A characterization of the Clifford torus*

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Abstract

The standard Clifford torus is characterized as the only connected compact orientable affine Lagrangian surface in $\mathbf{C}^2$ whose second fundamental tensor is parallel.

1 Introduction

The Clifford torus, similarly as round spheres, is one of the standard submanifolds. We shall deal with the 2-dimensional case. The 2-dimensional Clifford torus can be located, for instance, in the 3-dimensional sphere or in $\mathbf{C}^2$. As a submanifold of the sphere it is minimal. As a submanifold of $\mathbf{C}^2$ it is not minimal but its second fundamental tensor field is parallel relative to the induced connection. Moreover it is Lagrangian (in other words totally real) in the Kaehler space $\mathbf{C}^2$. The aim of this paper is to prove that the parallelism of the second fundamental tensor is a characteristic property of the Clifford torus within the class of affine Lagrangian surfaces of $\mathbf{C}^2$. The class of affine Lagrangian submanifolds is much larger than the class of Lagrangian ones in the metric sense. More precisely, we shall prove

Theorem 1.1. *Let M be a 2-dimensional connected compact orientable manifold. If a surface $f : M \longrightarrow \mathbf{C}^2$ is affine Lagrangian and its second fundamental tensor is parallel relative to the induced connection, then the surface is up to a complex affine transformation of $\mathbf{C}^2$ the Clifford torus. In particular, the surface is metric Lagrangian and the immersion is an embedding.*

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The main tools of the proof are results proved by the author in her previous papers. The necessary information is provided in the next section. In particular, we use a bilinear symmetric form g introduced in [3]. For the Clifford torus g is non-degenerate indefinite at each point. This property corresponds to the local strong convexity of hypersurfaces of affine spaces. We observe that there exist affine Lagrangian surfaces in $\mathbf{C}^2$ whose second fundamental tensor is parallel but g is not non-degenerate indefinite. Such examples can be found even if we additionally assume that the induced connection is flat (as it is on the Clifford torus). In general, the parallelism of the second fundamental tensor does not imply that the induced connection is flat. We provide suitable examples.

2 Preliminaries

Let $\mathbf{C}^2$ be equipped with the standard complex structure. By a Hermitian structure on $\mathbf{C}^2$ we mean any Hermitian product (definite or indefinite) G on the complex vector space $\mathbf{C}^2$. That is, a Hermitian scalar product G is characterized by the condition: $G(\mathrm{i}X, \mathrm{i}Y) = G(X,Y)$ for every vectors X, Y. If G is indefinite, then it is of type $(+,+,-,-)$. If D denotes the standard connection on $\mathbf{C}^2$, then G gives a Kahler structure on $\mathbf{C}^2$. Throughout the paper the complex structure on $\mathbf{C}^2$ and the connection D are fixed, but a Hermitian product G is not.

Let M be a 2-dimensional connected real manifold and $f : M \longrightarrow \mathbf{C}^2$ be an immersion. We say that f is affine Lagrangian (or purely real) if the bundle $\mathrm{i}f_*(TM)$ is transversal to the tangent bundle $f_*(TM)$. If $\mathbf{C}^2$ is equipped with a Hermitian structure, then f is called Lagrangian (or totally real) if $\mathrm{i}f_*(TM)$ is orthogonal to $f_*(TM)$. If for an affine Lagrangian submanifold there is a Hermitian metric relative to which it is Lagrangian, we shall say that it is metric Lagrangian.

An immersion f is affine Lagrangian if and only if $\omega_x \neq 0$ at every point of M, where ω is a complex valued real 2-form on M defined by

$$\omega(X,Y) = \det{}_{\mathbf{C}}(f_*X, f_*Y). \tag{1}$$

If M is oriented and f is affine Lagrangian, then we define a volume form ν on M by the condition

$$\nu(X,Y) = |\omega(X,Y)| \tag{2}$$

for a positively oriented basis X, Y of T_xM, for $x \in M$. If we change a basis X, Y to another positively oriented basis then in the expression $\omega(X,Y) = \mu e^{\mathrm{i}\theta}$, where $\mu \in \mathbf{R}^+, \theta \in \mathbf{R}$, the value of θ remains unchanged (up to $2k\pi$, $k \in \mathbf{Z}$). It is called the phase of the tangent space $f_*(T_xM)$. As a smooth function θ depending on $x \in M$ it is, in general, only locally well defined. If f is affine Lagrangian, then it is naturally equipped with a normal bundle. Namely, by the normal bundle we mean the bundle $\mathrm{i}f_*(TM)$. We can write the Gauss formula

$$D_X f_* Y = f_*(\nabla_X Y) + \mathrm{i}f_* Q(X,Y) \tag{3}$$

for vector fields X, Y on M. It turns out that ∇ is a torsion-free connection on M and Q is a symmetric $(1,2)$-tensor field on M. ∇ is called the induced connection and Q the second fundamental tensor for f. The Weingarten formula reduces to the Gauss formula. In particular, the normal connection corresponds to the induced connection and the parallelism of the second fundamental form iQ relative to the normal connection is equivalent to the parallelism of Q relative to ∇. The fundamental equations are the following

$$R(X,Y) = Q_X Q_Y - Q_Y Q_X, \tag{4}$$

$$(\nabla_X Q)(Y,Z) = (\nabla_Y Q)(X,Z) \tag{5}$$

for $X, Y, Z \in T_xM,\ x \in M$, where R is the curvature tensor of ∇ and $Q_X Y$ stands for $Q(X,Y)$. It follows, in particular, that if for some nonzero X the endomorphism Q_X is proportional to the identity, then the curvature tensor vanishes.

For an affine Lagrangian surface we define a 1- form τ and a symmetric bilinear 2-form g by the following formulas

$$\tau(X) = \operatorname{tr} Q_X, \tag{6}$$

$$g(X,X) = \det Q_X \tag{7}$$

for every $X \in T_xM$, $x \in M$. An important fact is that

$$\tau = d\theta, \tag{8}$$

where θ is the phase function. The bilinear form g was introduced and studied in [3]. In particular, if X, Y is a basis of T_xM and the matrices of Q_X and Q_Y relative to the basis are

$$Q_X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad Q_Y = \begin{bmatrix} c & e \\ d & k \end{bmatrix},$$

then $g(X,X) = \det Q_X$, $g(X,Y) = \frac{1}{2}\det \begin{bmatrix} a & e \\ b & k \end{bmatrix}$, $g(Y,Y) = \det Q_Y$.

If f is metric Lagrangian then the 1-form τ is dual to the mean curvature vector field multiplied by i. In the affine case we do not have a mean curvature vector field. We say that an affine Lagrangian immersion f is minimal if $\tau = 0$ on M. We shall say that an affine Lagrangian immersion is nowhere minimal if $\tau \neq 0$ at every point of M. Minimal affine Lagrangian submanifolds are studied in [2], [4].

We shall use the following facts proved in [1], [3], [4]:

Fact 2.1. ([3], Theorem 5.6) If M is compact and orientable, $f : M \longrightarrow \mathbf{C}^2$ is affine Lagrangian and $\operatorname{rk} g$ is constant on M, then M is a topological torus and g is nondegenerate indefinite on M.

Fact 2.2. ([4], Proposition 8) If $f : M \longrightarrow \mathbf{C}^2$ is affine Lagrangian, g is nowhere zero on M and the induced connection is flat, then f is nowhere minimal and M admits an almost product structure $(\ker \tau, \mathcal{D})$, where $\mathcal{D}$ is a distribution on which Q is proportional to the identity, that is, for $Y \in \mathcal{D}$, $Y \neq 0$, the endomorphism Q_Y is not zero and is proportional to the identity.

Fact 2.3. ([4], Lemma 9, Remark 10, formula (11)) Let $f : M \longrightarrow \mathbf{C}^2$ be an affine Lagrangian surface. If there is a 1-dimensional distribution $\mathcal{D}$ on which Q is proportional to the identity, then around each point of M there is a coordinate system (u, v) relative to which Q is given by the matrices

$$Q_U = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix}, \quad Q_V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{9}$$

where $U = \partial_u$, $V = \partial_v$ and b is a function of u. Moreover, for the induced connection we have

$$\nabla_U U = AU + BV, \quad \nabla_U V = CU + bEV, \quad \nabla_V V = EU + CV, \tag{10}$$

where B, C, E are functions. The vector field V is given uniquely and therefore globally on M. If $A_v = 0$, then by replacing U by $e^{\int_u A} U$ we get a new coordinate system which satisfies all the above conditions and additionally $A = 0$. If $\nabla_U U = 0$ the local vector field U is unique up to a constant on neighborhoods of points.

In particular, if the distribution $\ker \tau$ is ∇-parallel and so is the vector field V, then $B = E = C = 0$. Since ∇ is flat (because $Q_V = \mathrm{id}$), by formula (11) from [4] we have $A_v = 0$. Hence, in the coordinate system chosen as above, we have

$$\nabla_U U = \nabla_U V = \nabla_V V = 0 \tag{11}$$

Fact 2.4. ([1], fundamental theorem - Theorem 2.2) Let $f_1, f_2 : M \longrightarrow \mathbf{C}^2$ be affine Lagrangian immersions. If the induced connections and the second fundamental tensors for the two immersions are respectively equal, then the immersions are congruent modulo a complex affine transformation of $\mathbf{C}^2$.

3 Examples

The following example plays an essential role in our considerations.

Example 3.1. Let $\gamma(u)$ be a centroaffine curve in $\mathbf{R}^2$, that is, $\det(\gamma(u), \gamma'(u)) \neq 0$ for every $u \in \mathrm{I}$, where I is some open interval of $\mathbf{R}$. Then

$$\gamma'' = \alpha\gamma + \beta\gamma'.$$

The curve can be reparametrized in such a way that after reparametrization γ'' is proportional to γ. Hence we can assume that $\beta = 0$ on I. Let $\theta(v)$ be a real valued function on some open interval of $\mathbf{R}$ and $\theta'(v) \neq 0$ for every v. Define the following surface in $\mathbf{C}^2$:

$$f(u, v) = \mathrm{e}^{\mathrm{i}\theta(v)}\gamma(u). \tag{12}$$

We have

$$f_u = \mathrm{e}^{\mathrm{i}\theta}\gamma', \quad f_v = \mathrm{i}\theta'\mathrm{e}^{\mathrm{i}\theta}\gamma \tag{13}$$

and

$$\det{}_{\mathbf{C}}(f_u, f_v) = \mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+2\theta)}\theta' \det(\gamma, \gamma'). \tag{14}$$

It follows that f is an immersion, it is affine Lagrangian and its phase function equals to $\frac{\pi}{2} + 2\theta$. Since $\theta' \neq 0$, the surface is nowhere minimal.

Within the family described by (12) there are only two subclasses which are metric Lagrangian – one for a definite metric tensor field, another one for an indefinite metric. Namely, assume that G is a Hermitian metric tensor field (definite or indefinite) on $\mathbf{C}^2$ relative to which f is metric Lagrangian. We have $G(\mathrm{e}^{\mathrm{i}\theta}X, \mathrm{e}^{\mathrm{i}\theta}Y) = G(X,Y)$ for every $X, Y \in \mathbf{C}^2$ and every real number θ. f is metric Lagrangian if and only if $G(\mathrm{i}f_v, f_u) = 0$. This condition is equivalent to the condition $G(\gamma, \gamma') = 0$. Since $(G(\gamma,\gamma))' = 2G(\gamma,\gamma')$, we have that $G(\gamma,\gamma') = 0$ if and only $G(\gamma,\gamma)$ is constant, that is, if and only if the curve γ is a piece of a circle or a hyperbola.

We now go back to general considerations. For the surface f we have

$$
\begin{aligned}
f_{uu} &= \mathrm{e}^{\mathrm{i}\theta}\gamma'' = -\mathrm{i}\frac{\alpha}{\theta'}(\mathrm{i}\theta'\mathrm{e}^{\mathrm{i}\theta}\gamma) = -\mathrm{i}\frac{\alpha}{\theta'}f_v, &(15)\\
f_{uv} &= \mathrm{i}\theta' f_u, &(16)\\
f_{vv} &= \mathrm{i}\theta''\mathrm{e}^{\mathrm{i}\theta}\gamma + \mathrm{i}\theta'\mathrm{i}\theta'\mathrm{e}^{\mathrm{i}\theta}\gamma = \frac{\theta''}{\theta'}f_v + \mathrm{i}\theta' f_v. &(17)
\end{aligned}
$$

Therefore, if ∇ is the induced connection and Q the second fundamental tensor for f, then

$$\nabla_U U = 0, \quad \nabla_U V = 0, \quad \nabla_V V = \frac{\theta''}{\theta'}V, \tag{18}$$

$$Q(U,U) = -\frac{\alpha}{\theta'}V, \qquad Q(U,V) = \theta' U, \qquad Q(V,V) = \theta' V, \tag{19}$$

where $U = \partial_u$, $V = \partial_v$. We have two ∇-parallel distributions spanned by U and V respectively. Since $\operatorname{tr} Q_U = 0$, the vector field U spans $\ker\tau$. On the distribution spanned by V Q is proportional to the identity, namely $Q_V = \theta'\mathrm{id}$. In particular, by (4), the connection ∇ is flat.

Compute now the tensor field ∇Q. We have

$$
\begin{aligned}
(\nabla_V Q)(V,V) &= (\nabla_V Q)(U,V) = (\nabla_V Q)(U,U) = 0,\\
(\nabla_U Q)(U,U) &= -\tfrac{\alpha'}{\theta'}V.
\end{aligned} \tag{20}
$$

It follows that $\nabla Q = 0$ if and only if $\alpha' = 0$, i.e. $\gamma'' = \alpha\gamma$, where α is constant. In such a case

$$\gamma(u) = b(\cos au, \sin au) \tag{21}$$

if $\alpha < 0$ or

$$\gamma(u) = b(\cosh au, \sinh au) \tag{22}$$

if $\alpha > 0$, where $a^2 = \mid \alpha \mid$ and b is a constant. When α is constant, we define a Hermitian scalar product G along γ by the following formulas

$$G(\gamma,\gamma) = 1, \quad G(\gamma,\gamma') = 0, \qquad G(\gamma',\gamma') = -\alpha. \tag{23}$$

One easily checks that $DG = 0$ along γ. Thus G can be extended to the whole $\mathbf{C}^2$. For this G the space $\mathbf{R}^2$, in which the curve γ is located, is a metric Lagrangian subspace of $\mathbf{C}^2$.

Observe that in the example under consideration

$$\tau(U) = 0, \quad \tau(V) = 2\theta',$$

$$g(U,U) = \alpha, \quad g(U,V) = 0, \quad g(V,V) = \theta'^2.$$

By a straightforward computation one checks that $\nabla\tau = 0$. Moreover, $\nabla g = 0$ if and only if α is constant.

If $\gamma(u) = (\cos u, \sin u)$ and $\theta(v) = v$ the surface is the Clifford torus. For the Clifford torus we have

$$\nabla_U U = \nabla_U V = \nabla_V V = 0,$$
$$Q_U U = V, \quad Q_U V = U, \quad Q_V V = V.$$

In particular g is non-degenerate indefinite.

We shall now give examples showing that the there exist non-compact affine Lagrangian submanifolds satisfying the condition $\nabla Q = 0$ having local invariants different than the Clifford torus.

Example 3.2. **1.** The surface given by the parametrization

$$f(u,v) = (e^{2\mathrm{i}v}, ue^{(1+\mathrm{i})v}) \tag{24}$$

is affine Lagrangian and has the following induced objects:

$$Q_U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q_V = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \tag{25}$$

$$\nabla_U U = 0, \quad \nabla_U V = U, \quad \nabla_V V = 2uU. \tag{26}$$

One easily sees that $\nabla Q = 0$, $\mathrm{Ric}\,(U,U) = \mathrm{Ric}\,(U,V) = 0$, $\mathrm{Ric}\,(V,V) = 1$. It follows that the induced connection is non-flat and non-metrizable and therefore the surface is not metric Lagrangian relative to any Kaehler structure on $\mathbf{C}^2$. The rank of the tensor field g is one.

2. Consider the following family of flat examples. Let us define ∇ and Q on $\mathbf{R}^2$ as follows

$$\nabla_U U = \nabla_U V = \nabla_V V = 0, \tag{27}$$

$$Q_U = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad Q_V = \begin{bmatrix} c & e \\ d & k \end{bmatrix}, \tag{28}$$

where $U = \partial_u$, $V = \partial_v$, (u,v) is a coordinate system on $\mathbf{R}^2$ and a, b, c, d, e, k are constants such that $Q_U Q_V = Q_V Q_U$. The last condition is equivalent to the following system of equations

$$cd = eb, \;\; d(d-a) + b(c-k) = 0, \quad c(c-k) + e(d-a) = 0. \tag{29}$$

The objects satisfy the Gauss and the Codazzi equations and hence, by the existence part of the fundamental theorem from [1], they can be realized as the induced objects on some affine Lagrangian surface in $\mathbf{C}^2$. It is easy to find solutions of the above system of equations for which $g = 0$ or $\mathrm{rk}\, g = 1$ or g is positive definite. For instance, if we take $d = e = 0$, $b = -c$, $k = c$ and $a < 2c$, then g is positive definite.

4 Proof of Theorem 1.1

We first prove the following lemma.

Lemma 4.1. *If $f : M \longrightarrow \mathbf{C}^2$ is affine Lagrangian, g is non-degenerate indefinite on M and $\nabla Q = 0$, then ∇ is flat.*

Proof. Since $\nabla Q = 0$, we have $\nabla \tau = 0$ and $\nabla g = 0$. Let $X = \partial_x, Y = \partial_y$, where (x, y) is a local coordinate system on M, be vector fields spanning the asymptotic distributions of g. The distributions are parallel relative to ∇. Since $[X, Y] = 0$, we have

$$\nabla_X X = AX, \quad \nabla_X Y = 0, \quad \nabla_Y Y = FY. \tag{30}$$

The curvature tensor R of ∇ is given by the formulas

$$R(X, Y)X = -(YA)X, \qquad R(X, Y)Y = (XF)Y.$$

Hence, it suffices to show that $YA = 0$ and $XF = 0$. Since

$$(\nabla_X g)(X, Y) = X(g(X, Y)) - Ag(X, Y), \quad (\nabla_Y g)(X, Y) = Y(g(X, Y)) - Fg(X, Y)$$

and $\nabla g = 0$, we have $A = X \ln |g(X, Y)|$, $F = Y \ln |g(X, Y)|$. Consequently

$$YA = XF. \tag{31}$$

Let

$$Q_X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \qquad Q_Y = \begin{bmatrix} c & e \\ d & k \end{bmatrix},$$

where a, b, c, d, e, k are functions on the domain of X, Y. Since $0 \neq g(X, Y) = \frac{1}{2}\det \begin{bmatrix} a & e \\ b & k \end{bmatrix}$, the vectors (a, b), (e, k) are linearly independent. In particular, they are non-zero. Since $0 = g(X, X) = \det Q_X$, we have $(c, d) = \lambda(a, b)$. We now have $0 = g(Y, Y) = \det \begin{bmatrix} \lambda a & e \\ \lambda b & k \end{bmatrix}$. Hence $\lambda = 0$ and

$$Q_X = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \qquad Q_Y = \begin{bmatrix} 0 & e \\ 0 & k \end{bmatrix}. \tag{32}$$

One has

$$0 = (\nabla_X \tau)(X) = Xa - aA, \quad 0 = (\nabla_Y \tau)(X) = Ya. \tag{33}$$

It follows that if $a_{x_0} \neq 0$, then around x_0 we have $A = \frac{Xa}{a}$, and consequently, since $Ya = 0$, we obtain $YA = 0$. Combining this with (31) we get the desired result. If $a_{x_0} = 0$, then $e \neq 0$ in a neighborhood of x_0. Formulas (30) and (32) yield

$$\begin{aligned} (\nabla_X Q)(Y, Y) &= (Xe + eA)X + (Xk)Y \\ (\nabla_Y Q)(Y, Y) &= (Ye - 2eF)X + (Yk - kF)Y. \end{aligned}$$

Since $\nabla Q = 0$ and $e \neq 0$, we obtain

$$\begin{aligned} A &= -\tfrac{Xe}{e} = -X \ln |e| \\ F &= \tfrac{Ye}{2e} = \tfrac{1}{2} Y \ln |e|. \end{aligned} \tag{34}$$

Using now (31) one sees that $XY \ln |e| = 0$. Thus $YA = XF = 0$. The proof of the lemma is completed.

We can now complete the proof of the theorem. Since $\nabla g = 0$, the rank of g is constant. By using Fact 2.1 we know that M is a topological torus and g is nondegenerate indefinite on M. The universal covering space for the torus is $\tilde{M} = \mathbf{R}^2$. We lift the immersion f to the immersion $\tilde{f} : \tilde{M} \longrightarrow \mathbf{C}^2$. The immersion $\tilde{f}$ has the same local properties as f, i.e. it is affine Lagrangian and its second fundamental tensor $\tilde{Q}$ is parallel relative to the induced connection $\tilde{\nabla}$. Therefore, if $\tilde{g}$ and $\tilde{\tau}$ are determined by $\tilde{f}$, then $\tilde{\nabla}\tilde{\tau} = 0$ and $\tilde{\nabla}\tilde{g} = 0$. Moreover $\tilde{g}$ is nondegenerate indefinite on $\tilde{M}$. By the above lemma, $\tilde{\nabla}$ is flat. Using now Facts 2.2 and 2.3 we know that around each point there exist vector fields $U = \partial_u$, $V = \partial_v$, where (u, v) is a coordinate system, such that $\tilde{Q}_V = \mathrm{id}$, $\operatorname{tr} \tilde{Q}_U = 0$. Moreover $\tilde{Q}(U, U) = bV$. Since $\tilde{\nabla}\tilde{Q} = 0$, the vector field V, for which $\tilde{Q}_V = \mathrm{id}$, is parallel relative to $\tilde{\nabla}$. Using again Fact 2.3, we know that around each point there is a coordinate system (u, v) in which all Christoffel symbols of $\tilde{\nabla}$ vanish. Since $(\tilde{\nabla}_U \tilde{Q})(U, U) = (Ub)V$, we have that b is constant.

The vector field V is uniquely and globally given on $\tilde{M}$. Let us fix an orientation on $\mathbf{R}^2$. We have the volume form $\tilde{\nu}$ on $\tilde{M}$ defined by (2) for $\tilde{f}$. The vector field U is given locally and on its connected domains it is unique up to a constant. We can now define U uniquely and hence globally by imposing the conditions $\tilde{\nu}(U, V) > 0$ and $\tilde{Q}(U, U) = V$. We now have two uniquely given global vector fields U, V such that $[U, V] = 0$. Since $\tilde{M}$ is simple connected, there is a global coordinate system (u, v) on $\tilde{M}$ such that $U = \partial_u$, $V = \partial_v$. The Clifford torus defined as in Example 3.1 gives the same induced connection and second fundamental tensor as $\tilde{f}$. Hence, by the fundamental theorem (Fact 2.4) $\tilde{f}$ is complex affine congruent to the Clifford torus.

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