2/n}}{(vol\ (\Omega))^{2/n}} j_{n/2-1}^2,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and j_ν is the first positive zero of the Bessel function J_ν of order ν. The second result concerns convex domains of finite inradius δ_0 defined as

$$\delta_0 = \delta_0(\Omega) = \sup_{x \in \Omega} \ \delta,$$

where

$$\delta = \ dist(x, \partial\Omega).$$

Namely, for any n-dimensional convex domain there is the sharp inequality

$$\lambda_1(\Omega) \geq \frac{\pi^2}{4\delta_0^2(\Omega)}. \tag{1}$$

For $n = 1$ it follows from the Poincaré estimate $\lambda_1(\Omega) \geq \pi^2/(diam\ (\Omega))^2$, for $n = 2$ the inequality (1) is due to J. Hersch [9], for $n \geq 3$ it is proved by L. E. Payne and I. Stakgold [16]. The inequality (1) means that

$$\int_\Omega |\nabla f|^2\, dx \ \geq \ \frac{\pi^2}{4\delta_0^2(\Omega)} \int_\Omega |f|^2\, dx, \quad \forall f \in H_0^1(\Omega), \tag{2}$$

where Ω is an open and convex set in $\mathbb{R}^n$, the space $H_0^1(\Omega)$ is the closure of the family $C_0^1(\Omega)$ of smooth functions $f : \Omega \to \mathbb{R}$ with finite Dirichlet integral and supported in Ω. On the other hand, for n-dimensional convex domains there are the following Hardy-type inequalities

$$\int_\Omega |\nabla f|^2\, dx \ \geq \ \frac{1}{4} \int_\Omega \frac{|f|^2}{\delta^2}\, dx, \quad \forall f \in H_0^1(\Omega), \tag{3}$$

and

$$\int_\Omega |\nabla f|^2\, dx \;\geq\; \frac{1}{4}\int_\Omega \frac{|f|^2}{\delta^2}\, dx + \frac{1}{4\,(diam\,(\Omega))^2}\int_\Omega |f|^2\, dx, \quad \forall f \in H_0^1(\Omega). \tag{4}$$

It is well known that the constant $1/4$ in (3) is sharp for any convex subdomain of $\mathbb{R}^n$ although there is no function $f \not\equiv 0,\ f \in H_0^1(\Omega)$ for which equality in (3) is actually attained. The sharpness of $1/4$ is proved by Hardy for $n = 1$ (see [8] and [12]) and by T. Matskewich and P. E. Sobolevskii [15] and by M. Marcus, V. J. Mitzel and Y. Pinchover [14] for $n \geq 2$. The inequality (4) is due to H. Brezis and M. Marcus [6] (see also E. B. Davies [7] for inequalities of this type, and M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and A. Laptev [10] for a generalization to convex domains of finite volume).

In [4] we proved a new sharp form of the inequality (4). Namely, for any convex domain Ω of finite inradius δ_0 it was proved that

$$\int_\Omega |\nabla f|^2\, dx \;\geq\; \frac{1}{4}\int_\Omega \frac{|f|^2}{\delta^2}\, dx + \frac{\lambda_0^2}{\delta_0^2}\int_\Omega |f|^2\, dx, \quad \forall f \in H_0^1(\Omega), \tag{5}$$

where $\lambda_0 = 0.940...$ is a Lamb constant defined as the first zero in $(0, +\infty)$ of the function $J_0(x) - 2xJ_1(x)$, J_0 and J_1 being the Bessel functions of order 0 and 1, respectively. The inequality (5) is sharp for all dimensions $n \geq 1$.

Let Ω be an n-dimensional convex domain. Suppose that $p \in (0, +\infty)$ and $q \in (0, +\infty)$. The main aim of this paper is to obtain a new Hardy-type inequality with two sharp constants $h \in [0, +\infty)$ and $\lambda \in [0, +\infty)$ such that

$$\int_\Omega \frac{|\nabla f|^2}{\delta^{p-1}}\, dx \;\geq\; h\int_\Omega \frac{|f|^2}{\delta^{p+1}}\, dx + \frac{\lambda^2}{\delta_0^q}\int_\Omega \frac{|f|^2}{\delta^{p-q+1}}\, dx \tag{6}$$

for all differentiable functions $f : \Omega \to \mathbb{R}$ vanishing at the boundary of the domain. More precisely, we will suppose that f belongs to the space $H_0^1(\Omega,\ \delta^{1/2-p/2})$ that is the closure of smooth functions supported in Ω and having finite integral $\int_\Omega |\nabla f|^2 \delta^{1-p}\, dx$.

If $\lambda = 0$ in (6) then it is known that the sharp value of h is $p^2/4$ (see [8] for $n = 1$, $p > 0$ and $\Omega = (0, +\infty)$, the case $p = 1$ and $n \geq 1$ corresponds to the inequality (3), for $p > 0$ and $n \geq 2$ it is proved in [2] and [3]). Consequently, in the general case we have to consider h such that

$$0 \leq h \leq p^2/4.$$

We will use the term "extremal domain Ω_0" for an inequality (6) with two sharp constants $h \in [0, +\infty)$ and $\lambda \in [0, +\infty)$ in the usual sense: *For any $\varepsilon > 0$ there exist functions $f_\varepsilon \in H_0^1(\Omega,\ \delta^{1/2-p/2})$ and $g_\varepsilon \in H_0^1(\Omega,\ \delta^{1/2-p/2})$ such that*

$$\int_{\Omega_0} \frac{|\nabla f_\varepsilon|^2}{\delta^{p-1}}\, dx \;<\; (h+\varepsilon)\int_{\Omega_0} \frac{|f_\varepsilon|^2}{\delta^{p+1}}\, dx + \frac{\lambda^2}{\delta_0^q}\int_{\Omega_0} \frac{|f_\varepsilon|^2}{\delta^{p-q+1}}\, dx$$

and

$$\int_{\Omega_0} \frac{|\nabla g_\varepsilon|^2}{\delta^{p-1}}\, dx \;<\; h\int_{\Omega_0} \frac{|g_\varepsilon|^2}{\delta^{p+1}}\, dx + \frac{\lambda^2+\varepsilon}{\delta_0^q}\int_{\Omega_0} \frac{|g_\varepsilon|^2}{\delta^{p-q+1}}\, dx.$$

2 Main results

We will fix $h \geq 0$ and consider λ as the constant best possible in (6) for the set of all n-dimensional convex domains with fixed inradius δ_0. It will be shown that such a constant satisfies the inequalities $0 < \lambda \leq j_{p/q-1}\, q/2$, where j_ν be the first positive zero of the Bessel function J_ν of order ν. The upper estimate for λ is a corollary of our first theorem which deals with the case $h = 0$.

Theorem 1. *Suppose that $p \in (0, +\infty)$ and $q \in (0, +\infty)$. If Ω is an n-dimensional convex domain of finite inradius δ_0, then the sharp inequality*

$$\int_\Omega \frac{|\nabla f|^2}{\delta^{p-1}}\, dx \;\geq\; \frac{q^2\, j^2_{p/q-1}}{4\delta_0^q} \int_\Omega \frac{|f|^2}{\delta^{p-q+1}}\, dx, \quad \forall f \in H^1_0(\Omega,\ \delta^{1/2-p/2}),$$

is valid. Finite intervals $(a, b) \subset \mathbb{R}$ for $n = 1$ and domains of the form $(a, b) \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$ for $n \geq 2$ are extremal domains.

The known asymptotic formula 9.5.14 in [1] implies that $j_{\nu-1}/\nu = 1 + O(\nu^{-2/3})$ as $\nu \to +\infty$. Hence, $q\, j_{p/q-1} \to p$ as $q \to 0$. Thus, Theorem 1 presents the mentionned above inequality

$$\int_\Omega \frac{|\nabla f|^2}{\delta^{p-1}}\, dx \;\geq\; \frac{p^2}{4} \int_\Omega \frac{|f|^2}{\delta^{p+1}}\, dx, \quad \forall f \in H^1_0(\Omega,\ \delta^{1/2-p/2}),$$

as a limit case as $q \to 0$.

Also, taking $p = 1$ in Theorem 1 gives an inequality from our paper [4]. Next, as a corollary we give two cases that correspond to the equations

$$p/q = 3/2 \quad \text{and} \quad p/q = 1/2$$

using the known facts

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi}}\, \frac{\sin x}{\sqrt{x}}, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi}}\, \frac{\cos x}{\sqrt{x}}.$$

and, consequently, $j_{-1/2} = \pi/2$ and $j_{1/2} = \pi$ (see, for instance, [11], p. 439).

Corollary 1. *For any $p \in (0, +\infty)$ and n-dimensional convex domains Ω of finite inradius δ_0 there are the following sharp inequalities*

$$\int_\Omega \frac{|\nabla f|^2}{\delta^{p-1}}\, dx \;\geq\; \frac{(\pi/3)^2}{\delta_0^{2p/3}}\, p^2 \int_\Omega \frac{|f|^2}{\delta^{p/3+1}}\, dx, \quad \forall f \in H^1_0(\Omega,\ \delta^{1/2-p/2}),$$

and

$$\int_\Omega \frac{|\nabla f|^2}{\delta^{p-1}}\, dx \;\geq\; \frac{\pi^2 p^2}{4\delta_0^{2p}} \int_\Omega \frac{|f|^2}{\delta^{1-p}}\, dx, \quad \forall f \in H^1_0(\Omega,\ \delta^{1/2-p/2}).$$

Since $H^1_0(\Omega, 1) = H^1_0(\Omega)$, the latter inequality for $p = 1$ gives the Poincaré-Hersch-Payne-Stakgold inequality (2).

To formulate the next theorem we need the Lamb constant $z = \lambda_\nu(p)$ defined as the first positive root of the equation

$$pJ_\nu(z) + 2zJ'_\nu(z) = 0 \qquad (\nu \geq 0). \tag{7}$$

The zeros of $2zJ'_\nu(z) + pJ_\nu(z)$ for $\nu > 0$ have been studied by H. Lamb in [13] (see also G.N. Watson [17], p.502). For this reason we shall call $\lambda_\nu(p)$ Lamb's constant. It is clear that $0 < \lambda_\nu(p) < j_\nu$. According to H. Lamb (compare [13], p. 272), for large p one has the approximation

$$\lambda_\nu(p) \approx (1 - 2/p)j_\nu.$$

The main result of this paper is the following theorem on the Hardy-type inequality (6) with two sharp constants h and λ.

Theorem 2. *Suppose that $p \in (0, +\infty)$, $q \in (0, +\infty)$, $\nu \in [0, p/q]$, and $\lambda_\nu(p)$ is the Lamb constant. For any n-dimensional convex domain Ω of finite inradius δ_0 the sharp inequality*

$$\int_\Omega \frac{|\nabla f|^2}{\delta^{p-1}}\, dx \;\geq$$

$$\frac{p^2 - \nu^2 q^2}{4} \int_\Omega \frac{|f|^2}{\delta^{p+1}}\, dx + \frac{q^2 \lambda_\nu^2(2p/q)}{4\delta_0^q} \int_\Omega \frac{|f|^2}{\delta^{p-q+1}}\, dx, \quad \forall f \in H^1_0(\Omega,\ \delta^{1/2-p/2}),$$

is valid. Finite intervals $(a, b) \subset \mathbb{R}$ for $n = 1$ and domains of the form $(a, b) \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$ for $n \geq 2$ are extremal domains.

Clearly, one can write the inequality of Theorem 2 as follows.

If Ω is an n-dimensional convex domain of finite inradius, p and q are positive numbers, then for any $f \in H^1_0(\Omega,\ \delta^{1/2-p/2})$ and any $h \in [0, p^2/4]$

$$\int_\Omega \frac{|\nabla f|^2}{\delta^{p-1}}\, dx \;\geq\; h \int_\Omega \frac{|f|^2}{\delta^{p+1}}\, dx + \frac{q^2}{4\delta_0^q} \lambda_\nu^2(2p/q) \int_\Omega \frac{|f|^2}{\delta^{p-q+1}}\, dx,$$

where

$$\nu = \frac{\sqrt{p^2 - 4h}}{q}.$$

Since $\lambda_0(1) = \lambda_0 = 0.940...$, Theorem 2 implies our inequality (5) (see [4]).

Letting $\nu = p/q$ in Theorem 2 gives that the first constant $h = 0$. In this case the Lamb equation (7) is equivalent to the equation $J_{\nu-1}(z) = 0$ because of the identity (see, for instance, E. Kamke [11], p. 439)

$$\nu\, J_\nu(x) + x\, J'_\nu(x) = x\, J_{\nu-1}(x), \quad x > 0, \nu > 0.$$

Hence, one has

$$\lambda_{p/q}(2p/q) = j_{p/q-1}.$$

Thus, Theorem 2 contains Theorem 1 as a particular case and we have to prove Theorem 2, only. The proof will be considered in Section 3.

In Section 4 we will examine properties of the Lamb constant $z = \lambda_\nu(p)$ considering it as a function which is defined implicitly by the equation (7) for any $p > 0$. It is interesting that $z = \lambda_\nu(p)$ may be found as the solution of the initial value problem :

$$\frac{dz}{dp} = \frac{2z}{p^2 - 4\nu^2 + 4z^2}, \quad z\big|_{p=2\nu} := \lambda_\nu(2\nu) = j_{\nu-1} \tag{8}$$

in the case $\nu > 0$, and

$$\frac{dz}{dp} = \frac{2z}{p^2 + 4z^2}, \quad z\big|_{p=1} := \lambda_0(1) = \lambda_0 = 0.940... \tag{9}$$

for $\nu = 0$. Using these differential equations we easily obtain bounds and asymptotic formulas for the monotonic increasing function $z = \lambda_\nu(p)$ of the variable $p \in (0, \infty)$.

3 Proof of Theorem 2

In the sequel we will use the Bessel function

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \; x^{2k+\nu}}{2^{2k+\nu} \; k! \; \Gamma(k+1+\nu)}$$

for some $\nu \geq 0$, and the function

$$y = F_{\nu,p,q}(x) = x^{\frac{p}{2}} J_\nu\left(\lambda_\nu(2p/q) \; x^{\frac{q}{2}}\right), \quad x \in [0,1].$$

One has

$$4x^{1-\frac{p}{2}} F'_{\nu,p,q}(x)/q = \frac{2p}{q} J_\nu\left(\lambda_\nu(2p/q) \; x^{\frac{q}{2}}\right) + 2x^{\frac{q}{2}} \lambda_\nu(2p/q) J'_\nu\left(\lambda_\nu(2p/q) \; x^{\frac{q}{2}}\right).$$

Using this equation and the definition of the Lamb constant and the facts that $F_{\nu,p,q}(x) > 0$ and $F'_{\nu,p,q}(x) > 0$ for small positive x, we obtain

$$F'_{\nu,p,q}(1) = 0, \;\; F_{\nu,p,q}(x) > 0, \;\; x \in (0,1] \;\text{ and }\; F'_{\nu,p,q}(x) > 0, \;\; x \in (0,1). \tag{10}$$

We need some preparatory assertions. Namely, in two lemmas we examine our basic one-dimensional inequality that is an improvement of the original Hardy inequality for finite intervals.

Lemma 1. *Let $\lambda_\nu(p)$ be the Lamb constant. If $p \in (0, +\infty)$, $q \in (0, +\infty)$ and $\nu \in [0, p/q]$ and f is an absolutely continuous function in $[0,1]$ such that $f(0) = 0$ and $f'/t^{p/2-1/2} \in L^2[0,1]$, then*

$$\int_0^1 \frac{f'^2(t)}{t^{p-1}}\, dt \geq \frac{p^2 - \nu^2 q^2}{4} \int_0^1 \frac{f^2(t)}{t^{p+1}}\, dt + \frac{q^2 \lambda_\nu^2(2p/q)}{4} \int_0^1 \frac{f^2(t)}{t^{p-q+1}}\, dt. \tag{11}$$

If $\nu = 0$, then there is no admissible function $f \not\equiv 0$ for which equality in (11) is actually attained. If $0 < \nu \le p/q$, then equality in (11) occurs if and only if $f(t) = CF_{\nu,p,q}(t)$, where C is a constant.

Proof of Lemma 1. If $0 < \nu \le p/q$, then one may prove the inequality (11) using the classical variational calculus. This is not possible for the case $\nu = 0$, when the Hardy term dominates. We will give an unified proof.

Clearly, we will need some properties of Bessel's functions. As is known (see E. Kamke [11], p. 440), the function $y = F_{\nu,p,q}(x)$ is a solution of the differential equation

$$x^2y'' + (1-p)xy' + \left(\frac{p^2 - \nu^2q^2}{4} + \frac{q^2\lambda_\nu^2(2p/q)}{4x^{-q}}\right) y = 0, \quad x \in \mathbb{R}. \tag{12}$$

Using the expansion for the Bessel function it is easy to obtain that

$$\lim_{t\to 0^+} \frac{tF'_{\nu,p,q}(t)}{F_{\nu,p,q}(t)} = c_1, \quad \frac{F'^2_{\nu,p,q}(t)}{t^{p-1}} = \frac{c_2}{t^{1-\nu q}}\,(1 + O(t^q)) \quad \text{as} \quad t \to 0^+, \tag{13}$$

where

$$c_1 = \frac{p+\nu q}{2} > 0, \qquad c_2 = \frac{\lambda_\nu^{2\nu}(2p/q)(p+\nu q)^2}{4q^{2\nu}\Gamma^2(1+\nu)} > 0.$$

For an absolutely continuous function $f : [0,1] \to \mathbb{R}$ with properties $f(0) = 0$ and $f'/t^{p/2-1/2} \in L^2[0,1]$ one has that

$$f^2(t) \le \left(\int_0^t |f'(x)|\,dx\right)^2 \le \int_0^t x^{p-1}\,dx \int_0^t \frac{f'^2(x)}{x^{p-1}}\,dx \le \frac{t^p}{p}\int_0^t \frac{f'^2(x)}{x^{p-1}}\,dx.$$

This together with the the first equation from (13) imply that

$$\lim_{t\to 0^+} \frac{f^2(t)F'_{\nu,p,q}(t)}{t^{p-1}F_{\nu,p,q}(t)} = 0. \tag{14}$$

We have

$$0 \le P := \int_0^1 \frac{1}{t^{p-1}}\left(f'(t) - \frac{F'_{\nu,p,q}(t)}{F_{\nu,p,q}(t)}\,f(t)\right)^2\,dt$$

$$= \int_0^1 \frac{f'^2(t)}{t^{p-1}}\,dt - \int_0^1 \frac{F'_{\nu,p,q}(t)}{t^{p-1}F_{\nu,p,q}(t)}\,df^2(t) + \int_0^1 \frac{F'^2_{\nu,p,q}(t)f^2(t)}{t^{p-1}F^2_{\nu,p,q}(t)}\,dt.$$

Integrating by parts and using the asymptotic behavior (14) and the differential equation (12) one easily obtains

$$0 \le P = \int_0^1 \frac{f'^2(t)}{t^{p-1}}\,dt + \int_0^1 \frac{t^2F''_{\nu,p,q}(t) + (1-p)tF'_{\nu,p,q}(t)}{t^{p+1}F_{\nu,p,q}(t)} f^2(t)\,dt$$

$$= \int_0^1 \frac{f'^2(t)}{t^{p-1}}\,dt - \int_0^1 \left[\frac{p^2 - \nu^2q^2}{4t^{p+1}} + \frac{q^2\lambda_\nu^2(2p/q)}{4t^{p-q+1}}\right] f^2(t)\,dt,$$

which is the inequality to prove.

Clearly, $P = 0$ if and only if $f(t) = CF_{\nu,p,q}(t)$, where C is a constant, in particular, $C = 0$. According to the second formula in (13), the function $F'_{\nu,p,q}/t^{p/2-1/2} \in L^2[0,1]$ for $\nu > 0$, only. Hence, for $\nu = 0$ we have to put $C = 0$. For $\nu > 0$ any constant C is admissible.

This completes the proof of Lemma 1.

For $\nu > 0$ both constants in (11)

$$\frac{p^2 - \nu^2 q^2}{4} \qquad \text{and} \qquad \frac{q^2 \lambda_\nu^2(2p/q)}{4}$$

are sharp because of the existence of the extremal functions $F_{\nu,p,q}(t) \not\equiv 0$. In the next lemma we will prove that the constants are sharp in the case $\nu = 0$, too.

Lemma 2. *If $p \in (0,+\infty)$ and $q \in (0,+\infty)$ and $\lambda_0(2p/q)$ is the Lamb constant, then for any $\varepsilon > 0$ there exist two functions f_ε and g_ε that satisfy the conditions of Lemma 1 and the following inequalities*

$$\int_0^1 \frac{f_\varepsilon'^2(t)}{t^{p-1}}\,dt < \left(\frac{p^2}{4} + \varepsilon\right) \int_0^1 \frac{f_\varepsilon^2(t)}{t^{p+1}}\,dt + \frac{q^2\lambda_0^2(2p/q)}{4} \int_0^1 \frac{f_\varepsilon^2(t)}{t^{p-q+1}}\,dt, \tag{15}$$

and

$$\int_0^1 \frac{g_\varepsilon'^2(t)}{t^{p-1}}\,dt < \frac{p^2}{4} \int_0^1 \frac{g_\varepsilon^2(t)}{t^{p+1}}\,dt + \left(\frac{q^2\lambda_0^2(2p/q)}{4} + \varepsilon\right) \int_0^1 \frac{g_\varepsilon^2(t)}{t^{p-q+1}}\,dt. \tag{16}$$

Proof of Lemma 2. Let $\varepsilon > 0$. Without loss of generality we suppose that $\varepsilon \le 1$. We will define functions f_ε and g_ε explicitly.

Consider first the function $f_\varepsilon(t) := t^{(p+\varepsilon/(p+1))/2}$. Straightforward computations give that

$$\int_0^1 \frac{f_\varepsilon'^2(t)}{t^{p-1}}\,dt = \left(p + \frac{\varepsilon}{p+1}\right)^2 \frac{p+1}{4\varepsilon} < (p^2 + 4\varepsilon)\frac{p+1}{4\varepsilon} = \frac{p^2+4\varepsilon}{4} \int_0^1 \frac{f_\varepsilon^2(t)}{t^{p+1}}\,dt,$$

which implies the inequality (15).

Next, we consider the function

$$g_\varepsilon(t) = t^{\alpha/2} F_{0,p,q}(t) = t^{\frac{p+\alpha}{2}} J_0\left(\lambda_0(2p/q)\ t^{\frac{q}{2}}\right)$$

for some $\alpha = \alpha(\varepsilon) \in (0, q]$. By computations as in the proof of Lemma 1 one has

$$P_\varepsilon := \frac{p^2}{4} \int_0^1 \frac{g_\varepsilon^2(t)}{t^{p+1}}\,dt + \left(\frac{q^2\lambda_0^2(2p/q)}{4} + \varepsilon\right) \int_0^1 \frac{g_\varepsilon^2(t)}{t^{p-q+1}}\,dt - \int_0^1 \frac{g_\varepsilon'^2(t)}{t^{p-1}}\,dt$$

$$= \varepsilon \int_0^1 \frac{g_\varepsilon^2(t)}{t^{p-q+1}}\,dt - \int_0^1 \frac{1}{t^{p-1}} \left(g_\varepsilon'(t) - \frac{F'_{0,p,q}(t)}{F_{0,p,q}(t)}\, g_\varepsilon(t)\right)^2 dt$$

Since

$$g_\varepsilon'(t) - \frac{F_{0,p,q}'(t)}{F_{0,p,q}(t)}\, g_\varepsilon(t) = \frac{\alpha}{2} t^{\frac{p+\alpha}{2}-1} J_0\left(\lambda_0(2p/q)\ t^{\frac{q}{2}}\right)$$

one easily gets

$$P_\varepsilon = \varepsilon \int_0^1 t^{\alpha+q-1} J_0^2\left(\lambda_0(2p/q)\ t^{\frac{q}{2}}\right)\, dt - \frac{\alpha^2}{4}\int_0^1 t^{\alpha-1} J_0^2\left(\lambda_0(2p/q)\ t^{\frac{q}{2}}\right)\, dt$$

$$\geq \varepsilon \int_0^1 t^{2q-1} J_0^2\left(\lambda_0(2p/q)\ t^{\frac{q}{2}}\right)\, dt - \frac{\alpha}{4}\max_{0\leq t\leq j_0} J_0^2(t).$$

Clearly, $P_\varepsilon > 0$ for sufficiently small α. This implies the inequality (16).

The proof of Lemma 2 is complete.

Proof of Theorem 2. During this proof we suppose that

$$h = \frac{p^2 - \nu^2 q^2}{4} \quad \text{and} \quad \lambda = \frac{q}{2}\lambda_\nu(2p/q), \tag{17}$$

where $\lambda_\nu(p)$ is the Lamb constant.

Consider first the case $n = 1$. If Ω is a finite interval (a, b), then $\delta_0 = (b-a)/2$ and $\delta = \delta(x) = \min\{x - a,\ b - x\}$. We have to prove the inequality

$$\int_a^b \frac{|f'(x)|^2}{\delta^{p-1}(x)}\, dx \ \geq\ h\int_a^b \frac{|f(x)|^2}{\delta^{p+1}(x)}\, dx + \frac{\lambda^2}{\delta_0^q}\int_a^b \frac{|f(x)|^2}{\delta^{p-q+1}(x)}\, dx \tag{18}$$

for all functions $f \in H_0^1((a,b),\ \delta^{1/2-p/2})$.

On the one hand, by the change $\tau = yt$ of variables for any constant $y > 0$ the inequality (11) of Lemma 1 implies that

$$\int_0^y \frac{|f'(\tau)|^2}{\tau^{p-1}}\, d\tau \ \geq\ h\int_0^y \frac{|f(\tau)|^2}{\tau^{p+1}}\, d\tau + \frac{\lambda^2}{y^q}\int_0^y \frac{|f(\tau)|^2}{\tau^{p-q+1}}\, d\tau \tag{19}$$

for all functions $f \in H_0^1((0, 2y),\ \delta^{1/2-p/2})$.

On the other hand, the inequality (18) is the sum of the inequalities

$$\int_a^{a+\delta_0} \frac{|f'(x)|^2}{(x-a)^{p-1}}\, dx \ \geq\ h\int_a^{a+\delta_0} \frac{|f(x)|^2}{(x-a)^{p+1}}\, dx + \frac{\lambda^2}{\delta_0^q}\int_a^{a+\delta_0} \frac{|f(x)|^2}{(x-a)^{p-q+1}}\, dx$$

and

$$\int_{b-\delta_0}^b \frac{|f'(x)|^2}{(b-x)^{p-1}}\, dx \ \geq\ h\int_{b-\delta_0}^b \frac{|f(x)|^2}{(b-x)^{p+1}}\, dx + \frac{\lambda^2}{\delta_0^q}\int_{b-\delta_0}^b \frac{|f(x)|^2}{(b-x)^{p-q+1}}\, dx,$$

each of them is equivalent to the inequality (19) with $y = \delta_0$ by the changes $x - a = \tau$ and $b - x = \tau$ of variables.

Clearly, Lemmas 1 and 2 imply that the constants (17) in the inequality (18) are sharp. In particular, for $0 < \nu \leq p/q$ equality in (18) holds if and only if $f(x) = CG(x)$, where C is a constant and the extremal function G is defined by the equations

$$G(a + \delta_0 t) = F_{\nu,p,q}(t) = G(b - \delta_0 t), \qquad t \in [0, 1].$$

This completes the proof of Theorem 2 in the case $n = 1$.

Now, let $n \geq 2$. We will use the way from [2] to extend one-dimensional inequalities to convex domains in $\mathbb{R}^n$, $n \geq 2$. More precisely, we will use the following assertion (see [2] , Section 6 for a proof):

Let Ω be an open and convex set in $\mathbb{R}^n$ with finite inradius $\delta_0 := \delta_0(\Omega)$, let $\delta = dist(x, \partial\Omega)$ and let p, q, h and λ be some non-negative constants. If the inequality (19) *is valid for any $y \in (0, \delta_0]$ and any $f \in H_0^1((0, 2y), t^{1/2-p/2})$ then*

$$\int_\Omega \frac{|\nabla f|^2}{\delta^{p-1}} dx \ \geq \ h \int_\Omega \frac{|f|^2}{\delta^{p+1}} dx + \frac{\lambda^2}{\delta_0^q} \int_\Omega \frac{|f|^2}{\delta^{p+q-1}} dx$$

for any $f \in H_0^1(\Omega, \delta^{1/2-p/2})$.

Thus, the inequality of Theorem 2 is implied by Lemma 1.

To complete the proof of Theorem 2 one has to prove that the constants are sharp in the case $n \geq 2$, too. Since the constants are invariant under linear transformation of domains it is sufficient to consider the domains

$$\Omega_1 = (0, 2) \qquad \text{and} \qquad \Omega_n = (0, 2) \times \mathbb{R}^{n-1} \ \ (n \geq 2)$$

and to prove the following assertion:

For any $\varepsilon > 0$ there exist functions $f_{n,\varepsilon}$ and *$g_{n,\varepsilon}$ that belong to $H_0^1(\Omega_n, \delta^{1/2-p/2})$ and satisfy the inequalities*

$$A_n := \int_{\Omega_n} \frac{|\nabla f_{n,\varepsilon}|^2}{\delta^{p-1}} dx - (h + \varepsilon) \int_{\Omega_n} \frac{|f_{n,\varepsilon}|^2}{\delta^{p+1}} dx - \frac{\lambda^2}{\delta_0^q} \int_{\Omega_n} \frac{|f_{n,\varepsilon}|^2}{\delta^{p+q-1}} dx < 0$$

and

$$B_n := \int_{\Omega_n} \frac{|\nabla g_{n,\varepsilon}|^2}{\delta^{p-1}} dx - h \int_{\Omega_n} \frac{|g_{n,\varepsilon}|^2}{\delta^{p+1}} dx - \frac{\lambda^2 + \varepsilon}{\delta_0^q} \int_{\Omega_n} \frac{|g_{n,\varepsilon}|^2}{\delta^{p+q-1}} dx < 0.$$

As in our paper [4] we proceed by mathematical induction on the dimension n. For $n = 1$ the assertion follows from Lemmas 1 and 2. Suppose that the inequalities $A_n < 0$ and $B_n < 0$ are valid for n, $n \geq 1$. We define functions $f_{n+1,\varepsilon}$ and $g_{n+1,\varepsilon}$ for any $\alpha = \alpha(\varepsilon) > 0$ as the products

$$f_{n+1,\varepsilon}(x) = f_{n,\varepsilon}(x')\, \varphi_\alpha(x_{n+1}) \quad \text{and} \quad g_{n+1,\varepsilon}(x) = g_{n,\varepsilon}(x')\, \varphi_\alpha(x_{n+1}),$$

where $x = (x', x_{n+1})$, $x' \in \Omega_n$, $x_{n+1} \in \mathbb{R}$, and $\varphi_\alpha : \mathbb{R} \to [0, 1]$ is the even function, defined in $[0, \infty)$ by equations

$$\varphi_\alpha(t) = 1, \quad t \in [0, 1/\alpha]; \quad \varphi_\alpha(t) = 0, \quad t \in [1 + 1/\alpha, +\infty),$$

and

$$\varphi_\alpha(t) = \Big(1 - (t - 1/\alpha)^2\Big)^2, \quad t \in (1/\alpha, 1 + 1/\alpha).$$

Using the function $\varphi(t) = (1 - t^2)^2$ and the equations

$$\delta = dist(x, \partial\Omega_{n+1}) \equiv dist(x', \partial\Omega_n), \quad \int_{-\infty}^{+\infty} \varphi_\alpha^2(t)\, dt = \frac{2}{\alpha} + 2 \int_0^1 \varphi^2(t)\, dt$$

and straightforward computations one gets that

$$A_{n+1} = \frac{2}{\alpha} A_n + A'_n - A''_n \quad \text{and} \quad B_{n+1} = \frac{2}{\alpha} B_n + B'_n - B''_n,$$

where

$$A'_n = 2 \int_0^1 dt \int_{\Omega_n} \frac{|\nabla f_{n,\varepsilon}(x')|^2 \varphi^2(t) + f^2_{n,\varepsilon}(x')\varphi'^2(t)}{\delta^{p-1}} dx',$$

$$B'_n = 2 \int_0^1 dt \int_{\Omega_n} \frac{|\nabla g_{n,\varepsilon}(x')|^2 \varphi^2(t) + g^2_{n,\varepsilon}(x')\varphi'^2(t)}{\delta^{p-1}} dx,'$$

$$A''_n = 2 \int_0^1 \varphi^2(t)\, dt \left((h+\varepsilon) \int_{\Omega_n} \frac{|f_{n,\varepsilon}(x')|^2}{\delta^{p+1}}\, dx' + \frac{\lambda^2}{\delta_0^q} \int_{\Omega_n} \frac{|f_{n,\varepsilon}(x')|^2}{\delta^{p+q-1}}\, dx' \right)$$

and

$$B''_n = 2 \int_0^1 \varphi^2(t)\, dt \left(h \int_{\Omega_n} \frac{|g_{n,\varepsilon}(x')|^2}{\delta^{p+1}}\, dx' + \frac{\lambda^2+\varepsilon}{\delta_0^q} \int_{\Omega_n} \frac{|g_{n,\varepsilon}(x')|^2}{\delta^{p+q-1}}\, dx' \right).$$

The quantities A_n, A'_n, A''_n, B_n, B'_n and B''_n are not dependent on α. Since $A_n < 0$ and $B_n < 0$, it is clear that $A_{n+1} < 0$ and $B_{n+1} < 0$ for sufficiently small positive α.

This completes the proof of Theorem 2.

4 Lamb's constant as a function

Let

$$\Phi(p,z) := pJ_\nu(z) + 2zJ'_\nu(z).$$

We consider the Lamb equation (7) with $z \in (0, j_\nu)$ as the identity

$$\Phi(p,z) = 0 \qquad (0 < p < \infty)$$

which implicitly defines the function $z = \lambda_\nu(p),\ 0 < p < \infty$. Using the identity $\Phi(p,z) = 0$ and the Bessel differential equation

$$z^2 J''_\nu(z) + zJ'_\nu(z) + (z^2 - \nu^2)J_\nu(z) = 0$$

one easily derives that

$$\frac{\partial \Phi}{\partial p}\, dp + \frac{\partial \Phi}{\partial z}\, dz = J_\nu(z)\, dp - \frac{p^2 - 4\nu^2 + 4z^2}{2z}\, J_\nu(z)\, dz = 0$$

which implies differential equations (8) and (9).

Case $\nu = 0$. It is obvious from (7) and (9) that z is a positive and monotonic increasing function of the variable $p > 0$. Further, the formula (7) implies that $\lambda_0(p) J'_0(\lambda_0(p)) \to 0$ as $p \to 0$. As j'_0, the first positive zero of J'_0, is bigger than

λ_0, we conclude that $\lambda_0(p) \to 0$ as $p \to 0$. Using the Taylor expansion of J_0 at the origin and (7) we get

$$\lambda_0(p) = \sqrt{p}\left(1 - \frac{1}{16}p + O(p^2)\right) \quad \text{as} \quad p \to 0.$$

For $p \to \infty$, formula (7) implies that $J_0(\lambda_0(p)) \to 0$ as $p \to \infty$, therefore $\lambda_0(p) \to j_0$ as $p \to \infty$.

Thus, *for any* $p \in (0, \infty)$ *one has*

$$\lim_{p\to 0} \lambda_0(p) = 0 < \lambda_0(p) < j_0 = \lim_{p\to\infty} \lambda_0(p). \tag{20}$$

To get an asymptotic expansion of $\lambda_0(p)$ near the point at infinity, we may use (9) and get

$$\frac{1}{2}\frac{d}{dp}\ln z = \frac{1}{p^2} - \frac{4j_0^2}{p^4}\left(1 + O\left(\frac{1}{p^2}\right)\right) \quad \text{as} \quad p \to \infty.$$

By integration and exponentiation we conclude that

$$\lambda_0(p) = j_0\left(1 - \frac{2}{p} + \frac{2}{p^2} - \frac{4(1-2j_0^2)}{3p^3} + O\left(\frac{1}{p^4}\right)\right) \quad \text{as} \quad p \to \infty.$$

Case $\nu > 0$. According to the differential equation (8), $dz/dp > 0$ whenever $p \geq 2\nu$. To prove that the function λ_ν is monotonic increasing in $(0, \infty)$ we proceed as follows. We get from (7) that

$$\lim_{p\to\infty} \lambda_\nu(p) = j_\nu, \quad \text{and} \quad \lim_{p\to 0} \lambda_\nu(p) = j'_\nu,$$

where j'_ν is the first positive zero of J'_ν. To obtain the second lim we have used the Taylor expansion of J_ν at the origin.

Since $j_\nu > j'_\nu > \nu$ for $\nu > 0$ (see [17], p. 485), we derive from (8) that $dz/dp > 0$ for any $p \in (0, \infty)$. Consequently, λ_ν is a monotonic increasing function of the variable $p \in (0, \infty)$ and one has the following assertion.

If $\nu > 0$ *then for any* $p \in [2\nu, \infty)$

$$\lambda_\nu(2\nu) = j_{\nu-1} \leq \lambda_\nu(p) < j_\nu = \lim_{p\to\infty} \lambda_\nu(p). \tag{21}$$

and for any $p \in (0, 2\nu]$

$$\lim_{p\to 0} \lambda_\nu(p) = j'_\nu < \lambda_0(p) \leq j_{\nu-1} = \lambda_\nu(2\nu), \tag{22}$$

where j'_ν *is the first positive zero of the derivative* J'_ν*of the Bessel function.*

It is clear from (8) that

$$\lim_{p\to 0} \frac{\lambda_\nu(p) - j'_\nu}{p} = \frac{j'_\nu}{2((j'_\nu)^2 - \nu^2)}.$$

For $p \to \infty, \nu > 0$, we may proceed analogously to the case $\nu = 0$ and we can improve Lamb's asymptotic expansion to any desirable extent, for example

$$\lambda_\nu(p) = j_\nu \left(1 - \frac{2}{p} + \frac{2}{p^2} - \frac{4(2\nu^2 + 1 - 2j_\nu^2)}{3p^3} + O\left(\frac{1}{p^4}\right) \right) \quad \text{as} \quad p \to \infty.$$

Clearly, in addition to estimates (20), (21) and (22) one can derive several new estimates for $\lambda_\nu(p)$ using the differential equations (8) and (9) together with the fact that $\ln z$ is a concave function in both cases $\nu = 0$ and $\nu > 0$ because of the inequality

$$\frac{d^2 \ln z}{dp^2} = -\frac{4p + 16zz'}{(p^2 - 4\nu^2 + 4z^2)^2} < 0.$$

Finally, we attract reader's attention to some facts on the bounds for the Lamb constants, i.e. on the quantities j_ν and j'_ν:

$$j_0 = 2.4048... \quad \text{and} \quad \sqrt{\nu(\nu+2)} < j'_\nu < j_\nu < \sqrt{2(\nu+1)(\nu+3)}$$

for any positive ν (see G.H. Watson [17], pp. 485-486).

5 A remark

In the proof of the inequality of Theorem 2 we do not use the restriction $4h = p^2 - \nu^2 q^2 \geq 0$. Consequently, the inequality holds for any positive ν, but $h < 0$ in the case $\nu > p/q$ and this changes the type of the inequality. For instance, letting $p \to 0^+$ gives the following inequality for convex domains of finite inradius and all positive numbers q and ν : *For functions f vanishing at the boundary of the domain*

$$\int_\Omega \delta |\nabla f|^2 \, dx \; + \; \frac{\nu^2 q^2}{4} \int_\Omega \frac{|f|^2}{\delta} \, dx \geq \frac{q^2 j_\nu'^2}{4\delta_0^q} \int_\Omega \frac{|f|^2}{\delta^{1-q}} \, dx,$$

where j'_ν is the first positive zero of the derivative J'_ν of Bessel's function J_ν.

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