Continuous linear decomposition of analytic functions

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Abstract

We prove that continuous linear decomposition operators exist on the space $A(J)$ of real analytic germs and on the space $A(I)$ of real analytic functions where J is a compact interval (and I is an open interval). We then characterize when $A(J)$ and $A(I)$ contain the space $A_{per}(\mathbb{R})$ of $2\pi-$periodic real analytic functions as a complemented subspace. As a further application we present new formulas for continuous linear right inverses for convolution operators on real analytic functions.

1 Introduction

Continuous linear decomposition operators are a standard tool in analysis which is important especially in the structure theory of classical spaces of (generalized) functions (see [18]) and for explicit formulas for continuous linear right inverses of convolution operators (see [13] and [12]). For spaces of non quasianalytic functions these decomposition operators are usually defined by multiplication operators with suitable functions with compact support which ansatz is clearly not possible in the quasianalytic case, especially for real analytic germs or functions. We will prove in the present paper that nevertheless there exist linear and continuous decomposition operators also for spaces of real analytic functions if the support

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condition is relaxed to a one sided exponential decrease of the decomposition factors. To state the basic result more precisely, let $P_*([a,\infty[) := \mathrm{ind}_k\, P_{*,k}([a,\infty[)$ for $a \in \mathbb{R}$, where

$$P_{*,k}([a,\infty[) := \{f \in \mathcal{H}(U_{[a,\infty[,k}) \mid |f|_k := \sup_{z \in U_{[a,\infty[,k}} |f(z)|e^{|z|/k} < \infty\}$$

$$\text{for } U_{[a,\infty[,k} := \{z \in \mathbb{C} \mid |\operatorname{Im} z| < 1/k, \operatorname{Re} z > a - 1/k\}.$$

The space $P_*(]\infty, a])$ is defined similarly. Our main result is the following

Theorem. There is a linear continuous (decomposition) operator

$$D := (D_-, D_+) : A([a,b]) \to P_*(]\infty, b]) \times P_*([a,\infty[) \text{ such that}$$

$$D_-(f)\big|_{[a,b]} - D_+(f)\big|_{[a,b]} = f \text{ for any } f \in A([a,b]).$$

The proof is based on tame splitting theory for power series spaces of finite type (see [6] and [16]). The corresponding notions and results are recalled in section 2.

Our first application concerns the space $A_{per}(\mathbb{R})$ of $2\pi-$periodic real analytic functions. Using the splitting theory for the $\overline{\partial}-$complex developed in [7, 8] we have shown in [9] that $A_{per}(\mathbb{R})$ is a complemented subspace of $A(\mathbb{R})$. With a new proof based on the theorem above we can extend this result considerably: by restriction, $A_{per}(\mathbb{R})$ is canonically embedded in $A(J)$ (and in $A(I)$) for any non void compact interval J (and any open interval I, respectively). We will show that $A_{per}(\mathbb{R})$ is complemented in $A(J)$ (and in $A(I)$, respectively) if and only if the length of J (and I) is strictly larger than 2π.

We then show that a suitable variant of the above theorem also holds for real analytic functions on open intervals (see Theorem 5.1 for the precise formulation). This is applied to the right inverse problem for convolution operators on real analytic functions. In the non quasianalytic case the classical formulas use elementary solutions and cut off functions both supported in half rays (see [13]). Our decomposition operators now can be used to obtain a similar explicit formula also in the case of real analytic functions (see (6.4)).

Finally, we prove the existence of decomposition operators for entire functions in section 7. Here the technical background is splitting theory for power series spaces of infinite type (see [14]).

2 The basic tame theory

In this section we introduce the basic notions and tools from the structure theory of Fréchet spaces which are needed in the sequel. We will have to use precise (so called tame) estimates in large parts of this paper, so we will recall some basic related notions first: a Fréchet space E with a fixed increasing system $(|\ \ |_j)_{j \in \mathbb{N}}$ of seminorms defining the topology of E is called a graded Fréchet space.

A linear mapping

$$T : (E, |\ \ |_j) \to (F, |\ \ |_j)$$

between two graded $(F)-$spaces $(E, |\ \ |_j)$ and $(F, |\ \ |_j)$ is called (linearly) tame if there is $A \in \mathbb{N}$ such that for any $j \in \mathbb{N}$ there is $C_1 > 0$ such that for any $f \in E$

$$|T(f)|_j \le C_1 |f|_{Aj}.$$

T is called a tame isomorphism iff T is bijective and T and T^{-1} are tame.

The notion of graded spaces and tame linear mapping can easily be extended to $(DFS)-$spaces $F := \mathrm{ind}_k F_k$, i.e. to the inductive limit of a (fixed) compact injective spectrum of Banach spaces F_k.

Notice that we will always fix the grading of the spaces under consideration and that subspaces and quotients are always endowed with the canonically induced gradings.

A sequence

$$0 \to E \xrightarrow{i} F \xrightarrow{q} G \to 0$$

of graded Fréchet spaces E, F, and G is called tamely exact if and only if i is a tame isomorphism onto the subspace $i(E) \subset F$ and G is tamely isomorphic to the quotient $F/i(E)$ of F.

A main tool of this paper is the tame splitting theory of power series spaces of finite type (see [6] and [16]). Recall that power series spaces of finite type and their canonical gradings are defined as follows: Let $(a_k)_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers. Then

$$\Lambda_0(a_k) := \{(c_k)_{k\in\mathbb{N}} \mid \forall j \in \mathbb{N} : |(c_k)|_j := \sum_{k\in\mathbb{N}} |c_k| e^{-a_k/j} < \infty\}.$$

The arguments used in the present paper rely on the tame variants $(\overline{\Omega})_t$ and $(\underline{DN})_t$ (see [6]) of the conditions $(\overline{\Omega})$ and $(\underline{DN})$ of Vogt (see e.g. [14]). In fact, we do not need the precise definitions here but it is enough for our purposes that

$$(\overline{\Omega})_t \text{ and } (\underline{DN})_t \text{ are inherited to complemented subspaces} \tag{2.1}$$

and moreover

Theorem 2.1. *([6, Theorem 1.5]) A nuclear graded Fréchet space E is tamely isomorphic to a power series space of finite type (with its canonical grading) if E satisfies $(\overline{\Omega})_t$ and $(\underline{DN})_t$.*

The following result will a basic tool for our considerations (see also [16]):

Theorem 2.2. *([6, Theorem 1.6]) Let*

$$0 \to E \xrightarrow{i} F \xrightarrow{q} G \to 0$$

be a tamely exact sequence of graded nuclear Fréchet spaces E, F, and G. Then q has a tame right inverse (i.e. the sequence is tamely split) if $E, G \in (\overline{\Omega})_t \cap (\underline{DN})_t$.

3 Decomposition of real analytic germs

In this section we will prove the basic result of the present paper (see Theorem 3.1 below) stating that a linear continuous decomposition of real analytic germs is possible.

We start with the space $A([-1,1])$ of analytic germs near $[-1,1]$ and its canonical grading: let $W_{[-1,1],n}$ denote the ellipse with focusses in ± 1 and half axes

$\sqrt{1+1/n^2}e_1$ and $\frac{1}{n}e_2$ and let $\mathcal{H}^\infty(W_{[-1,1],n})$ be the space of bounded holomorphic functions on $W_{[-1,1],n}$. Then the canonical grading on $A([-1,1])$ is defined by $A([-1,1]) := \operatorname{ind}_n \mathcal{H}^\infty(W_{[-1,1],n})$.

It is well known that with this grading

$$A([-1,1]) \text{ is tamely isomorphic to } \Lambda_0(n)'_b. \tag{3.1}$$

To see this, we can use the mapping $g(z) := (z - 1/z)/(2i)$ which is a biholomorphic mapping from $\mathbb{D}_1 := \{z \in \mathbb{C} | 0 < |z| < 1\}$ onto $U := \mathbb{C} \setminus [-1,1]$ (see [1, p. 245]) which maps the punctured discs $\mathbb{D}_{1-1/n} := \{z \in \mathbb{C} | 0 < |z| < 1 - 1/n\}$ onto $U_n := \mathbb{C} \setminus W_{[-1,1],n}$. The sup-norms on $\mathbb{D}_{1-1/n}$ define the canonical grading on $\mathcal{H}_0(\mathbb{D}_1) := \{f \in \mathcal{H}(\mathbb{D}_1) \mid f(0) = 0\}$ which induces a tame isomorphism of $\mathcal{H}_0(\mathbb{D}_1)$ to $\Lambda_0(n)'_b$ by power series expansion. g defines a tame isomorphism of $\mathcal{H}_0(\mathbb{D}_1)$ to $\mathcal{H}_0(U) := \{f \in \mathcal{H}(U) \mid f(\infty) = 0\}$ where the grading of the latter space is defined by sup-norms on U_n. By the Köthe duality the latter space is tamely isomorphic to $A([-1,1])'_b$ with the grading defined above. This implies (3.1) by dualization.

By shift and dilation we define the canonical grading on $A([a,b]) := \operatorname{ind}_n \mathcal{H}^\infty(W_{[a,b],n})$ for $a < b$.

The less canonical auxiliary space is the space $P_* := \operatorname{ind}_k P_{*,k}$ of exponentially decreasing real analytic functions, where

$$P_{*,k} := \{f \in \mathcal{H}(U_k) \mid |f|_k := \sup_{z \in U_k} |f(z)| e^{|z|/k} < \infty\}$$

$$\text{for } U_k := \{z \in \mathbb{C} \mid |\operatorname{Im} z| < 1/k\},$$

and the half sided variants $P_*([a,\infty[) := \operatorname{ind}_k P_{*,k}([a,\infty[)$ where

$$P_{*,k}([a,\infty[) := \{f \in \mathcal{H}(U_{[a,\infty[,k}) \mid |f|_k := \sup_{z \in U_{[a,\infty[,k}} |f(z)| e^{|z|/k} < \infty\},$$

$$U_{[a,\infty[,k} := \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \frac{1}{k}, \operatorname{Re} z \geq \frac{a+b}{2}\} \cup \{z \in W_{[a,b],k} | \operatorname{Re} z < \frac{a+b}{2}\}$$

(and $P_*(]-\infty,b])$), respectively, which is defined similarly).

Notice that P_* is the space of test functions for the Fourier hyperfunctions (see [4]). For $a < b$ let

$$T : P_*(]-\infty,b]) \times P_*([a,\infty[) \to A([a,b]), T(f,g) := (f|_{[a,b]} - g|_{[a,b]}) \text{ and}$$

$$I : P_* \to P_*(]-\infty,b]) \times P_*([a,\infty[), I(f) := (f|_{]-\infty,b]}, f|_{[a,\infty[}).$$

Theorem 3.1. *The sequence*

$$0 \to P_* \xrightarrow{I} P_*(]-\infty,b]) \times P_*([a,\infty[) \xrightarrow{T} A([a,b]) \to 0 \tag{3.2}$$

is exact and split for $a < b$, i.e. there exists a continuous linear right inverse

$$D := (D_-, D_+) : A([a,b]) \to P_*(]-\infty,b]) \times P_*([a,\infty[)$$

of T.

Proof. By shift and dilation we may assume that $a = -1$ and $b = 1$. By the gradings induced above, the sequence (3.2) consists of nuclear spaces and is tame exact at the first and second place. Moreover, T is tame. To show that T is tame open let $f \in \mathcal{H}^\infty(W_{[-1,1],n})$. Choose $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi = 0$ on $]-\infty, 0]$ and $\varphi = 1$ on $[1/4, \infty[$ and set

$$F(z) := \overline{\partial}(\varphi(\text{Re } z)f(z)) \text{ if } z \in]-1/2, 1/2[+]-i/(2n), i/(2n)[$$

$$\text{and } F(z) := 0 \text{ otherwise.}$$

Let $G(z) := e^{-z^2}/(\pi z)$. Notice that G is a locally integrable elementary solution for the $\overline{\partial}-$ operator. Hence $\phi := G * F$ is defined. Set

$$h_-(z) := \varphi(\text{Re } z)f(z) - \phi(z) \text{ if } z \in U_{]-\infty,1],4n})$$

$$\text{and } h_+(z) := -(1-\varphi(\text{Re } z))f(z) - \phi(z) \text{ if } z \in U_{[-1,\infty[,4n}).$$

An easy calculation shows that the mapping

$$R : \mathcal{H}^\infty(W_{[-1,1],n}) \to P_{*,4n}(]-\infty, 1]) \times P_{*,4n}([-1,\infty[), R(f) := (h_-, h_+)$$

is defined, linear and continuous and $h_- - h_+ = f$, hence T is tame open. By dualization we have a tame exact sequence of nuclear Fréchet spaces

$$0 \to A([-1,1])'_b \overset{^tT}{\to} P_*(]-\infty, b])'_b \times P_*([a,\infty[)'_b \overset{^tI}{\to} P'_* \to 0$$

where the first and the last space are isomorphic to a power series space of finite type by (3.1) and [10], hence both have $(\overline{\Omega})_t$ and $(\underline{DN})_t$ by (the easy part of) Theorem 2.1. Thus the sequence is tamely split by Theorem 2.2 and also (3.2) splits tamely. ∎

Remark 3.2. *The decay of the decomposition factors in Theorem 3.1 may be essentially improved: let D be the operator from 3.1 and define $\widetilde{D}$ for $k \in \mathbb{N}$ by $\widetilde{D}(f)(x) := (D_-(fe^{\xi^{2k}})(x)e^{-x^{2k}}, D_+(fe^{\xi^{2k}})(x)e^{-x^{2k}}), f \in A([a,b])$. Then $\widetilde{D}$ is a decomposition operator with component functions decreasing faster than $e^{-x^{2k}}$.*

Corollary 3.3. *For $c < a < b < d$ the sequences*

$$0 \to A(\mathbb{R}) \overset{I}{\to} A(]-\infty, b]) \times A([a,\infty[) \overset{T}{\to} A([a,b]) \to 0$$

$$0 \to A([c,d]) \overset{I}{\to} A([c,b]) \times A([a,d]) \overset{T}{\to} A([a,b]) \to 0$$

$$0 \to A([a,b])'_b \overset{I}{\to} A([c,b])'_b \times A([a,d])'_b \overset{T}{\to} A([c,d])'_b \to 0$$

are exact and split.

Proof. The first statement is evident since the mapping D from Theorem 3.1 is also continuous as a mapping into $A(]-\infty, b]) \times A([a,\infty[)$ and it is a right inverse for $T : A(]-\infty, b]) \times A([a,\infty[) \to A([a,b])$. The second follows by the composition of D_- and D_+ with proper restrictions. The third sequence is the transposed of the second (with proper interpretation of the mappings I and T). ∎

We have proved in [10] that P_* is tamely isomorphic to $\Lambda_0(n^{1/2})'_b$. By Theorem 3.1 this also holds for the half sided variants:

Corollary 3.4. *For $a \in \mathbb{R}$ the spaces $P_*(]-\infty, a])$ and $P_*([a, \infty[)$ are tamely isomorphic to $\Lambda_0(n^{1/2})'_b$.*

Proof. Since $P_*(]-\infty, b])$ and $P_*([a, \infty[)$ clearly are tamely isomorphic for any $a, b \in \mathbb{R}$ we have the tame isomorphisms

$$\begin{aligned}\big(P_*(]-\infty, a]) \times P_*(]-\infty, a])\big) &\simeq_t \big(P_* \times A([0,1])\big)\\ &\simeq_t \big(\Lambda_0(n^{1/2})'_b \times \Lambda_0(n)'_b\big) \simeq_t \Lambda_0(n^{1/2})'_b. \end{aligned} \tag{3.3}$$

In fact, the first isomorphism follows from Theorem 3.1, the second holds since $P_* \simeq_t \Lambda_0(n^{1/2})'_b$ by [10] and $A([0,1]) \simeq_t \Lambda_0(n)'_b$ by (3.1), and the third follows from an increasing rearrangement. Therefore $P_*(]-\infty, a])$ is tamely isomorphic to some $\Lambda_0(\alpha_n)'_b$ (use also Theorem 2.1 and (2.1)). Using a standard argument via diametral dimensions and the stability of $\Lambda_0(n^{1/2})'_b$ we conclude by (3.3) that α_n is (equivalent to) $n^{1/2}$. ∎

4 Periodic real analytic functions

The space $A_{per}(\mathbb{R})$ of $2\pi-$periodic real analytic functions can be identified with a linear subspace of $A(I)$ where $\emptyset \neq I \subset \mathbb{R}$ is a compact or open interval. In fact, the identification is given by the restriction $r : A_{per}(\mathbb{R}) \to A(I), r(f) := f\big|_I$. Notice that r is continuous and injective by analyticity if I is non void. Using Theorem 3.1 we can precisely determine the intervals I such that $A_{per}(\mathbb{R})$ is (topologically isomorphic to) a complemented complemented subspace of $A(I)$ via the identification mapping r. We start with the compact case:

Theorem 4.1. *$A_{per}(\mathbb{R})$ is a complemented subspace of $A([a, b])$ via r if $b - a > 2\pi$.*

Proof. Since $A_{per}(\mathbb{R})$ is invariant by shifts we may assume that $a = 0$ and $b = 2\pi + \varepsilon > 2\pi$. Let

$$Q : A([0, 2\pi + \varepsilon]) \to A([0, \varepsilon]), Q(f)(x) := f(x + 2\pi) - f(x),$$

and consider the sequence

$$0 \to A_{per}(\mathbb{R}) \xrightarrow{r} A([0, 2\pi + \varepsilon]) \xrightarrow{Q} A([0, \varepsilon]) \to 0. \tag{4.1}$$

Clearly, $r(A_{per}(\mathbb{R})) = \ker(Q)$. To show that the sequence is (exact and) split we choose D from Theorem 3.1 (for $a := 0$ and $b := \varepsilon$) and set

$$R(f)(x) := \sum_{j=1}^{\infty} D_-(f)(x - 2\pi j) + \sum_{j=0}^{\infty} D_+(f)(x + 2\pi j), f \in A([0, \varepsilon]). \tag{4.2}$$

$R(f)$ defines a real analytic germ near $[0, 2\pi + \varepsilon]$ by the definition of the spaces $P_*(]\infty, \varepsilon])$ and $P_*([0, \infty[)$. $R : A([0, \varepsilon]) \to A([0, 2\pi + \varepsilon])$ is continuous and a short calculation (using Theorem 3.1) shows that R is a right inverse for Q. The corollary is proved. ∎

Formula (4.2) is based on the classical elementary solutions

$$E_1 := -\sum_{j=0}^{\infty} \delta_{-2\pi j} \text{ and } E_2 := \sum_{j=1}^{\infty} \delta_{2\pi j}$$

for the (convolution) operator Q on $C^\infty(\mathbb{R})$. Here δ_ξ denotes the point evaluation at $\xi \in \mathbb{R}$.

The statement of Theorem 4.1 is optimal:

Proposition 4.2. *$A_{per}(\mathbb{R})$ is not a complemented subspace of $A([a,b])$ via r if $0 \leq b - a \leq 2\pi$.*

Proof. We can again assume that $a = 0$ and $0 \leq b \leq 2\pi$. If there is a continuous linear projection $\Pi : A([0,b]) \to r_{[0,b]}(A_{per}(\mathbb{R})) \subset A([0,b])$ then a continuous projection $\Pi_1 : A([0,2\pi]) \to r_{[0,2\pi]}(A_{per}(\mathbb{R})) \subset A([0,2\pi])$ is given by the unique extension of $\Pi(f|_{[0,b]})$ to a function in $r_{[0,2\pi]}(A_{per}(\mathbb{R}))$. In fact, Π_1 clearly is a projection and $\Pi_1 : A([0,2\pi]) \to A([0,2\pi])$ is continuous by the closed graph theorem and the continuity of Π (and analyticity). We can thus assume that $b = 2\pi$. Let

$$Q : A([0,2\pi]) \to A(\{0\}), Q(f)(x) := f(x+2\pi) - f(x)$$

and consider the sequence

$$0 \to A_{per}(\mathbb{R}) \xrightarrow{r} A([0,2\pi]) \xrightarrow{Q} A(\{0\}) \to 0.$$

Clearly, $r(A_{per}(\mathbb{R})) = \ker(Q)$ and the sequence is exact since (4.1) is exact for any $\varepsilon > 0$. If $A_{per}(\mathbb{R})$ were complemented in $A([0,2\pi])$ the sequence were split, hence $A(\{0\})$ would be isomorphic to a complemented subspace of $A([0,2\pi])$, a contradiction, since the spaces are isomorphic to duals of power series spaces of different type. ∎

The complementation of $A_{per}(\mathbb{R})$ in $A(I)$ for open intervals I is an immediate consequence of Theorem 4.1 and Proposition 4.2:

Corollary 4.3. *Let $-\infty \leq a < b \leq \infty$. $A_{per}(\mathbb{R})$ is a complemented subspace of $A(]a,b[)$ via r if and only if $b - a > 2\pi$.*

Proof. Necessity. Let $b - a \leq 2\pi$ and let $\Pi : A(]a,b[) \to r_{]a,b[}(A_{per}(\mathbb{R})) \subset A(]a,b[)$ be a continuous projection. Then a slight variant of the proof of Proposition 4.2 shows that $\Pi(f|_{]a,b[}), f \in A([a,b])$, may be used to define a continuous projection in $A([a,b])$ onto $r_{[a,b]}(A_{per}(\mathbb{R}))$. Hence $b - a > 2\pi$ by Proposition 4.2, a contradiction.

Sufficiency. For $a < c < d < b$ and $d - c > 2\pi$ let $\Pi : A([c,d]) \to r_{[c,d]}(A_{per}(\mathbb{R}))$ $\subset A([c,d])$ be the continuous projection from Theorem 4.1. Then the unique extension of

$$T : A(]a,b[) \to A_{per}(\mathbb{R}), T(f) := r_{[c,d]}^{-1} \circ \Pi(f|_{[c,d]}), f \in A(]a,b[),$$

is a welldefined linear mapping. T is continuous by the closed graph theorem for webbed spaces (see [14, 24.31]) since $A(]a,b[)$ is ultrabornological and $A_{per}(\mathbb{R})$ is a $(DFS)-$space hence webbed. Thus, $\Pi_1 := r_{]a,b[} \circ T : A(]a,b[) \to r_{]a,b[}(A_{per}(\mathbb{R}))$ $\subset A(]a,b[)$ is continuous and a projection onto $r_{]a,b[}(A_{per}(\mathbb{R}))$. ∎

The fact that $A_{per}(\mathbb{R})$ is complemented in $A(\mathbb{R})$ has been shown already in [9] with an entirely different proof.

5 Decomposition of real analytic functions

The decomposition of real analytic germs in Theorem 3.1 leads to a decomposition of real analytic functions defined on open intervals into functions which are exponentially decreasing in one direction. More precisely, for $-\infty \leq c < d \leq \infty$ let $A_{exp,+}(]c,\infty[) := \operatorname{proj}_{m\to c} P_*([m,\infty[)$ (and $A_{exp,-}(]-\infty,d[) := \operatorname{proj}_{m\to d} P_*(]-\infty,m])$, respectively) and let

$$T_1 : A_{exp,-}(]-\infty,d[) \times A_{exp,+}(]c,\infty[) \to A(]c,d[), T_1(f,g) := f\big|_{]c,d[} - g\big|_{]c,d[}, \text{ and}$$

$$I_1 : P_* \to A_{exp,-}(]-\infty,d[) \times A_{exp,+}(]c,\infty[), I_1(f) := (f\big|_{]-\infty,d[}, f\big|_{]c,\infty[}).$$

The following result can also be viewed as a linear continuous separation of singularities.

Theorem 5.1. *For $-\infty \leq c < d \leq \infty$ the sequence*

$$0 \to P_* \stackrel{I_1}{\to} A_{exp,-}(]-\infty,d[) \times A_{exp,+}(]c,\infty[) \stackrel{T_1}{\to} A(]c,d[) \to 0$$

is exact and split, i.e. there exists a continuous linear right inverse

$$D_1 : A(]c,d[) \to A_{exp,-}(]-\infty,d[) \times A_{exp,+}(]c,\infty[)$$

of T_1.

Proof. With D from Theorem 3.1 (for $c < a < b < d$) we set

$$D_1(f) := (D_{1,-}(f), D_{1,+}(f)) := \big(D_-(f\big|_{[a,b]}), D_+(f\big|_{[a,b]})\big), f \in A(]c,d[).$$

Then

$$D_{1,-}(f)\big|_{[a,b]} = D_{1,+}(f)\big|_{[a,b]} + f\big|_{[a,b]}$$

and therefore $D_{1,-}(f)$ may be extended by the right hand side of this equation to a real analytic function on $]-\infty,d[$ (denoted also by $D_{1,-}(f)$ since the extension is unique by analyticity). Similarly, $D_{1,+}(f) \in A(]c,\infty[)$. In this way, $D_{1,-}(f) \in A_{exp,-}(]-\infty,d[)$ and $D_{1,+}(f) \in A_{exp,+}(]c,\infty[)$. The continuity of D and the closed graph theorem for webbed spaces (see [14, 24.31]) imply that D_1 is continuous (use analyticity again and notice that $A_{exp,-}(]-\infty,d[)$ and $A_{exp,+}(]c,\infty[)$ are $(PLS)-$spaces hence webbed). Also, D_1 is a right inverse for T_1 since D is a right inverse for T (analyticity is used here for the third time). ∎

Also Theorem 5.1 and a suitable version of formula (4.2) imply that $A_{per}(\mathbb{R})$ is complemented in $A(]a,b[)$ if $b-a > 2\pi$.

6 Convolution operators on real analytic functions

We will show in this section that the continuous linear decomposition of real analytic functions from Theorem 5.1 can be used to obtain new formulas for right inverses of convolution operators on real analytic functions (see (6.4) below).

Convolution operators are defined as follows: For $\mu \in A(\mathbb{R})'$ let $G := \operatorname{conv}(\operatorname{supp}\mu)$. Let $I \subset \mathbb{R}$ be an open interval. Then the convolution operator

$$T_\mu : A(I-G) \to A(I), T_\mu(f)(x) := \langle {}_y\mu, f(x-y)\rangle,$$

is defined, linear and continuous. The existence of continuous linear right inverses for T_μ was characterized in [9] as follows:

Theorem 6.1. *Let $I \subset \mathbb{R}$ be an open interval. Let $\mu \in A(\mathbb{R})'$ and suppose that $\operatorname{supp}(\mu) = \{0\}$ if $I \neq \mathbb{R}$. Then $T_\mu : A(I) \to A(I)$ admits a continuous linear right inverse if and only if there is a function $r(x) = o(x)$ on $\mathbb{R}_+$ such that*

$$|\operatorname{Im} z| \leq r(|z|) \text{ for any } z \in \mathbb{C} \text{ with } \widehat{\mu}(z) = 0 \tag{6.1}$$

and for any $x \in \mathbb{R}$ there is $t \in \mathbb{C}$ such that

$$|x-t| \leq r(x) \text{ and } |\widehat{\mu}(t)| \geq e^{-r(t)}. \tag{6.2}$$

The sufficiency of (6.1) and (6.2) was proved in [9] using the methods from [7, 8] providing continuous linear right inverses for the $\overline{\partial}-$operator. We will present here a new proof which is more natural and is based on elementary solutions supported in half lines i.e. on hyperbolic convolution operators.

To explain this we need some further notation: notice that μ also acts as a linear operator on the space $\mathcal{B}(\mathbb{R}) := \mathcal{H}(\mathbb{C}\setminus\mathbb{R})/H(\mathbb{C})$ of hyperfunctions on $\mathbb{R}$ in the following way

$$S_\mu : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathbb{R}), S_\mu([u]) := [\langle {}_y\mu, u(\,\cdot\, - y)\rangle].$$

$E \in \mathcal{B}(\mathbb{R})$ is called an elementary solution for S_μ if $S_\mu(E) = \delta$ where δ is Dirac's $\delta-$distribution considered as a hyperfunction. Our starting point is the following result:

Theorem 6.2. *([5, sections 6.1 and 6.2]) Let $\mu \in A(\mathbb{R})'$. Then S_μ admits elementary solutions $E_+, E_- \in \mathcal{B}(\mathbb{R})$ supported in $[c_+, \infty[$ (and $]-\infty, c_-]$, respectively) for some $c_+, c_- \in \mathbb{R}$ if μ satisfies* (6.1) *and* (6.2). *If* $\operatorname{supp}(\mu) = \{0\}$ *we have $c_- = c_+ = 0$.*

By [11, Corollary 4.2] we can extend E_+ and E_- to an exponentially decreasing Fourier hyperfunction, i.e. we may assume that E_+ is defined by $h_+ \in \mathcal{H}(\mathbb{C}\setminus[c_+,\infty[)$ satisfying for any $j \in \mathbb{N}$

$$|h_+(z)| \leq C_j e^{-j|z|} \text{ if } 1/j \leq |\operatorname{Im} z| \leq j \text{ and } \operatorname{dist}(z, [c_+,\infty[) \geq 1/j \tag{6.3}$$

(and similarly h_- can be chosen for E_-).

Let $S_\mu(h_\pm) =: u_\pm$. Then $u_\pm$ is a representing function for Dirac's distribution and also satisfies (6.3) by the definition of S_μ.

Using the decomposition operator $D_1 = (D_{1,-}, D_{1,+})$ from Theorem 5.1 we define a right inverse R for T_μ

$$R(f)(z) := \int_{\gamma_-} h_+(z-\xi)D_-(f)(\xi)d\xi - \int_{\gamma_+} h_-(z-\xi)D_+(f)(\xi)d\xi, f \in A(I), \tag{6.4}$$

where γ_- is a path around $]-\infty, c_+ + z]$ with positive orientation which is contained in the domain of $D_-(f)$ and is starting and ending as a line segment parallel to $\mathbb{R}$ (and similarly for γ_-). The integral is absolutely convergent for z near I by the definition of $A_{exp,\pm}$ and (6.3) (use also the fact that $c_+ = c_+ = 0$ if $I \neq \mathbb{R}$). It is independent of the choice of $\gamma_\pm$ by Cauchy's theorem and by the definition of $A_{exp,\pm}$ and (6.3) again. It defines a locally holomorphic function by fixing $\gamma_\pm$ and changing differentiation and integration. Summarizing, $R : A(I) \to A(I)$ is defined, linear and continuous. Finally we have

$$\begin{aligned}
&T_\mu(R(f))(x) \\
&\quad = \langle {}_y\mu, \int_{\gamma_-} h_+(x-y-\xi)D_-(f)(\xi)d\xi - \int_{\gamma_-} h_+(x-y-\xi)D_+(f)(\xi)d\xi\rangle \\
&\quad = \int_{\gamma_-} \langle {}_y\mu, h_+(x-y-\xi)\rangle D_-(f)(\xi)d\xi - \int_{\gamma_+} \langle {}_y\mu, h_-(x-y-\xi)\rangle D_+(f)(\xi)d\xi \\
&\quad = \int_{\gamma_-} S_\mu(h_+)(x-\xi)D_-(f)(\xi)d\xi - \int_{\gamma_+} S_\mu(h_-)(x-\xi)D_+(f)(\xi)d\xi \\
&\quad = \int_{\gamma_-} u_+(x-\xi)D_-(f)(\xi)d\xi - \int_{\gamma_+} u_-(x-\xi)D_+(f)(\xi)d\xi \\
&\quad = D_-(f)(x) - D_+(f)(x) = f(x) \text{ for } x \in \mathbb{R}
\end{aligned}$$

by Cauchy's formula and Theorem 5.1 since we may change the paths of integration to circles by Cauchy's theorem, (6.3) for $u_\pm$ and the definition of $A_{exp,\pm}$. By analyticity this shows that R is a right inverse for T_μ on $A(I)$.

7 Decomposition of entire functions

In this section we will treat the decomposition problem in the frame of entire functions. Since the space $\mathcal{H}(\mathbb{C})$ is a power series space of infinite type we will use here the technically easier splitting theory for exact sequences of power series spaces of infinite type developed by D. Vogt (see e.g. [14]). Especially, we will not need graded spaces and tame mappings in this section. The main auxiliary space is the Fréchet space P_{**} of exponentially decreasing entire functions defined by

$$P_{**} := \{f \in \mathcal{H}(\mathbb{C}) \mid \forall k \in \mathbb{N} : |f|_k := \sup_{|\operatorname{Im} z| < k} |f(z)| e^{k|z|} < \infty\}.$$

and the half sided variants

$$P_{**,+} := \{f \in \mathcal{H}(\mathbb{C}) \mid \forall k \in \mathbb{N} : \|f\|_k := \sup_{|\operatorname{Im} z| < k, \operatorname{Re} z > -k} |f(z)| e^{k|z|} < \infty\},$$

(and $P_{**,-}$, respectively, which is defined similarly).

Notice that P_{**} is the space of test functions for the Fourier ultra hyperfunctions (see [15] and [10]). Let

$$T_2 : P_{**,-} \times P_{**,+} \to \mathcal{H}(\mathbb{C}), T_2(f,g) := f - g \text{ and}$$

$$I_2 : P_{**} \to P_{**,-} \times P_{**,+}, I_2(f) := (f,f).$$

Theorem 7.1. *The sequence*

$$0 \to P_{**} \xrightarrow{I_2} P_{**,-} \times P_{**,+} \xrightarrow{T_2} \mathcal{H}(\mathbb{C}) \to 0$$

is exact and split.

Proof. P_{**} (and $\mathcal{H}(\mathbb{C})$) are isomorphic to power series spaces of infinite type by [10] (and Taylor series expansion, respectively). Hence the sequence splits by the splitting theory of Vogt (see [14]) if the sequence is exact.

Exactness is proved similarly as in Theorem 3.1: let $f \in \mathcal{H}(\mathbb{C})$. Choose $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi = 0$ on $]-\infty, -1]$ and $\varphi = 1$ on $[0, \infty[$ and extend φ to $\mathbb{C}$ by $\varphi(z) := \varphi(\text{Re } z)$. With $F := \overline{\partial}(\varphi f)$ we set

$$F_k(z) := F(z) \text{ if } z \in U_{k+1} := \{z \in \mathbb{C} \mid |\operatorname{Im} z| < k+1\}$$

and $F_k(z) := 0$ otherwise. With $G(z) := e^{-z^2}/(\pi z)$ as in the proof of 3.1 we set $\phi_k := G * F_k$. Then $\overline{\partial}\phi_k(z) = F(z)$ if $z \in U_{k+1}$ and

$$|\phi_k|_k := \sup_{z \in U_k} |f(z)| e^{k|z|} < \infty.$$

We have proved in [10, section 3] that for any $k \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that P_{**} is dense in $\{f \in \mathcal{H}(U_j) \mid |f|_j < \infty\}$ w.r.t. $|\ |_k$. Hence the Mittag-Leffler procedure implies that there is $\phi \in L^\infty_{loc}(\mathbb{C})$ such that

$$\overline{\partial}\phi = F \text{ on } \mathbb{C} \text{ and } |\phi|_k < \infty \text{ for any k.}$$

Let

$$h_- := \varphi f - \phi \text{ and } h_+ := -(1-\varphi) f - \phi.$$

Clearly, $h_\pm \in P_{**,\pm}$ and $h_- - h_+ = f$. The theorem is proved. ∎

Corollary 7.2. *$\mathcal{H}_{per}(\mathbb{C})$ is a complemented subspace of $\mathcal{H}(\mathbb{C})$.*

Proof. Formula (4.2) also applies in the present case if D is substituted by the right inverse of T_2. ∎

Using Fourier transformation the solution of a convolution equation on $\mathcal{H}(\mathbb{C})$ is translated into an interpolation problem for entire functions of exponential type. In this way, a more complicated formula for a right inverse of T_2 was given in the introduction of [17].

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