Characterization of different classes of convex bodies via orthogonality

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Abstract

Following the concept of normal to a convex body, which is due to Eggleston, we introduce the notion of affine orthogonality with respect to a convex body. In contrast to Eggleston's considerations, we do not need a metric and via this notion we are able to characterize convex bodies of constant width and further interesting classes of convex bodies.

1 Introduction

The notion of *normal* to a convex body was introduced by Eggleston (see [7], [6], and the survey [4]). It is a very useful notion, e.g., for the characterizations of convex bodies of constant width in Euclidean space as well as in any finite-dimensional real Banach space. Following this line, we introduce the notion of *affine orthogonality* with respect to a convex body. If the convex body is a circular disc, then our definition coincides with the usual Euclidean orthogonality. But for bodies of constant width the coincidence is with Eggleston's notion of normals. The advantage of our approach is that it allows one to characterize not only bodies of constant width but also other classes of convex bodies such as those which are centrally symmetric, ellipses, bodies whose boundary is a Radon curve, etc. Moreover, we contribute to the solution of the following problem posed by V. Soltan: to extend the characterization given by Makai Jr. and Martini

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in [12] of bodies of constant width in the Euclidean plane to normed planes by replacing Euclidean orthogonality by Birkhoff orthogonality. Note that, due to the counterexample constructed in [1], the mentioned characterization cannot be extended to all normed planes based on Birkhoff orthogonality. But if Euclidean orthogonality is replaced by affine orthogonality then such extension is possible (see Theorem 4.7). It should be noted that, unlike Eggleston's definition, our definition needs no metric. In other words, our considerations take place in an arbitrary affine space.

2 Preliminaries

By $\mathbb{R}^n$ we denote the n-dimensional real linear space and by $\mathbb{E}^n$ the n-dimensional Euclidean space. The linear space $\mathbb{R}^n$ equipped with an arbitrary norm $\|\cdot\|$ is called a (*Minkowski* or) *normed space*. For the *distance* between two points p, q in $\mathbb{E}^n$ we write $\|p-q\|$; the same notation will be used for the distance between two points in a normed space. The *line* through distinct points p and q is denoted by $\langle p, q\rangle$, and the *closed line segment* with endpoints p and q by $[p, q]$.

Let $\mathcal{K}$ be a *convex body* in $\mathbb{R}^n$, i.e., a compact, convex set with non-empty interior. For the *boundary* of $\mathcal{K}$ we write $\operatorname{bd}\mathcal{K}$. A convex body $\mathcal{K}$ is called *strictly convex* if $\operatorname{bd}\mathcal{K}$ does not contain segments; it is called *smooth* if any point in $\operatorname{bd}\mathcal{K}$ belongs to exactly one supporting hyperplane of $\mathcal{K}$. We call a closed line segment $[p_1, p_2]$ a *chord* of $\mathcal{K}$ if $p_1 \neq p_2$ and $p_1, p_2 \in \operatorname{bd}\mathcal{K}$, i.e., it is assumed that all chords considered in the sequel are non-degenerate. A chord $[p_1, p_2]$ of $\mathcal{K}$ is said to be an *affine diameter* if there exist two different parallel supporting hyperplanes, say H_1 and H_2, of $\mathcal{K}$ such that $p_1 \in H_1$ and $p_2 \in H_2$.

If $\mathcal{K}$ is centrally symmetric and $p \in \mathcal{K}$, then we denote by $\bar{p}$ the point opposite to p with respect to the center of $\mathcal{K}$. Thus, if $\mathcal{K}$ is centrally symmetric and $p \in \operatorname{bd}\mathcal{K}$, then $[p, \bar{p}]$ is an affine diameter. But if in the boundary of a centrally symmetric convex body $\mathcal{K}$ there is a segment that contains p_1 or p_2, then $[p_1, p_2]$ can be an affine diameter which does not pass through the center of $\mathcal{K}$.

A convex body $\mathcal{K}$ in $\mathbb{E}^n$ is said to be *of constant width* if all its affine diameters have equal lengths (cf. [6]). This definition makes also sense if $\mathcal{K}$ is a convex body in a Minkowski space, i.e., $\mathcal{K}$ is of constant Minkowskian width if all its affine diameters have equal lengths measured in the respective norm; see [7] and the survey [13].

The *diameter* of a set $\mathcal{K} \subseteq \mathbb{E}^n$ is defined by

$$\operatorname{diam} K := \sup\{\|x-y\| : x, y \in \mathcal{K}\}.$$

If $\mathcal{K}$ is a convex body in $\mathbb{E}^n$ and $p, q \in \operatorname{bd}\mathcal{K}$ are such that $\|p-q\| = \operatorname{diam}\mathcal{K}$, then the chord $[p, q]$ is also called a *diameter* of $\mathcal{K}$. Analogously, the notion of *Minkowskian diameter* in a Minkowski space is defined. It is easy to see, that in an Euclidean space, as well as in an arbitrary normed space, any diameter is also an affine diameter, but not vice versa.

For x and y in a normed space, x is said to be *Birkhoff orthogonal* to y, denoted by $x \perp_B y$, if $\|x+\lambda y\| \geq \|x\|$ for every $\lambda \in \mathbb{R}$. This notion of orthogonality coincides with the usual one if the space is Euclidean. Geometrically, $x \perp_B y$

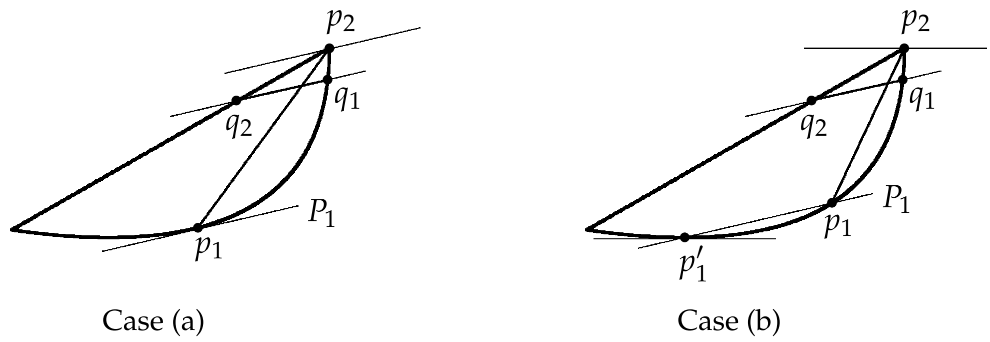

Figure 1: $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$.

means that the line through the point x and parallel to the vector y supports the sphere centered at the origin and having radius $\|x\|$ at x. Birkhoff orthogonality is always homogeneous, i.e., if $x \perp_B y$, then $\alpha x \perp_B \beta y$ for every $\alpha, \beta \in \mathbb{R}$. Thus, we can use the term of *Birkhoff orthogonal directions*. Given a direction L_1 in a strictly convex (respectively, smooth) normed plane there is only a direction L_2 such that $L_2 \perp_B L_1$ (respectively, $L_1 \perp_B L_2$).

3 Affine orthogonality in the two-dimensional case

Let $[p_1, p_2]$ and $[q_1, q_2]$ be two chords of a convex body $\mathcal{K}$ in $\mathbb{R}^2$, and let P_1 be the line through p_1 parallel to $\langle q_1, q_2 \rangle$. We say that $[p_1, p_2]$ *is affine orthogonal to* $[q_1, q_2]$ *through* p_1, denoted by

$$[p_1, p_2] \dashv_{p_1} [q_1, q_2],$$

if one of the following two conditions holds (see Figure 1):

a) P_1 supports $\mathcal{K}$ at p_1, and also the line through p_2 parallel to P_1 supports $\mathcal{K}$. Thus, $[p_1, p_2]$ is an affine diameter of $\mathcal{K}$.

b) $P_1 \cap \operatorname{bd} \mathcal{K} = \{p_1, p_1'\}$, $p_1 \neq p_1'$, and $[p_1', p_2]$ is an affine diameter of $\mathcal{K}$.

If $\mathcal{K}$ is a Euclidean disc, then the so-defined relation coincides with the usual Euclidean orthogonality. It is also clear that, in general, affine orthogonality is not symmetric with respect to the chords $[p_1, p_2]$ and $[q_1, q_2]$. It is also not symmetric with respect to the points p_1 and p_2. As we shall see later, these properties of symmetry will characterize special types of convex bodies. On the other hand, affine orthogonality is obviously symmetric with respect to q_1 and q_2.

The next proposition follows directly from the definition above. Property (1) shows that the position of the second chord $[q_1, q_2]$ does not matter: of importance is only the (non-oriented) direction of $[q_1, q_2]$. Property (2) recalls what happens if $\mathcal{K}$ is a Euclidean disc: if p, q and r are points in $\operatorname{bd} \mathcal{K}$ and $[q, r]$ is a diameter, then $[p, q]$ is orthogonal to $[p, r]$. This is, in fact, Thales' theorem, and was motivating for the definition of affine orthogonal chords.

Proposition 3.1. *Let $\mathcal{K}$ be a convex body in $\mathbb{R}^2$.*

(1) *If $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$, then for any chord $[q_1', q_2']$ of $\mathcal{K}$ that is parallel to $[q_1, q_2]$ the relation $[p_1, p_2] \dashv_{p_1} [q_1', q_2']$ holds.*

(2) *Let $[p_2, q_2]$ be an affine diameter of $\mathcal{K}$ and let the point $p_1 \in \operatorname{bd}\mathcal{K}$ be such that the segment $[p_1, q_2]$ does not belong to the boundary of $\mathcal{K}$. Then $[p_1, p_2] \dashv_{p_1} [p_1, q_2]$.*

It is immediate that for a given chord $[q_1, q_2]$ always another chord $[p_1, p_2]$ exists such that $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$. But Figure 2 points out that for a given chord $[p_1, p_2]$ there is not always a chord $[q_1, q_2]$ such that $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$. For that reason we introduce the following notion: a convex body $\mathcal{K}$ has the *orthogonal chords existence property* (o.c.e.p.) if it is possible to find for any chord $[p_1, p_2]$ another chord $[q_1, q_2]$ such that $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$. Proposition 3.2 shows that large classes of convex bodies have this property.

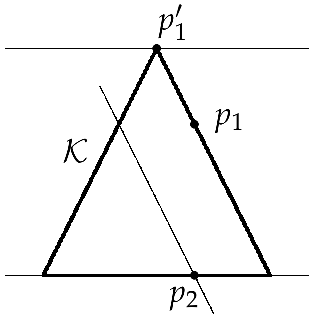

Figure 2: There is no $q \in \operatorname{bd}\mathcal{K}$ such that $[p_1, p_2] \dashv_{p_1} [p_1, q]$, because the only possible candidate is p_1', but $\langle p_1, p_1' \rangle$ supports $\mathcal{K}$ and the line parallel to $\langle p_1, p_1' \rangle$ through p_2 does not support $\mathcal{K}$.

Proposition 3.2. *Assume that $\mathcal{K}$ is a convex body in $\mathbb{R}^2$ that satisfies at least one of the following properties:*

(i) *It is strictly convex.*

(ii) *It is smooth.*

(iii) *It is centrally symmetric.*

Then $\mathcal{K}$ has o.c.e.p.

Proof. (i) If $\mathcal{K}$ is strictly convex, the result follows easily from Proposition 3.1, (2). (ii) Assume that $\mathcal{K}$ is smooth, and let $[p_1, p_2]$ be a chord of $\mathcal{K}$. Let P be the support line of $\mathcal{K}$ at p_2. If the support line at p_1 is parallel to P, then $[p_1, p_2]$ is affine orthogonal to any chord parallel to P. On the other hand, if the support line at p_1 is not parallel to P, let $q_1 \in \operatorname{bd}\mathcal{K}$ be such that the line Q parallel to P through q_1 supports $\mathcal{K}$ and p_1 is between P and Q. Then the line $\langle p_1, q_1 \rangle$ does not support $\mathcal{K}$ at p_1, and then $[p_1, p_2] \dashv_{p_1} [p_1, q_1]$. (iii) Assume finally that $\mathcal{K}$

is centrally symmetric and let $[p_1, p_2]$ be a chord of $\mathcal{K}$. Then $[p_2, \bar{p}_2]$ is an affine diameter. If $\langle p_1, \bar{p}_2 \rangle$ does not support $\mathcal{K}$, it follows that $[p_1, p_2] \dashv_{p_1} [p_1, \bar{p}_2]$. On the other hand, if $\langle p_1, \bar{p}_2 \rangle$ supports $\mathcal{K}$, then by symmetry $\langle p_2, \bar{p}_1 \rangle$ also supports $\mathcal{K}$ and both lines are parallel. This implies that $[p_1, p_2]$ is an affine diameter and $[p_1, p_2] \dashv_{p_1} [p_1, \bar{p}_2]$. ■

If $\mathcal{K}$ satisfies none of the properties (i)-(iii) in Proposition 3.2, then it can lack o.c.e.p.; see the example presented by Figure 2. Nevertheless, Figure 3 shows that there exist convex bodies satisfying o.c.e.p. but lacking properties (i)-(iii). Proposition 3.3 describes exactly the class of convex bodies possessing o.c.e.p.

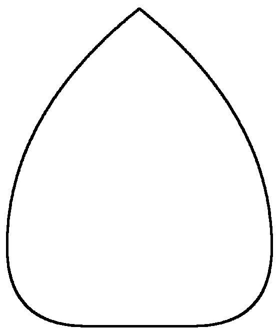

Figure 3: Example of convex body that lacks properties (i)-(iii) in Proposition 3.2, but satisfies o.c.e.p.

Proposition 3.3. *For a convex body $\mathcal{K}$ in $\mathbb{R}^2$ the following properties are equivalent:*

(i) *$\mathcal{K}$ has o.c.e.p.*

(ii) *For any chord $[p_1, p_2]$ that is not an affine diameter of $\mathcal{K}$ there exists an affine diameter $[p_1', p_2]$ such that $[p_1, p_1'] \not\subset \operatorname{bd} \mathcal{K}$.*

Proof. Assume that $\mathcal{K}$ satisfies o.c.e.p., and let $p_1, p_2 \in \operatorname{bd} \mathcal{K}$ be such that the chord $[p_1, p_2]$ is not an affine diameter. Let $[q_1, q_2]$ be such that $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$, and let P be the line through p_1 parallel to $[q_1, q_2]$. Since $[p_1, p_2]$ is not an affine diameter, $P \cap \operatorname{bd} \mathcal{K} = \{p_1, p_1'\}$ and $[p_1', p_2]$ is an affine diameter. Conversely, assume that property (ii) holds and let $p_1, p_2 \in \operatorname{bd} \mathcal{K}$. If $[p_1, p_2]$ is an affine diameter, then $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$ for any chord $[q_1, q_2]$ parallel to a supporting line at p_1. If $[p_1, p_2]$ is not an affine diameter, then there exists $p_1' \in \operatorname{bd} \mathcal{K}$ such that $[p_1', p_2]$ is an affine diameter (and then $p_1' \neq p_1$), and $\langle p_1, p_1' \rangle \cap \operatorname{bd} \mathcal{K} = \{p_1, p_1'\}$. This implies $[p_1, p_2] \dashv_{p_1} [p_1, p_1']$. Thus, $\mathcal{K}$ has o.c.e.p. ■

4 Characterization of special types of convex bodies

Now we will see that affine orthogonality allows to characterize special types of convex bodies. For more information about such bodies we refer, e.g., to [9].

4.1 Centrally symmetric, strictly convex bodies

Example (A) in Figure 4 shows that, in general, $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$ does not imply $[p_1, p_2] \dashv_{p_2} [q_1, q_2]$.

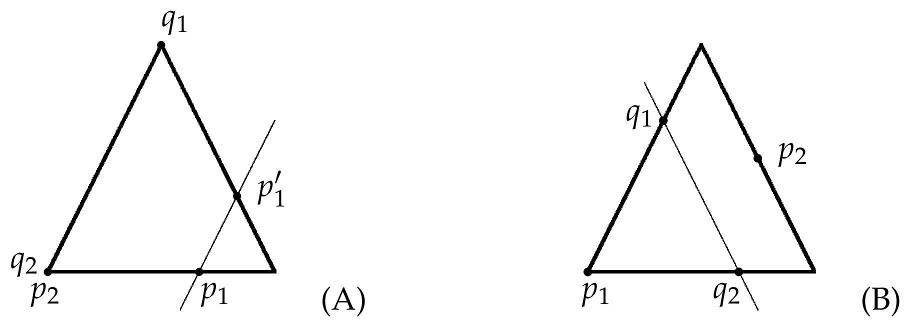

Figure 4: (A) $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$, but $[p_1, p_2] \not\dashv_{p_2} [q_1, q_2]$, (B) $[p_1, p_2] \dashv_{p_1, p_2} [q_1, q_2]$.

Theorem 4.1. *For a convex body $\mathcal{K} \subset \mathbb{R}^2$, the following properties are equivalent:*

(i) *$\mathcal{K}$ is centrally symmetric and strictly convex.*

(ii) *If $[p_1, p_2]$ and $[q_1, q_2]$ are two chords of $\mathcal{K}$ and $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$, then $[p_1, p_2] \dashv_{p_2} [q_1, q_2]$.*

Proof. (i)⇒(ii) Let x be the center of $\mathcal{K}$ and assume that $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$. Let P_1 be the line through p_1 parallel to $\langle q_1, q_2 \rangle$. If P_1 supports $\mathcal{K}$ at p_1, then trivially $[p_1, p_2] \dashv_{p_2} [q_1, q_2]$. Let now P_1 not support $\mathcal{K}$ at p_1, and $P_1 \cap \operatorname{bd} \mathcal{K} = \{p_1, p_1'\}$. Then $[p_1', p_2]$ is an affine diameter, and since $\mathcal{K}$ is strictly convex, p_1' and p_2 are opposite with respect to x, i.e., $p_1' = \bar{p}_2$. Since $\langle \bar{p}_1, p_2 \rangle$ is parallel to $\langle p_1, \bar{p}_2 \rangle = P_1$ and $[p_1, \bar{p}_1]$ is an affine diameter, we get $[p_1, p_2] \dashv_{p_2} [q_1, q_2]$.

(ii)⇒(i) Let us show first that property (ii) implies that $\mathcal{K}$ is strictly convex. Assume, on the contrary, that $\operatorname{bd} \mathcal{K}$ contains a segment $[p, q]$, and let p_1 and p_2 be two different points in the relative interior of that segment. Let $q_2 \in \operatorname{bd} \mathcal{K}$ be such that $[p_2, q_2]$ (and then also $[p_1, q_2]$) is an affine diameter. Then we have that $[p_1, p_2] \dashv_{p_1} [p_1, q_2]$. By (ii) also $[p_1, p_2] \dashv_{p_2} [p_1, q_2]$. Since the line Q through p_2 and parallel to $\langle p_1, q_2 \rangle$ does not support $\mathcal{K}$, we have that Q cuts $\operatorname{bd} \mathcal{K}$ in a point q_2' such that $[p_2, q_2']$ is an affine diameter and $\langle p_1, p_2 \rangle$ is parallel to $\langle q_2, q_2' \rangle$. But this implies that $[q_2, q_2']$ is a segment contained in $\operatorname{bd} \mathcal{K}$ and having the same length as $[p_1, p_2]$. Interchanging the roles of p_1 and p_2, the above argument shows that there exists another point q_2'' in $\operatorname{bd} \mathcal{K}$ such that q_2 is the midpoint of $[q_2', q_2'']$, and thus $\operatorname{bd} \mathcal{K}$ contains a segment having the double length of $[p_1, p_2]$. Since p_1 and p_2 are arbitrary points in the interior of $[p, q]$, we have proved that if $\operatorname{bd} \mathcal{K}$ has a segment, then it has another segment of double length, which is an absurdity.

Let us now show that $\mathcal{K}$ has the following property:

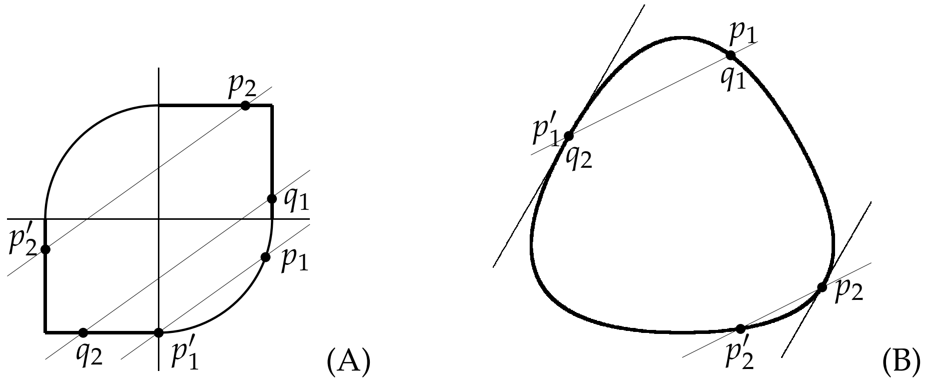

Figure 5: (A) Centrally symmetric but not strictly convex, (B) Strictly convex but not centrally symmetric. In both cases $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$ but $[p_1, p_2] \not\dashv_{p_2} [q_1, q_2]$.

$(*)$ If $[p,q]$ is an affine diameter and P is a line that supports $\mathcal{K}$ at p, then the line Q through q and parallel to P also supports $\mathcal{K}$.

Indeed, assume that $Q \cap \operatorname{bd}\mathcal{K} = \{q, q'\}$, where $q \neq q'$. Then $[p, q'] \dashv_{q'} [q, q']$ and by (ii), $[p, q'] \dashv_p [q, q']$. But this implies that Q supports $\mathcal{K}$ at q (recall (a) in the definition of affine orthogonality), which is impossible because $\mathcal{K}$ is strictly convex.

Fix now an affine diameter $[p_2, q_2]$ of $\mathcal{K}$. We shall see that $\mathcal{K}$ is centered at the midpoint of $[p_2, q_2]$. Denote by P_2 and Q_2 the parallel supporting lines at p_2 and q_2, respectively. Let p_1 be an arbitrary point in $\operatorname{bd}\mathcal{K}$, different from p_2 and q_2. By Proposition 3.1 we have that $[p_1, p_2] \dashv_{p_1} [p_1, q_2]$, and then by (ii) we have $[p_1, p_2] \dashv_{p_2} [p_1, q_2]$. This, together with property $(*)$, implies that the line P_2' through p_2 and parallel to $[p_1, q_2]$ cuts $\operatorname{bd}\mathcal{K}$ in a point $p_2' \neq p_2$, with $[p_1, p_2']$ being an affine diameter. Moreover, we have $[p_2', p_2] \dashv_{p_2'} [p_2', q_2]$, and by (ii) we get $[p_2', p_2] \dashv_{p_2} [p_2', q_2]$. Again by $(*)$, the line P_2'' through p_2 and parallel to $[p_2', q_2]$ cuts $\operatorname{bd}\mathcal{K}$ in a point $p_2'' \neq p_2$, and $[p_2', p_2'']$ is an affine diameter. Since $[p_1, p_2']$ and $[p_2', p_2'']$ are both affine diameters, property $(*)$ and the strict convexity of $\mathcal{K}$ imply that $p_2'' = p_1$. Hence the points p_1, p_2, p_2' and q_2 form a parallelogram and p_2' is the symmetric point of p_1 with respect to the midpoint of $[p_2, q_2]$. ∎

In Figure 5 we see examples confirming that if in Theorem 4.1 property (i) fails, then also (ii) fails. But, as Proposition 4.1 below shows, if the chord $[p_1, p_2]$ has a special position, then strict convexity is not necessary to obtain property (ii).

Proposition 4.1. *Let p, q_1, q_2 be three different points of the boundary of a centrally symmetric convex body $\mathcal{K}$. If $[p, \bar{p}] \dashv_p [q_1, q_2]$, then the line P through p and parallel to $[q_1, q_2]$ supports $\mathcal{K}$ at p, yielding $[p, \bar{p}] \dashv_{\bar{p}} [q_1, q_2]$.*

Proof. Assume that $[p, \bar{p}] \dashv_p [q_1, q_2]$ but $P \cap \operatorname{bd}\mathcal{K} = \{p, p'\}$, where $p \neq p'$. Then $\mathcal{K}$ has supporting lines at $\bar{p}$ and at p', that are parallel, and by the symmetry of $\mathcal{K}$

these lines are also parallel to a supporting line at p that, by the convexity of $\mathcal{K}$, coincides with P, a contradiction. ■

In view of the situations described in Theorem 4.1 and in Proposition 4.1, if $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$ and $[p_1, p_2] \dashv_{p_2} [q_1, q_2]$, we simply write $[p_1, p_2] \dashv [q_1, q_2]$.

Remark 4.1. *Let $\mathcal{K}$ be a centrally symmetric convex body in $\mathbb{R}^2$, taken as the unit ball of a norm. Then from Proposition* 4.1 *it follows that in this case $[p, \bar{p}] \dashv [q_1, q_2]$ if and only if $p - \bar{p} \perp_B q_1 - q_2$.*

4.2 Radon curves

Radon curves form a special class of centrally symmetric, closed, convex curves in the plane. They were introduced by Radon [16] in 1916 and independently rediscovered by Birkhoff [2]. A centrally symmetric, closed, convex curve $\mathcal{C}$ is called a *Radon curve* if it has the following property:

> *For $p \in \mathcal{C}$, let P be a supporting line of $\mathcal{C}$ at p and assume that the line parallel to P through the center of $\mathcal{C}$ cuts $\mathcal{C}$ at q and $\bar{q}$. Then the line through q parallel to $\langle p, \bar{p} \rangle$ supports $\mathcal{C}$.*

Any Radon curve centered at the origin defines a norm whose properties are "almost Euclidean". In fact, the boundary of a centrally symmetric convex body $\mathcal{K}$ in $\mathbb{R}^2$ is a Radon curve if and only if the Birkhoff orthogonality with respect to the induced norm is symmetric. For $n \geq 3$ the symmetry of the Birkhoff orthogonality characterizes the Euclidean spaces. In other words, a centrally symmetric convex body in $\mathbb{R}^n$ ($n \geq 3$) is an ellipsoid if and only if the boundaries of its two-dimensional sections through the center of symmetry are Radon curves. This characterization was obtained in gradual stages by G. Birkhoff [2], R. C. James [10, 11] and M. M. Day [5]. For further properties of Radon curves we refer, e.g., to [19, § 4.7], [15], and [14].

Theorem 4.2. *For a centrally symmetric convex body $\mathcal{K} \subset \mathbb{R}^2$, the following properties are equivalent:*

(i) *The boundary of $\mathcal{K}$ is a Radon curve.*

(ii) *If $p, q \in \operatorname{bd} \mathcal{K}$ with $[p, \bar{p}] \dashv [q, \bar{q}]$, then $[q, \bar{q}] \dashv [p, \bar{p}]$.*

Proof. (i)⇒(ii) This implication follows directly from Remark 4.1. (ii)⇒(i) Let $p \in \operatorname{bd} \mathcal{K}$, let the line P support $\mathcal{K}$ at p, and $q \in \operatorname{bd} \mathcal{K}$ be such that $[q, \bar{q}]$ is parallel to P. Then $[p, \bar{p}] \dashv [q, \bar{q}]$, and therefore $[q, \bar{q}] \dashv [p, \bar{p}]$. By Proposition 4.1 we obtain that the line Q through q and parallel to $[p, \bar{p}]$ supports $\mathcal{K}$ at q, which implies that $\operatorname{bd} \mathcal{K}$ is a Radon curve. ■

4.3 Ellipses

Theorem 4.2 above shows that the fact that affine orthogonality is symmetric over a particular class of chords of $\mathcal{K}$ is characteristic for Radon curves. The next theorem shows that if this symmetry is extended to a wider class of affine orthogonal chords, then it is even characteristic for ellipses.

Theorem 4.3. *For a centrally symmetric convex body $\mathcal{K} \subset \mathbb{R}^2$, the following properties are equivalent:*

(i) *The boundary of $\mathcal{K}$ is an ellipse.*

(ii) *If $p, q_1, q_2 \in \operatorname{bd}\mathcal{K}$ and $[p, \bar{p}] \dashv [q_1, q_2]$, then $[q_1, q_2] \dashv_{q_1} [p, \bar{p}]$ or $[q_1, q_2] \dashv_{q_2} [p, \bar{p}]$.*

Proof. (i)⇒(ii) This is evident. (ii)⇒(i) First we show that $\mathcal{K}$ is strictly convex. Assume the contrary, namely that $[s, t]$ is a segment contained in $\operatorname{bd}\mathcal{K}$ and that there is no larger segment containing it. Assume, without loss of generality, that the center of $\mathcal{K}$ is the origin of $\mathbb{R}^2$ and consider the points of $\mathbb{R}^2$ as vectors. Let $p = \frac{1}{4}s + \frac{3}{4}t$, $q_1 = \frac{4}{5}\bar{s} + \frac{1}{5}\bar{t}$ and $q_2 = \frac{5}{6}\bar{s} + \frac{1}{6}\bar{t}$. Then $p \in [s, t]$, $q_1, q_2 \in [\bar{s}, \bar{t}]$ and $[p, \bar{p}] \dashv [q_1, q_2]$. If $[q_1, q_2] \dashv_{q_1} [p, \bar{p}]$, then $p + q_1 - \bar{p} = 2p + q_1 = \frac{-3}{10}s + \frac{13}{10}t \in \operatorname{bd}\mathcal{K}$, which implies $[s, t] \subsetneq [s, \frac{-3}{10}s + \frac{13}{10}t] \subset \operatorname{bd}\mathcal{K}$, contradicting the hypothesis. If $[q_1, q_2] \dashv_{q_2} [p, \bar{p}]$, we obtain a similar result. Thus, by Theorem 4.1, our assumption (ii) can be rewritten as: *If* $p, q_1, q_2 \in \operatorname{bd}\mathcal{K}$ *and* $[p, \bar{p}] \dashv [q_1, q_2]$, *then* $[q_1, q_2] \dashv [p, \bar{p}]$.

Now we shall see that the midpoints of every family of parallel chords lie in a line, which is a well known characterization of ellipses; see, e.g., [10]. Let $q \in \operatorname{bd}\mathcal{K}$, and let P be a line parallel to $[q, \bar{q}]$ that supports $\mathcal{K}$ at a point, say p. Then $[p, \bar{p}] \dashv [q_1, q_2]$ for any chord $[q_1, q_2]$ parallel to $[q, \bar{q}]$. By (ii), $[q_1, q_2] \dashv [p, \bar{p}]$. Let Q_i, $i = 1, 2$, denote the lines parallel to $[p, \bar{p}]$ and passing through q_i. By the definition of affine orthogonality it follows that if Q_1 supports $\mathcal{K}$, then Q_2 also supports $\mathcal{K}$, and since $\mathcal{K}$ is strictly convex we get $q_2 = \bar{q}_1$. This implies that the midpoint of $[q_1, q_2]$ is the center of $\mathcal{K}$, thus lying in $[p, \bar{p}]$. On the other hand, if $Q_1 \cap \operatorname{bd}\mathcal{K} = \{q_1, q_1'\}$, then $[q_1', q_2]$ is an affine diameter and its midpoint is again the center of $\mathcal{K}$. Since Q_1 is parallel to $[p, \bar{p}]$, this chord cuts $[q_1, q_2]$ in its midpoint. ■

Theorem 4.4. *For a centrally symmetric convex body $\mathcal{K} \subset \mathbb{R}^2$, the following properties are equivalent:*

(i) *The boundary of $\mathcal{K}$ is a circular disc.*

(ii) *For $p, q_1, q_2 \in \operatorname{bd}\mathcal{K}$, if $[p, \bar{p}] \dashv [q_1, q_2]$, then $[p, \bar{p}]$ is orthogonal to $[q_1, q_2]$ in the Euclidean sense.*

Proof. (i)⇒(ii) This is evident. (ii)⇒(i) First we show that $\mathcal{K}$ is smooth. Assume, on the contrary, that at a point p of $\operatorname{bd}\mathcal{K}$ there are two different supporting lines, say L_1 and L_2. Let $q_1, q_2 \in \operatorname{bd}\mathcal{K}$ be such that $[q_1, \bar{q}_1]$ is parallel to L_1 and $[q_2, \bar{q}_2]$ is parallel to L_2. Then $[p, \bar{p}] \dashv [q_1, \bar{q}_1]$ and $[p, \bar{p}] \dashv [q_2, \bar{q}_2]$, which implies that $[p, \bar{p}]$ is orthogonal to $[q_1, \bar{q}_1]$ and to $[q_2, \bar{q}_2]$ in the Euclidean sense, which is absurd. Without loss of generality we can assume that $\mathcal{K}$ is centered at the origin. Since it is smooth, we can consider $\mathcal{K}$ as the unit ball of a Gateaux differentiable norm and parameterize bd $\mathcal{K}$ via a function

$$\theta \in [0, 2\pi] \to x(\theta) = \rho(\theta)(\cos\theta, \sin\theta) \in \operatorname{bd}\mathcal{K},$$

where $\rho(\theta) = \|(\cos\theta, \sin\theta)\|^{-1}$ is thus a positive differentiable function. Then, for each $\theta \in [0, 2\pi]$, the line through $x(\theta)$ parallel to $x'(\theta)$ supports $\mathcal{K}$ at $x(\theta)$, which implies that $[x(\theta), -x(\theta)]$ is affine orthogonal (and then also orthogonal in the Euclidean sense) to that line. Therefore, the scalar product $x(\theta) \cdot x'(\theta)$ is zero and then

$$0 = \rho(\theta)\big(\cos\theta, \sin\theta\big) \cdot \big(\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta, \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta\big) = \rho(\theta)\rho'(\theta),$$

which implies that $\rho'(\theta) = 0$ for $\theta \in [0, 2\pi]$. Consequently bd $\mathcal{K}$ is the circle of center zero and radius $\rho(0)$. ■

4.4 Convex bodies of constant width

Let $(\mathbb{M}, \|\cdot\|)$ be a normed plane, and let $\mathcal{K} \subset \mathbb{M}$ be a convex body. Following Eggleston [7, p. 166], a chord $[p, q]$ of $\mathcal{K}$ is called a *normal of $\mathcal{K}$ at p* if $[p, q]$ is Birkhoff orthogonal to a supporting line of $\mathcal{K}$ at p.

Theorem 4.5. ([4, (VI')]) *In a strictly convex and smooth normed plane, a convex body $\mathcal{K}$ is of constant width if and only if every chord $[p, q]$ of $\mathcal{K}$ that is a normal of $\mathcal{K}$ at p is also a normal of $\mathcal{K}$ at q.*

Our next theorem characterizes bodies of constant width by relating affine orthogonal chords to normal chords.

Theorem 4.6. *Let $\mathcal{K}$ be a convex body in a strictly convex and smooth normed plane. The following properties are equivalent:*

(i) *$\mathcal{K}$ is of constant width.*

(ii) *If $[p_1, p_2]$ is a normal chord of $\mathcal{K}$ at p_1 and $[p_1, p_2]$ is Birkhoff orthogonal to the chord $[q_1, q_2]$, then $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$.*

Proof. (i)⇒(ii) Assume that $[p_1, p_2]$ is a normal chord of $\mathcal{K}$ at p_1. Then $[p_1, p_2]$ is Birkhoff orthogonal to a line L that supports $\mathcal{K}$ at p_1. By Theorem 4.5, the line through p_2 parallel to L supports $\mathcal{K}$ at p_2. If $[p_1, p_2]$ is Birkhoff orthogonal to the chord $[q_1, q_2]$, then $[q_1, q_2]$ is parallel to L, from which it follows that $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$.
(ii)⇒(i) Let $[p_1, p_2]$ be a chord of $\mathcal{K}$ that is a normal of $\mathcal{K}$ at p_1, and let L be a supporting line of $\mathcal{K}$ at p_1 such that $[p_1, p_2]$ is Birkhoff orthogonal to L. Let $[q_1, q_2]$ be any chord parallel to L. Then $[p_1, p_2]$ is Birkhoff orthogonal to $[q_1, q_2]$, and by (ii) we have $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$, which implies that the line through p_2 parallel to L supports $\mathcal{K}$ at p_2. Thus $[p_1, p_2]$ is also a normal of $\mathcal{K}$ at p_2 and, by Theorem 4.5, $\mathcal{K}$ is of constant width. ■

In [12] Martini and Makai Jr. gave the following characterization of bodies of constant width in the Euclidean plane: a convex body of diameter 1 in $\mathbb{E}^2$ is of constant width 1 if and only if any two perpendicular chords of it have total length greater than or equal to 1. V. Soltan posed the question of extending this characterization to normed planes by replacing the usual Euclidean orthogonality by Birkhoff orthogonality. But as the counterexample constructed in [1] shows, in general this cannot be done. Now we prove that such an extension is possible if Euclidean orthogonality is replaced by affine orthogonality. For that purpose we need the following lemma.

Lemma 4.1 ([18, Property 3.2]). *Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex body. Then for any line L there exist $q_1, q_2 \in \operatorname{bd} \mathcal{K}$ such that $[q_1, q_2]$ is an affine diameter of $\mathcal{K}$ and $\langle q_1, q_2 \rangle$ is parallel to L.*

Let $[q_1, q_2]$ be a chord of a convex body $\mathcal{K} \subset \mathbb{R}^2$, and let $p \in \operatorname{bd} \mathcal{K}$. We say that *$p$ is in the neighbourhood of* $[q_1, q_2]$ if there exists an affine diameter $[q_1', q_2']$ which is parallel to $[q_1, q_2]$ such that $[p, q_1'] \cap [q_1, q_2] \neq \emptyset$. And we say that a convex body $\mathcal{K}$ in a normed plane has the *affine orthogonality property* if for any two intersecting chords $[p_1, p_2]$ and $[q_1, q_2]$, with p_1 in the neighbourhood of $[q_1, q_2]$ and $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$, the inequality

$$\|p_1 - p_2\| + \|q_1 - q_2\| \geq \operatorname{diam} \mathcal{K}$$

holds.

Theorem 4.7. *A convex body $\mathcal{K}$ in a normed plane is of constant width if and only if it has the affine orthogonality property.*

Proof. Assume that $\mathcal{K}$ is of constant width, and let $[p_1, p_2]$ and $[q_1, q_2]$ be two intersecting chords of $\mathcal{K}$ such that $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$ and p_1 is in the neighbourhood of $[q_1, q_2]$. Let P_1 be the line through p_1 and parallel to $[q_1, q_2]$. Then there exists $p_1' \in P_1 \cap \operatorname{bd} \mathcal{K}$ (it is possible that $p_1' = p_1$) such that $[p_1', p_2]$ is an affine diameter. Since p_1 is in the neighbourhood of $[q_1, q_2]$, we have that $\|p_1 - p_1'\| \leq \|q_1 - q_2\|$, and then

$$\operatorname{diam} \mathcal{K} = \|p_2 - p_1'\| \leq \|p_2 - p_1\| + \|p_1 - p_1'\| \leq \|p_1 - p_2\| + \|q_1 - q_2\|.$$

Conversely, assume that $\mathcal{K}$ has the affine orthogonality property and that there exists an affine diameter $[x, y]$ with $\|x - y\| < \operatorname{diam} \mathcal{K}$. Let P be a supporting line of $\mathcal{K}$ at x, and let us first assume that P touches $\operatorname{bd} \mathcal{K}$ only at x. Then there exists a chord $[q_1, q_2]$ of $\mathcal{K}$ parallel to P and such that $\|q_1 - q_2\| < \operatorname{diam} \mathcal{K} - \|x - y\|$, with x in the neighbourhood of $[q_1, q_2]$. But then $[x, y] \dashv_x [q_1, q_2]$ and $\|q_1 - q_2\| + \|x - y\| < \operatorname{diam} \mathcal{K}$, which contradicts the affine orthogonality property. On the other hand, if P touches $\operatorname{bd} \mathcal{K}$ at a segment that contains x then, taking a point x' in that segment such that $\|x - x'\| < \operatorname{diam} \mathcal{K} - \|x - y\|$, we get also a contradiction, because $[x, y] \dashv_x [x, x']$. ■

5 Concluding remarks

The notion of affine orthogonality defined for a two-dimensional convex body can be extended to higher dimensions in a natural way. Let $\mathcal{K}$ be a convex body in $\mathbb{R}^n$ ($n \geq 3$). The intersection of $\mathcal{K}$ with a two-dimensional flat α is a plane convex body (*plane section*) that will be denoted by $\mathcal{K}_\alpha$. Let $[p_1, p_2]$ and $[q_1, q_2]$ be two intersecting chords of $\mathcal{K}$ and let α be the 2-flat that contains them. We say that $[p_1, p_2]$ is affine orthogonal through p_1 to $[q_1, q_2]$, denoted again $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$, if $[p_1, p_2]$ is affine orthogonal through p_1 to $[q_1, q_2]$ with respect to $\mathcal{K}_\alpha$. For defining affine orthogonality it is not necessary that $[p_1, p_2]$ and $[q_1, q_2]$ intersect, but sufficient that there exists a 2-flat containing both chords. If α passes through the origin, we call $\mathcal{K}_\alpha$ a *main plane section of* $\mathcal{K}$, and if two chords of $\mathcal{K}$ determine a 2-flat passing through the origin, then we call them *main chords*.

Theorem 5.1 below follows directly from next lemma and Theorems 4.1, 4.3, 4.4, and 4.7.

Lemma 5.1 ([8, § 7.1]). *Let $\mathcal{K}$ be a convex body in the Euclidean space $\mathbb{E}^n$ and $2 \leq k \leq n-1$. Denote by $\mathcal{G}(n,k)$ the k-dimensional subspaces of $\mathbb{E}^n$. Then:*

1. *If $\mathcal{K} \cap S$ is centrally symmetric for each $S \in \mathcal{G}(n,k)$, then $\mathcal{K}$ is either symmetric about the origin or an ellipsoid.*
2. *If $\mathcal{K}$ contains the origin in its interior and $\mathcal{K} \cap S$ is an ellipsoid for each $S \in \mathcal{G}(n,k)$, then $\mathcal{K}$ is an ellipsoid.*
3. *If $\mathcal{K} \cap S$ is a ball for each $S \in \mathcal{G}(n,k)$, then $\mathcal{K}$ is a ball.*
4. *If $\mathcal{K} \cap S$ is of constant with in S for each $S \in \mathcal{G}(n,k)$, then $\mathcal{K}$ is a ball.*

Theorem 5.1. *For a convex body $\mathcal{K}$ in the Euclidean space $\mathbb{E}^n$, $n \geq 3$, the following statements holds.*

1a. *Assume that $\mathcal{K}$ is strictly convex and symmetric about the origin. If $[p_1, p_2]$ and $[q_1, q_2]$ are two main chords with $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$, then $[p_1, p_2] \dashv_{p_2} [q_1, q_2]$.*

1b. *Assume that for any two main chords $[p_1, p_2]$ and $[q_1, q_2]$ the implication*

$$[p_1, p_2] \dashv_{p_1} [q_1, q_2] \Longrightarrow [p_1, p_2] \dashv_{p_2} [q_1, q_2]$$

holds. Then $\mathcal{K}$ is strictly convex and symmetric about the origin or an ellipsoid.

2. *Assume that $\mathcal{K}$ is symmetric about the origin. The following properties are equivalent:*

   (i) *$\mathcal{K}$ is an ellipsoid.*

   (ii) *If p, q_1 and q_2 belong to the boundary of a main plane section of $\mathcal{K}$ and $[p, \bar{p}] \dashv [q_1, q_2]$, then $[q_1, q_2] \dashv_{q_1} [p, \bar{p}]$ or $[q_1, q_2] \dashv_{q_2} [p, \bar{p}]$.*

3. *Assume that $\mathcal{K}$ is symmetric about the origin. The following properties are equivalent:*

(i) *$\mathcal{K}$ is a ball.*

(ii) *If p, q_1 and q_2 belong to a main plane section of $\mathcal{K}$ and $[p, \bar{p}] \dashv [q_1, q_2]$ then $[p, \bar{p}]$ is orthogonal to $[q_1, q_2]$ in the Euclidean sense.*

4. *Assume that for any two main intersecting chords $[p_1, p_2]$, $[q_1, q_2]$, with p_1 in the neighbourhood of $[q_1, q_2]$ and $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$, the inequality $\|p_1 - p_2\| + \|q_1 - q_2\| \geq \operatorname{diam} \mathcal{K}$ holds. Then $\mathcal{K}$ is a ball.*

As we mentioned before, the notion of affine orthogonality with respect to circular discs in the Euclidean plane coincides with the usual Euclidean orthogonality. Now we consider two types of non-metrical affine planes, namely the Lorentzian and the isotropic plane, and we shall see that again the affine orthogonality with respect to corresponding circles in such planes coincides with the usual Lorentzian or isotropic orthogonality. Note that in this framework we are not working anymore with the boundary of a convex body. We consider closed convex (in the sense that at any point there is a supporting line) curves which involve points at infinity. Thus the line at infinity can be also a supporting line. A chord $[p_1, p_2]$ of such closed convex curve is an affine diameter if there are supporting lines at p_1 and p_2 having an infinity point in common.

Let $\mathbb{L}^2$ be the vector space $\mathbb{R}^2$ equipped with the *Lorentzian inner product*

$$x \cdot y = x_1 y_1 - x_2 y_2 \quad \text{for} \quad x = (x_1, x_2),\ y = (y_1, y_2).$$

Two vectors x and y are said to be *Lorentzian orthogonal* if $x \cdot y = 0$. A vector $x \in \mathbb{L}^2$ is said to be *timelike* if $x \cdot x < 0$, *spacelike* if $x \cdot x > 0$, and *null* if $x \cdot x = 0$. The affine plane associated to the Lorentzian vector space $\mathbb{L}^2$ is called the *Lorentzian plane*; see, e.g., [3] and [20, § 11 and § 12]. It should be noticed that the terms "pseudo-Euclidean" or "Minkowski" are also used for this plane. Indeed, Minkowski introduced this geometry but Lorentz pioneered the notion of the Lorentzian inner product. We also note that the Minkowski plane in this sense should be distinguished from Minkowski planes in the sense of normed planes. A (timelike) *Lorentzian circle* is said to be a curve with equation

$$(x_1 - p_1)^2 - (x_2 - p_2)^2 = \lambda^2.$$

From the viewpoint of Klein's concept of geometry the absolute of the Lorentzian geometry consists of two points, for example $f_1 = (1, 1, 0)$ and $f_2 = (1, -1, 0)$ with respect to an affine coordinate system with homogeneous coordinates such that the line at infinity has the equation $x_3 = 0$. Let us consider a Lorentzian circle that is the rectilinear hyperbola $\mathcal{H} : x_1^2 - x_2^2 = x_3^2$. For any point q of $\mathcal{H}$ with $x_3 \neq 0$ the vector $\overrightarrow{Oq}$ is Lorentzian orthogonal to the tangent line at q. The hyperbola $\mathcal{H}$ (together with f_1 and f_2) is a closed convex curve. One can imagine this curve gluing the two branches of $\mathcal{H}$ at its infinite points. A chord of $\mathcal{H}$ is an affine diameter if and only if p_1 and p_2 are finite points and $p_1 = -p_2$. Thus we have that for two chords $[p_1, p_2]$, $[q_1, q_2]$ of $\mathcal{H}$ the relation $[p_1, p_2] \dashv_{p_1} [q_1, q_2]$ holds if and only if p_1, p_2 are finite, $p_1 = -p_2$, and $[q_1, q_2]$ is parallel to the tangent of $\mathcal{H}$ at p_1. In other words, this holds if and only if the vector $\overrightarrow{0p_1}$ is Lorentzian orthogonal to the direction determined by $[q_1, q_2]$.

The *isotropic plane* is defined as a projective plane with absolute (in the sense of Klein) consisting of a line F and a point f on this line; see, e.g, [17] and [20, Chapter 1 and Chapter 2]. This plane is also called Galilean plane, since its group of motions describes Galileo's principle of relativity. This principle says that all properties studied in mechanics are preserved under transformations of the physical system obtained by imparting to it a velocity which is constant in magnitude and direction, i.e., under so-called Galilean transformations. An *isotropic circle* is a conic touching F at f. Any line through f is *isotropic orthogonal* to an arbitrary line; see again [17]. Choosing again an affine coordinate system such that $F : x_3 = 0$ and $f = (0, 1, 0)$, we have that isotropic circles are parabolas with diameter parallel to the second coordinate axis, and a line parallel to the second coordinate axis is isotropic orthogonal to any straight line. We can consider an isotropic circle as a convex closed curve, and then all affine diameters are the chords through f. Thus we have that two chords $[p_1, p_2]$, $[q_1, q_2]$ of an isotropic circle are affine orthogonal if and only if they are isotropic orthogonal.

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