Almost Kenmotsu manifolds with conformal Reeb foliation

Anna Maria Pastore Vincenzo Saltarelli

Abstract

We consider almost Kenmotsu manifolds with conformal Reeb foliation. We prove that such a foliation produces harmonic morphisms, we study the k-nullity distributions and we discuss the isometrical immersion of such a manifold M as hypersurface in a real space form $\widetilde{M}(c)$ of constant curvature c proving that $c \leq -1$ and, if $c < -1$, M is totally umbilical, Kenmotsu and locally isometric to the hyperbolic space of constant curvature -1. Finally, the Einstein and η-Einstein conditions are discussed.

1 Introduction

Manifolds known as Kenmotsu manifolds have been studied by K. Kenmotsu in 1972 ([14]). They set up one of the three classes of almost contact Riemannian manifolds whose automorphism group attains the maximum dimension ([23]). Consider an almost contact metric manifold M^{2n+1}, with structure (φ, ξ, η, g) given by a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g satisfying $\varphi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$, and $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$ for any vector fields X and Y. The fundamental 2-form Φ is defined by $\Phi(X,Y) = g(X, \varphi Y)$ for any vector fields X and Y. It is well known that the normality of an almost contact metric manifold is expressed by the vanishing of the tensor field $N = [\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ ([3]). For more details, we refer to Blair's books [3, 5]. A Kenmotsu manifold

Received by the editors March 2010 - In revised form in September 2010.
Communicated by L. Vanhecke.
2000 *Mathematics Subject Classification :* 53C15; 53C25.
Key words and phrases : Almost Kenmotsu manifolds, harmonic morphisms, nullity distributions, real space forms, η-Einstein conditions.

can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. It is well known that Kenmotsu manifolds can be characterized, through their Levi-Civita connection, by $(\nabla_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi(X)$, for any vector fields X, Y. Moreover, Kenmotsu proved that such a manifold M^{2n+1} is locally a warped product $]-\varepsilon, \varepsilon[\times_f N^{2n}$, N^{2n} being a Kähler manifold and $f = ce^t$, c a positive constant. More recently in [13, 19, 15, 7], almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold.

In section 2 we recall briefly some results we need, referring to the cited sources for details. In contact geometry, it is a well-known fact that the vanishing of $h = \frac{1}{2}\mathcal{L}_\xi \varphi$, where $\mathcal{L}$ denotes the Lie differentiation, is equivalent to say that ξ is a Killing vector field. As we will see, in an almost Kenmotsu manifold $h = 0$ means that the canonical foliation $\mathcal{F}$, generated by the involutive 1-dimensional distribution $\ker(\varphi)$ and called Reeb foliation, is conformal. Some properties of such manifolds are investigated in section 3, showing, in particular, that $\mathcal{F}$ produces harmonic morphisms. In section 4, we prove that an almost Kenmotsu manifold, whose characteristic vector field belongs to the k-nullity distribution, has a conformal Reeb foliation; moreover, k is necessarily equal to -1. In [21], contact metric hypersurfaces in a real space form are studied and in this paper we study an analogous problem in the context of the almost Kenmotsu manifolds. If M^{2n+1} is an almost Kenmotsu manifold with (-1)-nullity condition and it is isometrically immersed as hypersurface in a real space form $\widetilde{M}(c)$ of constant curvature c, we prove that $c \leq -1$ and, if $c < -1$, M^{2n+1} is totally umbilical and Kenmotsu; while, if $c = -1$, under the additional assumption that M^{2n+1} is Einstein, M^{2n+1} is totally geodesic and Kenmotsu. In any case M^{2n+1} is locally isometric to the hyperbolic space of constant curvature -1. In the final section 5, the Einstein and η-Einstein conditions on almost Kenmotsu manifolds with conformal Reeb foliation are discussed.

Finally, all manifolds are assumed to be connected; $\mathcal{D}$ will denote the distribution $\ker(\eta)$; the notation for curvature and Ricci tensor fields is that used in [16].

2 Almost Kenmotsu manifolds

We recall from [15, 7] the results we need. Let M^{2n+1} be an almost Kenmotsu manifold with structure (φ, ξ, η, g). Since the 1-form η is closed, we have $\mathcal{L}_\xi \eta = 0$ and $[X, \xi] \in \mathcal{D}$ for any $X \in \mathcal{D}$. Then, using the expression of the Levi-Civita connection for an almost contact metric manifold ([3]), we have:

$$2g((\nabla_X \varphi)Y, Z) = 2g(g(\varphi X, Y)\xi - \eta(Y)\varphi X, Z) + g(N(Y, Z), \varphi X), \tag{1}$$

for any vector fields X, Y, Z. We obtain $\nabla_\xi \varphi = 0$, which implies that $\nabla_\xi \xi = 0$ and $\nabla_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$. The tensor fields $h = \frac{1}{2}\mathcal{L}_\xi \varphi$, is a symmetric operator anticommuting with φ and $h(\xi) = 0$. We remark that since $\nabla_X \xi = -\varphi^2 X - \varphi h X$ for any $X \in \Gamma(TM^{2n+1})$, then $h = 0$ if and only if $\nabla \xi = -\varphi^2$. By direct computation we have

$$R_{\xi X}\xi - \varphi R_{\xi \varphi X}\xi = -2\varphi^2 X + 2h^2 X\,, \tag{2}$$

$$R_{XY}\xi = \eta(X)(Y - \varphi hY) - \eta(Y)(X - \varphi hX) + (\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y\,, \tag{3}$$

$$(\mathcal{L}_\xi g)(X,Y) = 2g(X, Y - \eta(Y)\xi + h\varphi Y)\,. \tag{4}$$

Proposition 2.1. ([15]) *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold. The integral submanifolds of $\mathcal{D}$ are almost Kähler manifolds with mean curvature vector field $H = -\xi$. They are totally umbilical submanifolds if and only if $h = 0$.*

Proposition 2.2. ([15, 7]) *An almost Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold if and only if $h = 0$ and the integral submanifolds of $\mathcal{D}$ are Kähler.*

As a consequence of an analogous result proved in [9] for globally framed almost Kenmotsu f-manifolds, the distribution $\mathcal{D}$ of an almost Kenmotsu manifold has Kähler leaves if and only if, for any $X, Y \in \Gamma(TM^{2n+1})$,

$$(\nabla_X \varphi)(Y) = g(\varphi X + hX, Y)\xi - \eta(Y)(\varphi X + hX)\,.$$

Theorem 2.1. ([7]) *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h = 0$. Then, M^{2n+1} is locally a warped product $M' \times_f N^{2n}$, where N^{2n} is an almost Kähler manifold, M' is an open interval with coordinate t, and $f = ce^t$, for some positive constant c.*

Theorem 2.2. ([7]) *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold. Then, M^{2n+1} is a Kenmotsu manifold if and only if $h = 0$. Moreover, if any of the above equivalent conditions holds, then M^{2n+1} has constant sectional curvature $K = -1$.*

Furthermore, as proved in [7], an almost Kenmotsu manifold of constant curvature K is a Kenmotsu manifold and $K = -1$.

3 Conformal Reeb foliation and harmonic morphisms

As it is well known in the contact case the vanishing of the tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ means that the Reeb vector field is Killing. We shall see that for almost Kenmotsu manifolds $h = 0$ means that the Reeb foliation is conformal (in fact homothetic) and we shall investigate some properties of such manifolds.

We recall the following definition ([1, 11, 12]).
Let $\mathcal{F}$ be an n-dimensional foliation on a Riemannian manifold (N, g) of dimension m and denote by $\mathcal{V}$ the corresponding involutive distribution and by $\mathcal{H}$ its orthogonal distribution, which is not integrable in the general case. The foliation $\mathcal{F}$ is said to be
a) Riemannian if the metric g is bundle-like (with respect to $\mathcal{F}$) i.e.

$$\forall X, Y \in \Gamma(\mathcal{H})\,,\ \forall V \in \Gamma(\mathcal{V})\ \ (\mathcal{L}_V g)(X,Y) = 0\,,$$

b) conformal if

$$\forall X, Y \in \Gamma(\mathcal{H}),\ \forall V \in \Gamma(\mathcal{V}) \ \ (\mathcal{L}_V g)(X,Y) = \lambda(V) g(X,Y)$$

where λ is a function depending on V.
c) homothetic if it is conformal and the 1-form dual to the mean curvature vector of $\mathcal{H}$, $H^{\mathcal{H}}$, is closed.

Moreover, $\mathcal{F}$ Riemannian means that the distribution $\mathcal{H}$ has totally geodesic leaves if it is involutive and $\mathcal{F}$ conformal means that $\mathcal{H}$ has totally umbilical leaves if it is involutive.

Now, let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold. Then the distributions $\ker \varphi =< \xi >$ and $\mathcal{D}$ are both integrable. Since a non vanishing h admits eigenvalues with opposite sign, (4) implies that the Reeb foliation $\mathcal{F}$ is conformal if and only if $h = 0$ and this in turn means that the integral submanifolds of $\mathcal{D}$ are totally umbilical. A first property of an almost Kenmotsu manifold with conformal Reeb foliation is related to harmonic morphisms.

We recall that a harmonic morphism is a smooth map $f : N \to N'$ between two Riemannian manifolds (N, g) and (N', g') which pulls back germs of harmonic functions on N' to germs of harmonic functions on N. It is well known that the connected components of a submersive harmonic morphism form a conformal foliation. Then, we say that a foliation $\mathcal{F}$ of a smooth Riemannian manifold (N, g) produces harmonic morphisms if each point has a neighborhood U which supports a submersive harmonic morphism with associated foliation $\mathcal{F}_{|U}$, ([1]). The following results are proved in [1, 6, 24].

Theorem 3.1. *Let (N, g) be a Riemannian manifold of dimension m.*
1) A foliation of codimension $q = 2$ produces harmonic morphisms if and only if it is conformal and has minimal leaves.
2) A foliation of codimension $q \neq 2$ produces harmonic morphisms if and only if it is conformal and $\mathrm{d}\omega = 0$, where $\omega = (q-2)(H^{\mathcal{H}})^\flat - (m-q)(H^{\mathcal{V}})^\flat$, $\flat$ denoting the musical isomorphism.

Remark 3.1. A homothetic foliation with codimension $q \neq 2$ produces harmonic morphisms if and only if $(H^{\mathcal{H}})^\flat$ is closed.

Since in an almost Kenmotsu manifold with conformal Reeb foliation the mean curvature vector field of $\mathcal{D}$ is $H = -\xi$ and its dual 1-form $-\eta$ is closed, Theorem 3.1 implies that if $n > 1$, then $\mathcal{F}$ produces harmonic morphisms and the same happens if $n = 1$ since $\mathcal{F}$ has totally geodesic leaves.

4 The low compatibility with $N(\kappa)$ distributions

We recall that the κ-nullity distribution, $\kappa \in \mathbb{R}$, is defined as the distribution given by putting for each $p \in M^{2n+1}$

$$N_p(\kappa) = \{Z \in T_p M^{2n+1} \mid R_{XY}Z = \kappa(g(Y,Z)X - g(X,Z)Y)\}.$$

As usual, when ξ belongs to a nullity distribution, we say that the related nullity condition holds.

Proposition 4.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold with conformal Reeb foliation. Then, for any vector fields X and Y, one has:*

$$R_{XY}\xi = \eta(X)Y - \eta(Y)X \tag{5}$$

$$R_{X\xi}\xi = \varphi^2(X). \tag{6}$$

$$Ric(X, \xi) = -2n\eta(X) \tag{7}$$

$$R_{\xi X}Y = -g(X, Y)\xi + \eta(Y)X. \tag{8}$$

Proof. Since $h = 0$, the first formula follows immediately from (3). Moreover, (6) follows from (5) putting $Y = \xi$. We consider an orthonormal basis $(E_1, \dots, E_{2n}, \xi)$ and computing $Ric(X, \xi)$, we get

$$Ric(X, \xi) = \sum_{i=1}^{2n} g(E_i, R_{E_i X}\xi) = \sum_{i=1}^{2n} g(E_i, -\eta(X)E_i) = -2n\eta(X)$$

that is (7). Finally by the symmetries of the curvature tensor field we get (8). Moreover (6) also implies that the ξ-sectional curvatures are $K(X, \xi) = -1$. ■

We notice that if $(M^{2n+1}, \varphi, \xi, \eta, g)$ is an almost Kenmotsu manifold with conformal Reeb foliation, then (5) implies that ξ belongs to the (-1)-nullity distribution. Now, as a consequence of the following theorem, the converse holds so that the property of the Reeb foliation to be conformal and the (-1)-nullity condition become equivalent.

Theorem 4.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold. Let us suppose that ξ belongs to the κ-nullity distribution, $\kappa \in \mathbb{R}$. Then, the Reeb foliation is conformal and $\kappa = -1$. Therefore, M^{2n+1} is locally a warped product of an almost Kähler manifold and an open interval. Finally, assuming the local symmetry, M^{2n+1} is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ of constant curvature -1.*

Proof. Let X be a unit vector field orthogonal to ξ. Since ξ belongs to the κ-nullity distribution we have $R_{X\xi}\xi = \kappa X$ and (2) implies that $h^2 X = -(\kappa + 1)X$. Now, if X is a unit eigenvector of h with eigenvalue λ, we get $\lambda^2 X = -(\kappa + 1)X$ and thus $-(\kappa + 1) = \lambda^2 \geq 0$. It follows that $\kappa \leq -1$ and $Spec(h) = \{0, \lambda, -\lambda\}$, with λ constant. Computing $R_{X\xi}\xi$ by means of (3), we get

$$\kappa X = \nabla_\xi \varphi h X - \varphi h \nabla_\xi X - X + 2\lambda\varphi X - \lambda^2 X,$$

and taking the scalar product with φX, we obtain $\lambda = 0$, $\kappa = -1$, and thus $K(X, \xi) = -1$. Being $h = 0$, Theorem 2.1 ensures that M^{2n+1} is locally a warped product. Obviously, if $n = 1$, M^3 is a Kenmotsu manifold, by Proposition 2.2. Furthermore, if M^{2n+1} is locally symmetric, by Theorem 2.2, it is a Kenmotsu manifold locally isometric to $\mathbb{H}^{2n+1}(-1)$. ■

Now, before looking at an almost Kenmotsu manifold, with conformal Reeb foliation, as hypersurface of a real space form $\widetilde{M}(c)$, we recall from [22, 25] the following result in the case of $\mathbb{H}^{2n+2}(c)$, $c < 0$. Let $\rho = \sqrt{-1/c}$ and consider

$$\mathbb{H}^{2n+2}(c) = \{(x_1, \dots, x_{2n+3}) \in \mathbb{R}^{2n+3} \mid x_{2n+3} > 0, \\ x_1^2 + \dots + x_{2n+2}^2 - x_{2n+3}^2 = -\rho^2\}.$$

Lemma 4.1 ([25]). *Let M^{2n+1} be a complete hypersurface in $\mathbb{H}^{2n+2}(c)$. Suppose that, under a suitable choice of a local orthonormal tangent frame field of TM^{2n+1}, the shape operator over TM^{2n+1} is expressed as a matrix A. If M^{2n+1} has at most two distinct constant principal curvatures, then it is congruent to one of the following:*

(1) $M_1 = \{x \in \mathbb{H}^{2n+2}(c) \mid x_1 = 0\}$. *In this case, $A = 0$, and M_1 is totally geodesic. Hence M_1 is isometric to $\mathbb{H}^{2n+1}(c)$;*

(2) $M_2 = \{x \in \mathbb{H}^{2n+2}(c) \mid x_1 = r > 0\}$. *In this case, $A = \frac{1/\rho^2}{\sqrt{1/\rho^2+1/r^2}} I_{2n+1}$ where I_{2n+1} denotes the identity matrix of degree $2n+1$, and M_2 is isometric to $\mathbb{H}^{2n+1}(-1/(\rho^2+r^2))$;*

(3) $M_3 = \{x \in \mathbb{H}^{2n+2}(c) \mid x_{2n+3} = x_{2n+2} + \rho\}$. *In this case, $A = \frac{1}{\rho} I_{2n+1}$ and M_3 is isometric to a Euclidean space E^{2n+1};*

(4) $M_4 = \{x \in \mathbb{H}^{2n+2}(c) \mid \sum_{i=1}^{2n+2} x_i^2 = r^2 > 0\}$. *Then $A = \sqrt{1/r^2 + 1/\rho^2} I_{2n+1}$ and M_4 is isometric to a round sphere $S^{2n+1}(r)$ of radius r;*

(5) $M_5 = \{x \in \mathbb{H}^{2n+2}(c) \mid \sum_{i=1}^{k+1} x_i^2 = r^2 > 0, \sum_{j=k+2}^{2n+2} x_j^2 - x_{2n+3}^2 = -\rho^2 - r^2\}$. *In this case, $A = \lambda I_k \oplus \mu I_{2n+1-k}$, where $\lambda = \sqrt{1/r^2 + 1/\rho^2}$ and $\mu = \frac{1/\rho^2}{\sqrt{1/\rho^2+1/r^2}}$. M_5 is isometric to $S^k(r) \times \mathbb{H}^{2n+1-k}(-1/(r^2+\rho^2))$.*

Remark 4.1. We notice that, except for case (1), the shape operator can not admit zero as eigenvalue. Furthermore, assuming $c < -1$, the shape operator of $M_2 = \{x \in \mathbb{H}^{2n+2}(c) \mid x_1 = r = \sqrt{1 + 1/c} > 0\}$ has only the eigenvalue $\sqrt{-(1+c)}$ and M_2 in $\mathbb{H}^{2n+2}(c)$ is a totally umbilical hypersurface, isometric to $\mathbb{H}^{2n+1}(-1)$, which carries the standard Kenmotsu structure.
Finally, if $c = -1$ then $M_1 = \{x \in \mathbb{H}^{2n+2}(-1) \mid x_1 = 0\}$ has vanishing shape operator and it is totally geodesic, isometric to $\mathbb{H}^{2n+1}(-1)$.

Theorem 4.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold with conformal Reeb foliation and assume that M^{2n+1} is a hypersurface of a real space form $\widetilde{M}(c)$ of constant curvature c. Then*

i) $c \leq -1$,

ii) *If $c < -1$, then M^{2n+1} is Kenmotsu of constant sectional curvature $K = -1$ and then locally isometric to $\mathbb{H}^{2n+1}(-1)$.*

Proof. The curvature R of the immersed M^{2n+1} in $\widetilde{M}(c)$ reads

$$R(X,Y)Z = c(X \wedge Y)Z + (A(X) \wedge A(Y))Z \tag{9}$$

for any $X, Y, Z \in \Gamma(TM^{2n+1})$, being $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ and A denoting the Weingarten operator.

Now, computing $R_{XY}\xi$ and $R_{\xi X}Y$ by means of (9) and comparing with (5), (8) respectively, we obtain

$$(c+1)(\eta(Y)X - \eta(X)Y) + \eta(AY)AX - \eta(AX)AY = 0 \tag{10}$$

$$(c+1)(g(X,Y)\xi-\eta(Y)X)+g(AX,Y)A\xi-\eta(AY)AX=0 \tag{11}$$

Let $(E_1,\ldots,E_{2n+1})$ be a local orthonormal basis given by eigenvectors of the operator A, with eigenvalues $\mu_1,\ldots,\mu_{2n+1}$, non necessarily distinct. Clearly, there exists at least an eigenvector E_k such that $\eta(E_k)\neq 0$, since dim ker $\eta=2n$.

Now, writing (10) for two distinct eigenvectors E_i,E_j, with eigenvalues μ_i,μ_j non necessarily distinct, we obtain

$$(c+1+\mu_i\mu_j)(\eta(E_j)E_i-\eta(E_i)E_j)=0 \tag{12}$$

Furthermore, writing (11) for $E_i\neq E_j$ and for $X=Y=E_i$, we get

$$(c+1+\mu_i\mu_j)\eta(E_j)E_i=0 \tag{13}$$

and

$$-(c+1+\mu_i^2)\eta(E_i)E_i+(c+1)\xi+\mu_iA(\xi)=0 \tag{14}$$

We discuss equation (12) distinguishing two cases:

(I) $c+1+\mu_i\mu_j=0$, for any i,j

(II) there exist $E_i\neq E_j$ such that $c+1+\mu_i\mu_j\neq 0$.

In the first case, we have $\mu_i\mu_j=-(c+1)$, for any i,j which obviously implies $\mu_1=\mu_2=\ldots=\mu_{2n+1}=\mu$ with $\mu^2=-(c+1)$ and then $c\leq -1$.

Now we prove that in the case (II), we have $c=-1$ and ξ is an eigenvector of A with eigenvalue $\mu_\xi=0$.
Supposing that there exist $E_i\neq E_j$ such that $c+1+\mu_i\mu_j\neq 0$, we get

$$\eta(E_i)=\eta(E_j)=0,\quad E_i\neq\xi,\quad E_j\neq\xi \tag{15}$$

and applying (14) to E_i

$$(c+1)\xi+\mu_iA(\xi)=0\,,\quad (c+1,\mu_i)\neq(0,0). \tag{16}$$

It follows that $A(\xi)\in<\xi>$, so ξ is an eigenvector of A with eigenvalue μ_ξ. Note that $c+1=0$ if and only if $\mu_\xi=0$. To end the proof, we show that $c+1\neq 0$ gives a contradiction. From (16) we obtain $\xi=-\frac{\mu_i}{c+1}A(\xi)$, so that $\mu_i\neq 0$ and $\mu_\xi=-\frac{c+1}{\mu_i}$. Analogously, $\mu_\xi=-\frac{c+1}{\mu_j}$ so that we have $\mu_i=\mu_j=\mu$ and (16) implies $\mu_\xi=-\frac{c+1}{\mu}\neq\frac{\mu^2}{\mu}$. Clearly the other eigenvectors verify $\eta(E_k)=0$ and writing (13) with $E_j=\xi$ we get $\mu_k\mu_\xi=-c-1=\mu\mu_\xi$ from which $\mu_k=\mu$ follows. Moreover, we have $A(X)=\mu X$ for any $X\in\mathcal{D}$ and μ constant along any integral submanifold of $\mathcal{D}$. Being $c+1+\mu^2\neq 0$ there exists a point $p\in M^{2n+1}$ such that $c+1+\mu^2(p)\neq 0$. Since $h=0$, M^{2n+1} is locally, around p, a warped product $I\times_f\bar{M}$, being $\bar{M}$ an integral submanifold of the distribution $\mathcal{D}$ through p, I an interval and $f=c'e^t$, c' a positive constant.

Using (9) we compute some sectional curvatures of M^{2n+1}, obtaining for any orthonormal vector fields $X,Y\in\mathcal{D}$,

$$R(X,\xi)\xi=c(X\wedge\xi)\xi+(A(X)\wedge A(\xi))\xi=cX-\frac{c+1}{\mu}\mu X=-X,$$

$$R(X,Y)Y = (c+\mu^2)X,$$

and $K(X,\xi) = -1$, $K(X,Y) = c+\mu^2$.

Then, owing to the structure of warped product, for the sectional curvature on $\bar{M}$ we get $\bar{K}(X,Y) = K(X,Y)+1 = c+\mu^2+1$. Then, being $\bar{M}$ an almost Kähler manifold of constant curvature, it has to be Kähler and flat, contradicting $c+\mu^2(p)+1 \neq 0$ ([4, 17, 18]).

Therefore in any case we get $c \leq -1$ and i) is proved.

Now suppose that $c < -1$. Then, for any other choice of a local orthonormal basis given by eigenvectors of the operator A, the case (I) occurs. It follows that M^{2n+1} is totally umbilical and A has eigenvalue $\sqrt{-c-1}$. Then M^{2n+1} has constant sectional curvature $K = -1$, it is Kenmotsu, locally isometric to $\mathbb{H}^{2n+1}(-1)$. ∎

Remark 4.2. Under the same hypotheses of the previous theorem, if $c = -1$ and case (I) always occurs, then M^{2n+1} is Kenmotsu, locally isometric to $\mathbb{H}^{2n+1}(-1)$ and totally geodesic. By the contrary, if $c = -1$ and the two cases occur in different local orthonormal bases given by eigenvectors of the operator A, then $A(\xi) = 0$ and, as follows by Remark 4.1, A can not admit two constant eigenvalues.

Now, to continue about the case $c = -1$, we recall the following result of Fialkow ([10]) quoted as Theorem 4 in [21].

Theorem 4.3. *Let M be a hypersurface of dimension $n \geq 3$ in a real space form $\widetilde{M}(c)$ of constant curvature c. If M is Einstein with Ricci curvature $\rho = (n-1)c$, then M is either a totally geodesic hypersurface or a developable hypersurface in $\widetilde{M}(c)$. In particular M is a space of constant curvature c.*

Theorem 4.4. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold with conformal Reeb foliation and assume that M^{2n+1} is a hypersurface of a space form $\widetilde{M}(c)$ of constant curvature c. Then*

a) *if $c = -1$ and M^{2n+1} is Einstein, then M^{2n+1} is totally geodesic and Kenmotsu,*

b) *if M^{2n+1} is totally geodesic, then $c = -1$.*

Proof. Let us suppose $c = -1$ and M^{2n+1} Einstein. Then, from (7) it follows that $Ric = -2ng = 2ncg$ and by Theorem 4.3, M^{2n+1} has constant curvature $K = c = -1$ which implies that M^{2n+1} is totally geodesic and Kenmotsu. Finally if M^{2n+1} is totally geodesic then it has constant curvature $K = c$ and, being $K(X,\xi) = -1$ for $X \in \mathcal{D}$, we get $K = -1$ and $c = -1$. Again M^{2n+1} is Kenmotsu. In any case M^{2n+1} and $\widetilde{M}(c)$ are locally isometric to $\mathbb{H}^{2n+1}(-1)$ and $\mathbb{H}^{2n+2}(-1)$, respectively. ∎

5 The η-Einstein condition

In this section we discuss the η-Einstein condition in an almost Kenmotsu manifold with conformal Reeb foliation. Thus we assume $h = 0$ and

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{17}$$

or equivalently

$$Q(X) = aX + b\eta(X)\xi. \tag{18}$$

where $g(Q(X), Y) = Ric(X, Y)$ and a, b are smooth functions on M^{2n+1}.

As proved in Proposition 8 and Corollary 9 of [14] if a Kenmotsu manifold is η-Einstein with a or b constant, then $b = 0$ and the manifold is Einstein.

Now, we consider an almost Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ with conformal Reeb foliation and η-Einstein. From (17) we obtain $Ric(\xi, \xi) = a + b$ and $Ric(X, X) = ag(X, X)$ for any $X \in \mathcal{D}$. Since $h = 0$ we have also $Ric(\xi, \xi) = -2n$. It follows that $a + b = -2n$ and the scalar curvature is given by $Sc = -2n + 2na$. Obviously b and a must be both constant or both non constant.

We begin considering the case of dimension $2n + 1 \geq 5$.

Theorem 5.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$, $n > 1$, be an almost Kenmotsu manifold with conformal Reeb foliation and η-Einstein. Then one of the following cases occurs: either i): $b = 0$ or ii): b is not constant. They are characterized as follows:*

i) M^{2n+1} is Einstein. Moreover the integral submanifolds of $\mathcal{D}$ are Einstein almost Kähler Ricci-flat hypersurfaces.

ii) Locally, we have $b = ce^{-2t}$, $c \neq 0$ and $Q(X) = -2nX + ce^{-2t}\varphi^2(X)$.

Proof. We consider an integral submanifold $\bar{M}$ of the distribution $\mathcal{D}$, which is an almost Kähler manifold. By direct computation, since the Weingarten operator is given by $A(X) = -X$, we obtain

$$\begin{aligned} \bar{R}(X, Y, Z) &= R(X, Y, Z) + g(Y, Z)X - g(X, Z)Y \\ \bar{R}ic(X, Y) &= Ric(X, Y) + 2ng(X, Y) \\ \bar{K}(X, Y) &= K(X, Y) + 1. \end{aligned}$$

We notice that the above formulas can be obtained also using the properties of the warped product structure $M' \times_f N^{2n}$ ensured by Theorem 2.1 ([2, 8, 20]).

We consider the well-known formula

$$\frac{1}{2}\nabla_Y Sc = trace[X \to (\nabla_X Q)Y)]$$

and putting $Y = \xi$, choosing a φ-basis $(\xi, E_1, \ldots, E_n, \varphi E_1, \ldots, \varphi E_n)$, we get

$$(\nabla_\xi Q)\xi = 0, \quad (\nabla_{E_i} Q)\xi = b\nabla_{E_i}\xi = bE_i, \quad (\nabla_{\varphi E_i} Q)\xi = b\nabla_{\varphi E_i}\xi = b\varphi E_i$$

which imply $\nabla_\xi Sc = 4nb$. On the other hand $\nabla_\xi Sc = \nabla_\xi(-2n + 2na) = 2n\xi(a)$, so we get $\xi(b) = -\xi(a) = -2b$.

Obviously b constant gives $b = 0$, $a = -2n$ and M^{2n+1} is Einstein. For the integral submanifold $\bar{M}$, we obtain $\bar{R}ic(X, Y) = (a + 2n)g(X, Y) = 0$, so $\bar{M}$ is almost Kähler Einstein and Ricci-flat.

Finally, if b, and then a, is not constant, putting $\bar{a} = a_{|\bar{M}}$, we obtain that $\bar{R}ic(X, Y) = (\bar{a} + 2n)g(X, Y)$. Being $n > 1$, we have $dim\,\bar{M} \geq 4$ and $\bar{M}$ is almost Kähler Einstein with constant scalar curvature $\bar{S}c = 2n(2n + \bar{a})$. It follows that $Z(a) = 0$, $Z(b) = 0$ for any $Z \in \mathcal{D}$, whereas $\xi(b) = -\xi(a) = -2b$. Again, being ξ tangent to M', locally, we can write $\xi = \frac{\partial}{\partial t}$ obtaining $b = ce^{-2t}$, $c \neq 0$ and $Q(X) = -2nX + ce^{-2t}\varphi^2(X)$. ∎

Remark 5.1. We observe that the proof of $i)$ does not depend on the condition $n > 1$. Namely, if $n = 1$, by a theorem of Schouten and Struik, M^3 is of constant curvature $K = \frac{a}{2} = -1$, hence Kenmotsu and locally isometric to $\mathbb{H}^3(-1)$.

Theorem 5.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold with conformal Reeb foliation. If $R_{X\xi}R = 0$, for any $X \in \Gamma(TM^{2n+1})$, then M^{2n+1} is Kenmotsu of constant curvature $K = -1$.*

Proof. We have to prove $R(X,Y,Z) = -g(Y,Z)X + g(X,Z)Y$ for any X, Y, Z vector fields. Indeed, since (5) and (8) ensure that the above formula holds when at least one of the vector fields coincides with ξ, we have to evaluate $R(U,V,W)$ for U, V, W orthogonal to ξ. By the hypothesis $R_{X\xi}R = 0$, for any $X \in \Gamma(TM^{2n+1})$ we get

$$R_{X\xi}R(U,V,W) = R(R_{X\xi}U,V,W) + R(U,R_{X\xi}V,W) + R(U,V,R_{X\xi}W) \tag{19}$$

and we can assume X orthogonal to ξ. Then, using (5) and (8) in (19) the statement easily follows. ∎

Finally, in the 3-dimensional case, as it is well known, one has

$$\begin{aligned} R(X,Y,Z) &= g(Y,Z)Q(X) - g(X,Z)Q(Y) + Ric(Y,Z)X \\ &\quad - Ric(X,Z)Y - \tfrac{Sc}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{20}$$

and we can state the following result.

Theorem 5.3. *Any 3-dimensional almost Kenmotsu manifold with conformal Reeb foliation is Kenmotsu and η-Einstein with $a = \frac{Sc}{2} + 1$ and $b = -(\frac{Sc}{2} + 3)$.*

Proof. Let $(M^3, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h = 0$. Then, the integral submanifolds of the distribution $\mathcal{D}$ have dimension 2, so they are Kähler manifolds. By virtue of Proposition 2.2, M^3 is Kenmotsu.
Now, computing $R(X,Y,\xi)$ according to Proposition 4.1, using (20), we get

$$\eta(Y)Q(X) - \eta(X)Q(Y) = \left(\frac{Sc}{2} + 1\right)\{\eta(Y)X - \eta(X)Y\}.$$

Setting $Y = \xi$ in this last equation, we obtain

$$Q(X) = \left(\frac{Sc}{2} + 1\right)X - \left(\frac{Sc}{2} + 3\right)\eta(X)\xi\,,$$

which means that M^3 is an η-Einstein manifold. This concludes the proof. ∎

Acknowledgement The authors wish to thank the referee for his remarks that have improved the paper.

References

[1] P. Baird and J.C. Wood, Harmonic Morphisms between Riemannian Manifold. Clarendon Press Oxford, 2003.

[2] A.L. Besse, Einstein manifolds, Springer-Verlag, Berlin, 1987.

[3] D.E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math. 509, Springer-Verlag, Berlin - New York, 1976.

[4] D.E. Blair, Nonexistence of 4-dimensional almost Kaehler manifolds of constant curvature, Proc. Amer.Math.Soc. **110** No. 4 (1990), 1033–1039.

[5] D.E. Blair, Riemannian geometry of contact and symplectic manifolds, Birkhäuser, Boston, 2002.

[6] R.L. Bryant, Harmonic morphisms with fibers of dimension one, Comm. Anal. Geom. **8** (2000), 219–265.

[7] G. Dileo and A.M. Pastore, Almost Kenmotsu manifolds and local symmetry, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 343–354.

[8] M. Falcitelli, S. Ianus and A.M. Pastore, Riemannian Submersions and related topics, World Scientific, 2004.

[9] M. Falcitelli and A.M. Pastore, Almost Kenmotsu f-manifolds, Balkan J. Geom. Appl. **12** No.1 (2007), 32–43.

[10] A. Fialkow, Hypersurfaces of a space of constant curvature, Ann. Math. **39** (1938), 762–785.

[11] S.Ianus and A.M. Pastore, Some foliations and harmonic morphisms, Revue Math. Pures et Appl. **50** (2005), fas. 5-6, 671–676.

[12] S. Ianus and A.M. Pastore, Some remarkable structures on Riemannian manifolds and harmonic maps, Tensor N.S. **69** (2008), 179–189.

[13] D. Janssens and L. Vanhecke, Almost contact structures and curvatures tensors, Kodai Math. J. **4** (1981), no. 1, 1–27.

[14] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tôhoku Math. J. **24** (1972), 93–103.

[15] T.W. Kim and H.K. Pak, Canonical foliations of certain classes of almost contact metric structures, Acta Math. Sin. (Engl. Ser.) **21** No. 4 (2005), 841–846.

[16] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. I, II, Interscience Publishers, New York, 1963, 1969.

[17] T. Oguro, On almost Kähler manifolds of constant curvature, Tsukuba J. Math. **21** No.1 (1997), 199–206.

[18] Z. Olszak, A note on almost Kaehler manifolds, Bulletin de l'Acadmie Polonaise des Sciences, **2**, no.2 (1978), 139–141.

[19] Z. Olszak, Locally conformal almost cosymplectic manifolds, Colloq. Math. **57** (1989), 73–87.

[20] B. O'Neill, Semi-Riemannian geometry, Academic Press, New York, 1983.

[21] C. Özgür, M.M. Tripathi and S. Hong, On contact metric hypersurfaces of a real space form, Kuwait Sci. Eng. **34** 2A (2007), 25–40.

[22] P.J. Ryan, Hypersurfaces with parallel Ricci tensor, Osaka J. Math. **8** (1971), 251–259.

[23] S. Tanno, The automorphism groups of almost contact Riemannian manifolds, Tôhoku Math. J. **21** (1969) 21–38.

[24] J.C. Wood, Harmonic morphisms, foliations and Gauss maps, Contem. Math. **49** (1986), 145–184.

[25] L. Ximin and S. Weihong, Hypersurfaces with constant scalar curvature in a hyperbolic space form, Balkan J. Geom. Appl. **7** No.1 (2002), 121–132.

Università di Bari, Dipartimento di Matematica
via E. Orabona, 4
70125 Bari, Italy
e-mail: pastore@dm.uniba.it; saltarelli@dm.uniba.it