Subharmonic solutions for some nonautonomous Hamiltonian systems with $p(t)$-Laplacian

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Abstract

In this paper, we deal with the existence of subharmonic solutions for the $p(t)$-Laplacian Hamiltonian system. Some new existence theorems are obtained by using minimax methods in critical point theory, and our results generalize and improve some existence theorems.

1 Introduction

Consider the second order $p(t)$-Laplacian Hamiltonian system

$$\frac{d}{dt}\big(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)\big) + \nabla F\big(t, u(t)\big) = 0 \quad \text{a.e. } t \in \mathbb{R} \tag{1.1}$$

where $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic $(T > 0)$ in t for all $x \in \mathbb{R}^N$, that is,

$$F(t + T, x) = F(t, x) \tag{1.2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in \mathbb{R}$, and satisfies the following assumption:

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(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$, such that

$$|F(t,x)| \le a(|x|)b(t), \quad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Moreover, we suppose that $p(t) \in C(\mathbb{R}, \mathbb{R}^+)$ satisfies the following assumption:

(A') $p(t) = p(t+T)$ for all $t \in \mathbb{R}$, where $p^+ := \max\limits_{0 \le t \le T} p(t)$, $p^- := \min\limits_{0 \le t \le T} p(t) > 1$, $q^+ > 1$ which satisfies $1/p^- + 1/q^+ = 1$.

If $p(t) \equiv p > 1$, system (1.1) reduces to the ordinary p-Laplacian system

$$\frac{d}{dt}\big(|\dot{u}(t)|^{p-2}\dot{u}(t)\big) + \nabla F\big(t, u(t)\big) = 0 \quad \text{a.e. } t \in \mathbb{R}. \tag{1.3}$$

When $p = 2$, system (1.3) reduces to the following second-order Hamiltonian system

$$\ddot{u}(t) + \nabla F(t, u(t)) = 0 \quad \text{a.e. } t \in \mathbb{R}. \tag{1.4}$$

To obtain the existence of periodic and subharmonic solution for system (1.4), P. Rabinowitz [15, 16] proposed the following subquadratic conditions, that is, there exist $\mu \in (0,2)$ and $L > 0$ such that

$$0 < \big(\nabla F(t,x), x\big) \le \mu F(t,x)$$

for all $|x| \ge L$ and a.e. $t \in [0,T]$, and the superquadratic condition, that is, there exist $\mu > 2$ and $L > 0$ such that

$$0 < \mu F(t,x) \le \big(\nabla F(t,x), x\big)$$

for all $|x| \ge L$ and a.e. $t \in [0,T]$, these two conditions are known as Ambrosetti-Rabinowitz conditions.

In recent years, considerable attention has been paid to the existence of periodic and subharmonic solutions for system (1.3) and (1.4) under Ambrosetti-Rabinowitz conditions (see [20, 27]), and many authors have devoted to the investigation to weaken the Ambrosetti-Rabinowitz conditions and some existence results on periodic and subharmonic solutions for (1.3) and (1.4) have been obtained under weak conditions (see [8, 12, 13, 21, 22]). Meanwhile, many solvability conditions not related to the Ambrosetti-Rabinowitz conditions are also given, such as the periodic potential condition (see [17]), the convex potential condition (see [2, 9, 26]), the bounded nonlinearity condition (see [1, 10, 11]), the even potential condition (see [23, 25]).

Specifically, when $F(t,x)$ is sublinear, that is, there exist $f, g \in L^1(0,T;R^+)$ and $\alpha \in [0,1)$ such that

$$|\nabla F(t,x)| \le f(t)|x|^\alpha + g(t) \tag{1.5}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$, and there exists a subset E of $[0,T]$ with $\text{meas}(E) > 0$ such that

$$|x|^{-2\alpha} F(t,x) \to +\infty \quad \text{as } |x| \to \infty \tag{1.6}$$

for a.e. $t \in E$, where α is the same in (1.5). Tang [22] obtained the existence of infinitely subharmonic solutions (kT-periodic solution for some positive integer k is called to be subharmonic) for system (1.4) under the conditions (1.5) and (1.6).

In the last decade, the study on problems of elliptic partial differential equations and variational problems with $p(x)$-growth conditions has attracted more and more interest in recent years (see, for example, [3-7, 28]). The ordinary $p(t)$-Laplacian system (1.1) has been studied by Fan (see [7]), then Wang (see [24]) obtained the existence and multiplicity of periodic solutions for ordinary $p(t)$-Laplacian system (1.1) under the generalized Ambrosetti-Rabinowitz conditions, but as far as we know, few papers discuss the subharmonic solutions for system (1.1).

The ordinary $p(t)$-Laplacian system can be applied to describe the physical phenomena with " pointwise different properties " which arose from the nonlinear elasticity theory (see [28]). The $p(t)$-Laplacian system possesses more complicated nonlinearity than that of the p-Laplacian, for example, it is not homogeneous, this causes many troubles, and some classical theories and methods, such as the theory of Sobolev spaces, are not applicable.

Inspired and motivated by the results mentioned above, we obtain some existence results of subharmonic solutions for system (1.1), we suppose that $F(t,x)$ is p^--sublinear, that is, there exist $f, g \in L^1(0,T,R^+)$ and $\alpha \in [0, p^- - 1)$ such that

$$|\nabla F(t,x)| \le f(t)|x|^{\alpha} + g(t) \tag{1.7}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$, and there exists a subset E of $[0,T]$ with $\text{meas}(E) > 0$ such that

$$|x|^{-q^+\alpha} F(t,x) \to +\infty \quad \text{as } |x| \to \infty \tag{1.8}$$

for a.e. $t \in E$, where p^- and q^+ are the same in the condition (A'), under the condition (1.7) and (1.8), the existence of subharmonic solutions for system (1.1), which generalizes Tang's results, are obtained by the minimax methods in critical point theory.

2 Preliminaries

In this section, we recall some known results in critical point theory, and the properties of space $W_T^{1,p(t)}$ are listed for the convenience of readers.

Definition 2.1. ([24]). Let $p(t)$ satisfies the condition (A'). Define

$$L^{p(t)}([0,T],\mathbb{R}^N) = \left\{ u \in L^1([0,T],\mathbb{R}^N); \int_0^T |u|^{p(t)} dt < \infty \right\}$$

with the norm
$$|u|_{p(t)} := \inf\left\{\lambda > 0; \int_0^T \left|\frac{u}{\lambda}\right|^{p(t)} dt \le 1\right\}.$$

For $u \in L^1_{loc}([0,T],\mathbb{R}^N)$, let u' denotes the weak derivative of u, if $u' \in L^1_{loc}([0,T],\mathbb{R}^N)$ and satisfies
$$\int_0^T u'\phi\, dt = -\int_0^T u\phi'\, dt,\ \ \forall \phi \in C_0^\infty([0,T],\mathbb{R}^N).$$

Define
$$W^{1,p(t)}([0,T],\mathbb{R}^N) = \left\{u \in L^{p(t)}([0,T],\mathbb{R}^N);\ u' \in L^{p(t)}([0,T],\mathbb{R}^N)\right\}$$
with the norm $\|u\|_{W^{1,p(t)}} := |u|_{p(t)} + |u'|_{p(t)}$.

Remark 2.1. If $p(t) = p$, where $p \in [1,\infty)$ is a constant, by the definition of $|u|_{p(t)}$, it is easy to get $|u|_p = (\int_0^T |u(t)|^p dt)^{1/p}$, which is the same with the usual norm in space L^p.

The space $L^{p(t)}$ is a generalized Lebesgue space, and the space $W^{1,p(t)}$ is a generalized Sobolev space. Because most of the following Lemmas have appeared in [3, 6, 14, 24], we omit their proofs.

Lemma 2.2. ([6]). $L^{p(t)}$ and $W^{1,p(t)}$ are both Banach spaces with the norms defined above, when $p^- > 1$, they are reflexive.

Definition 2.2. ([14]).
$$C_T^\infty = C_T^\infty(\mathbb{R},\mathbb{R}^N) = \left\{u \in C^\infty(\mathbb{R},\mathbb{R}^N) : u \text{ is } T\text{-periodic}\right\}$$
with the norm $\|u\|_\infty = \max\limits_{t\in[0,T]} |u(t)|$.

For a constant $p \in [1,\infty)$, using another conception of weak derivative which is called T-weak derivative, Mawhin and Willem gave the definition of the space $W_T^{1,p}$ by the following way.

Definition 2.3. ([14]). Let $u \in L^1([0,T],\mathbb{R}^N)$ and $v \in L^1([0,T],\mathbb{R}^N)$, if
$$\int_0^T v\phi\, dt = -\int_0^T u\phi'\, dt\ \ \forall \phi \in C_T^\infty,$$
then v is called a T-weak derivative of u and is denoted by $\dot{u}$.

Definition 2.4. ([14]). Define
$$W_T^{1,p}([0,T],\mathbb{R}^N) = \left\{u \in L^p([0,T],\mathbb{R}^N);\ \dot{u} \in L^p([0,T],\mathbb{R}^N)\right\}$$
with the norm $\|u\|_{W_T^{1,p}} := \left(|u|_p^p + |\dot{u}|_p^p\right)^{1/p}$.

Definition 2.5. ([3]). Define

$$W_T^{1,p(t)}([0,T],\mathbb{R}^N) = \{u \in L^{p(t)}([0,T],\mathbb{R}^N);\ \dot{u} \in L^{p(t)}([0,T],\mathbb{R}^N)\}$$

and $H_T^{1,p(t)}([0,T],\mathbb{R}^N)$ to be the closure of C_T^∞ in $W^{1,p(t)}([0,T],\mathbb{R}^N)$.

Remark 2.2. From Definition 2.5 , if $u \in W_T^{1,p(t)}([0,T],\mathbb{R}^N)$, it is easy to conclude that $u \in W_T^{1,p^-}([0,T],\mathbb{R}^N)$.

Lemma 2.3. ([3]).

(i) $C_T^\infty([0,T],\mathbb{R}^N)$ is dense in $W^{1,p(t)}([0,T],\mathbb{R}^N)$,

(ii) $W_T^{1,p(t)}([0,T],\mathbb{R}^N)$=$H_T^{1,p(t)}([0,T],\mathbb{R}^N)$=$\{u \in W^{1,p(t)}([0,T],\mathbb{R}^N); u(0) = u(T)\}$,

(iii) If $u \in H_T^{1,1}$, then the derivative u' is also the T-weak derivative $\dot{u}$, i.e. $u' = \dot{u}$.

Remark 2.3. In the following article, we use $\|u\|$ instead of $\|u\|_{W_T^{1,p(t)}}$ for convenience without clear indications.

Lemma 2.4. ([14]). Assume that $u \in W_T^{1,1}$, then

(i) $\int_0^T \dot{u} dt = 0$,

(ii) u has its continuous representation, which is still denoted by $u(t) = \int_0^t \dot{u}(s)ds + u(0)$, $u(0) = u(T)$,

(iii) $\dot{u}$ is the classical derivative of u, if $\dot{u} \in C([0,T],\mathbb{R}^N)$.

Since every closed linear subspace of a reflexive Banach space is also reflexive, we have

Lemma 2.5. ([3]). $H_T^{1,p(t)}([0,T],\mathbb{R}^N)$ is a reflexive Banach space if $p^- > 1$.

Obviously, there are continuous embeddings $L^{p(t)} \hookrightarrow L^{p^-}$, $W^{1,p(t)} \hookrightarrow W^{1,p^-}$ and $H_T^{1,p(t)} \hookrightarrow H_T^{1,p^-}$. By the classical Sobolev embedding theorem we obtain

Lemma 2.6. ([3]). There is a continuous embedding

$$W^{1,p(t)}(or\ H_T^{1,p(t)}) \hookrightarrow C([0,T],\mathbb{R}^N),$$

when $p^- > 1$, the embedding is compact.

Lemma 2.7. The space $W_T^{1,p(t)} = \tilde{W}_T^{1,p(t)} \oplus \mathbb{R}^N$, where

$$\tilde{W}_T^{1,p(t)} = \left\{u \in W_T^{1,p(t)}; \int_0^T u(t)dt = 0\right\},$$

there exist positive constants C, if $u \in \tilde{W}_T^{1,p(t)}$, such that

$$\|u\|_\infty \le 2C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{1/p^-} + 2CT^{1/p^-}.$$

Proof. Let $A = \{t \in [0,T] \mid |\dot{u}(t)| \geq 1\}$. From Remark 2.2 , $u \in W_T^{1,p^-}$, from the inequality in classical Sobolev space, there exists a positive constant $C_0 > 0$, such that

$$\begin{aligned}
\|u\|_\infty &\leq C\Big(\int_0^T |\dot{u}(t)|^{p^-} dt\Big)^{1/p^-} \\
&= C\Big(\int_A |\dot{u}(t)|^{p^-} dt + \int_{[0,T]\setminus A} |\dot{u}(t)|^{p^-} dt\Big)^{1/p^-} \\
&\leq C\Big(\int_A |\dot{u}(t)|^{p(t)} dt + \operatorname{meas}([0,T]\setminus A)\Big)^{1/p^-} \\
&\leq C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt + T\Big)^{1/p^-} \\
&\leq 2C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{1/p^-} + 2CT^{1/p^-},
\end{aligned}$$

This completes the proof of Lemma 2.7.

Lemma 2.8. ([3]). Each of the following two norms is equivalent to the norm in $W_T^{1,p(t)}$

(i) $|\dot{u}|_{p(t)} + |u|_q \; 1 \leq q \leq \infty$,

(ii) $|\dot{u}|_{p(t)} + |\bar{u}|$, where $\bar{u} = (1/T)\int_0^T u(t)dt$.

Lemma 2.9. ([7]). If we denote $\rho(u) = \int_0^T |u|^{p(t)} dt$, $\forall u \in L^{p(t)}$, then

(i) $|u|_{p(t)} < 1 \;(= 1; > 1) \iff \rho(u) < 1 \;(= 1; > 1)$;

(ii) $|u|_{p(t)} > 1 \implies |u|^{p^-}_{p(t)} \leq \rho(u) \leq |u|^{p^+}_{p(t)}$, $|u|_{p(t)} < 1 \implies |u|^{p^+}_{p(t)} \leq \rho(u) \leq |u|^{p^-}_{p(t)}$;

(iii) $|u|_{p(t)} \to 0 \iff \rho(u) \to 0$; $|u|_{p(t)} \to \infty \iff \rho(u) \to \infty$.

Proposition 2.1. In space $W_T^{1,p(t)}$, $\|u\| \to \infty \implies \big(\int_0^T |\dot{u}|^{p(t)} dt\big)^{1/p^-} + |\bar{u}| \to \infty$.

Proof. By Lemma 2.8, there exists a constant $C_0 > 0$, such that

$$\|u\| \leq C_0\Big(|\dot{u}|_{p(t)} + |\bar{u}|\Big), \tag{2.1}$$

if $|\dot{u}|_{p(t)} < 1$, it is easy to get

$$|\dot{u}|_{p(t)} < \Big(\int_0^T |\dot{u}|^{p(t)} dt\Big)^{1/p^-} + 1, \tag{2.2}$$

when $|\dot{u}|_{p(t)} \geq 1$, we conclude that

$$|\dot{u}|_{p(t)} \leq \Big(\int_0^T |\dot{u}|^{p(t)} dt\Big)^{1/p^-} \tag{2.3}$$

by Lemma 2.9, it follows (2.1), (2.2) and (2.3) that

$$\|u\| \le C_0\Big(\big(\int_0^T |\dot{u}|^{p(t)}dt\big)^{1/p^-} + |\bar{u}| + 1\Big) \tag{2.4}$$

which implies that

$$\|u\| \to \infty \Longrightarrow \Big(\int_0^T |\dot{u}|^{p(t)}dt\Big)^{1/p^-} + |\bar{u}| \to \infty.$$

Lemma 2.10. ([7]). If $u, u_n \in L^{p(t)}$ $(n = 1, 2, \cdots)$, then the following statements are equivalent to each other

(i) $\lim\limits_{n\to\infty} |u_n - u|_{p(t)} = 0$;

(ii) $\lim\limits_{n\to\infty} \rho(u_n - u) = 0$;

(iii) $u_n \to u$ in measure in $[0, T]$ and $\lim\limits_{n\to\infty} \rho(u_n) = \rho(u)$.

Lemma 2.11. ([18]). Suppose that F satisfies the assumption (A) and E is a measurable subset of $[0, T]$. Assume that

$$F(t, x) \to +\infty \quad \text{as } |x| \to \infty$$

for a.e. $t \in E$. Then for every $\delta > 0$, there exists a subset E_δ of E with $\text{meas}(E\backslash E_\delta) < \delta$ such that

$$F(t, x) \to +\infty \quad \text{as } |x| \to \infty$$

uniformly for all $t \in E_\delta$.

Lemma 2.12. ([24]). The functional on $W_T^{1,p(t)}$ given by

$$\varphi(u) = \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt - \int_0^T F(t, u(t))dt \tag{2.5}$$

is continuously differentiable on $W_T^{1,p(t)}$. Moreover, we have

$$\langle \varphi'(u), v\rangle = \int_0^T \Big[\Big(|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{v}(t)\Big) - \Big(\nabla F(t, u(t)), v(t)\Big)\Big]dt \tag{2.6}$$

for all $u, v \in W_T^{1,p(t)}$. It is well known that the critical points of φ correspond to the solutions for system (1.1).

Lemma 2.13. ([24]). J' is a mapping of (S_+), that is, if $u_n \rightharpoonup u$ weakly in $W_T^{1,p(t)}$ and $\limsup\limits_{n\to\infty} (J'(u_n) - J'(u), u_n - u)) \le 0$, then u_n has a convergent subsequence in $W_T^{1,p(t)}$, where J' is given by

$$\langle J'(u), v\rangle = \int_0^T \Big(|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{v}(t)\Big)dt. \tag{2.7}$$

3 Main Results and Proofs of Theorems

Let k be a positive integer. For $u \in W_{kT}^{1,p(t)}$, where

$$W_{kT}^{1,p(t)}([0,kT],\mathbb{R}^N) := \{u \in L^{p(t)}([0,kT],\mathbb{R}^N);\ \dot{u} \in L^{p(t)}([0,kT],\mathbb{R}^N)\}$$

is a reflexive Banach space with the norm defined by

$$\|u\|_{W_{kT}^{1,p(t)}} = |u|_{p(t)} + |\dot{u}|_{p(t)}$$

where

$$|u|_{p(t)} := \inf\left\{\lambda > 0; \int_0^{kT} \left|\frac{u}{\lambda}\right|^{p(t)} dt \le 1\right\}$$

and

$$|\dot{u}|_{p(t)} := \inf\left\{\lambda > 0; \int_0^{kT} \left|\frac{\dot{u}}{\lambda}\right|^{p(t)} dt \le 1\right\}$$

and

$$\|u\|_\infty = \max_{t\in[0,kT]} |u(t)|.$$

Let

$$\bar{u} = (kT)^{-1}\int_0^{kT} u(t)dt \quad \text{and} \quad \tilde{u}(t) = u(t) - \bar{u},$$

then we have

$$\|\tilde{u}\|_\infty \le 2C_k\Big(\int_0^{kT} |\dot{u}(t)|^{p(t)}dt\Big)^{1/p^-} + 2C_k(kT)^{1/p^-} \tag{3.1}$$

and

$$\|u\| \le C_{0k}\Big(\big(\int_0^{kT} |\dot{u}|^{p(t)}dt\big)^{1/p^-} + |\bar{u}| + 1\Big), \tag{3.2}$$

and

$$\|u\| \to \infty \Longrightarrow \Big(\int_0^{kT} |\dot{u}|^{p(t)}dt\Big)^{1/p^-} + |\bar{u}| \to \infty \tag{3.3}$$

by Lemma 2.7 and Proposition 2.1, where C_k and C_{0k} are two positive constants, which can be decided by k.

It follows from Lemma 2.12 and Lemma 2.13 that the functional φ_k on $W_{kT}^{1,p(t)}$ given by

$$\varphi_k(u) = \int_0^{kT} \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt - \int_0^{kT} F(t,u(t))dt$$

is continuously differentiable on $W_{kT}^{1,p(t)}$. Moreover, we have

$$\langle \varphi_k'(u), v\rangle = \int_0^{kT} \Big[\Big(|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{v}(t)\Big) - \Big(\nabla F(t,u(t)), v(t)\Big)\Big]dt$$

$$\langle J_k'(u), v\rangle = \int_0^{kT} \Big(|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{v}(t)\Big)dt$$

It is well known that the kT periodic solutions for system (1.1) correspond to the critical points of the functional φ_k.

Our main results are the following theorems.

Theorem 3.1. Suppose that $F(t,x)$ and $p(t)$ satisfy assumption (A), (A′), (1.2), (1.7). Assume that

$$|x|^{-q^+\alpha}F(t,x) \to +\infty \quad \text{as } |x| \to \infty \tag{3.4}$$

uniformly for a.e. $t \in [0,T]$, where α is the same in (1.7). Then system (1.1) has kT-periodic solution $u_k \in W^{1,p(t)}_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$.

We shall give a more general result than Theorem 3.1.

Theorem 3.2. Suppose that $F(t,x)$ and $p(t)$ satisfy assumption (A), (A′), (1.2), (1.7), (1.8). Assume that there exists $\gamma(t) \in L^1(0,T)$ such that

$$F(t,x) \geq \gamma(t) \tag{3.5}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Then system (1.1) has kT-periodic solution $u_k \in W^{1,p(t)}_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$.

Corollary 3.1. Suppose that F and $p(t)$ satisfies assumption (A), (A′), (1.2) and (3.5). Assume that there there exists a subset E of $[0,T]$ with $\text{meas}(E) > 0$ such that

$$F(t,x) \to +\infty \quad \text{as } |x| \to +\infty$$

for a.e. $t \in E$, and there exists $h(t) \in L^1(0,T,\mathbb{R}^+)$ such that

$$|\nabla F(t,x)| \leq h(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Then system (1.1) has kT-periodic solution $u_k \in W^{1,p(t)}_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$.

Remark 3.1. Corollary 3.1 is a special case of Theorem 3.2 corresponding to $\alpha = 0$. Without loss of generality, we can assume that the function $b(t)$ in assumption (A), $f(t)$, $g(t)$ in (1.7) and $\gamma(t)$ in (3.5) are also T-periodic, then assumption (A), (1.7) and (3.5) hold for a.e. $t \in \mathbb{R}$ by the periodicity of $F(t,x)$ in the first variable.

Because Theorem 3.2 is a more general result than Theorem 3.1, we only need to prove Theorem 3.2, and our steps to prove Theorem 3.2 are organized as follows. First, we show the functional φ_k satisfies the (PS) conditions; second, we prove that φ_k satisfies the other conditions of saddle point Theorem (see Theorem 4.6 in [14]); after these two steps, by saddle point theorem we know that φ_k has at least one critical point, which is a kT periodic solution for system (1.1), at last, we prove $\|u_k\|_\infty \to \infty$ as $k \to \infty$.

Proof of Theorem 3.2. It follows from (1.8) and (3.1) that

$$
\begin{aligned}
&\left|\int_0^{kT}(\nabla F(t,u(t)),\tilde{u}(t))\right| \\
&\qquad \le \int_0^{kT} f(t)|\bar{u}+\tilde{u}(t)|^{\alpha}|\tilde{u}(t)|dt + \int_0^{kT} g(t)|\tilde{u}(t)|dt \\
&\qquad \le 2^{p^- -1}\int_0^{kT} f(t)(|\bar{u}|^{\alpha}+|\tilde{u}|^{\alpha})|\tilde{u}(t)|dt + \int_0^{kT} g(t)|\tilde{u}(t)|dt \\
&\qquad \le 2^{p^- -1}(|\bar{u}|^{\alpha}+\|\tilde{u}\|_{\infty}^{\alpha})\|\tilde{u}\|_{\infty}\int_0^{kT} f(t)dt + \|\tilde{u}\|_{\infty}\int_0^{kT} g(t)dt \\
&\qquad = \Big((\frac{1}{2})^{1/p^-}\frac{\|\tilde{u}\|_{\infty}}{4C_k}\Big)\Big((2^{p^-+1})(2)^{1/p^-}C_k\int_0^{kT} f(t)dt\Big)|\bar{u}|^{\alpha} \\
&\qquad + 2^{p^- -1}\|\tilde{u}\|_{\infty}^{1+\alpha}\int_0^{kT} f(t)dt + \|\tilde{u}\|_{\infty}\int_0^{kT} g(t)dt \\
&\qquad \le \frac{1}{2}\int_0^{kT}|\dot{u}(t)|^{p(t)}dt + C_1|\bar{u}|^{q^+\alpha} + C_2\Big(\int_0^{kT}|\dot{u}(t)|^{p(t)}dt\Big)^{(\alpha+1)/p^-} \\
&\qquad + C_3\Big(\int_0^{kT}|\dot{u}(t)|^{p(t)}dt\Big)^{1/p^-} + C_4
\end{aligned}
$$

for all $u \in W_{kT}^{1,p(t)}$ and some positive constants C_1, C_2, C_3 and C_4 by Young inequality and (3.1), where C_k is the same as in (3.1).

Hence, we have

$$
\begin{aligned}
\|\tilde{u}_n\| &\ge |\langle \varphi_k'(u_n),\tilde{u}_n\rangle| \\
&\ge \int_0^{kT}|\dot{u}_n(t)|^{p(t)}dt - \int_0^{kT}\Big(\nabla F(t,u_n(t)),\tilde{u}_n(t)\Big)dt \\
&\ge \frac{1}{2}\int_0^{kT}|\dot{u}_n(t)|^{p(t)}dt - C_1|\bar{u}_n|^{q^+\alpha} - C_2\Big(\int_0^{kT}|\dot{u}_n(t)|^{p(t)}dt\Big)^{(\alpha+1)/p^-} \\
&- C_3\Big(\int_0^{kT}|\dot{u}_n(t)|^{p(t)}dt\Big)^{1/p^-} - C_4
\end{aligned}
\tag{3.6}
$$

for all large n.

It follows from (3.2) that

$$\|\tilde{u}_n\| \le C_{0k}\Big((\int_0^{kT}|\dot{u}_n(t)|^{p(t)}dt)^{1/p^-}+1\Big), \tag{3.7}$$

by (3.6) and (3.7), we have

$$\Big(\int_0^{kT}|\dot{u}_n(t)|^{p(t)}dt\Big)^{1/q^+} \le C_5|\bar{u}_n|^{\alpha} + C_6 \tag{3.8}$$

for some positive constants C_5, C_6 and all large n. which implies that

$$\|\tilde{u}_n\|_{\infty} \le C_7(|\bar{u}_n|^{q^+\alpha/p^-}+1)$$

for all large n and some positive constant C_7 by (3.1).

If $(|\bar{u}_n|)$ is unbounded, we may assume that going to a subsequence if necessary

$$|\bar{u}_n| \to \infty \ \text{ as } n \to \infty, \tag{3.9}$$

and we have

$$|u_n(t)| \geq |\bar{u}_n| - |\tilde{u}_n(t)| \geq |\bar{u}_n| - \|\tilde{u}_n\|_\infty \geq |\bar{u}_n| - C_7(|\bar{u}_n|^{q^+\alpha/p^-} + 1)$$

for all large n and every $t \in [0, kT]$, which implies that

$$|u_n(t)| \geq \frac{1}{2}|\bar{u}_n| \tag{3.10}$$

for all large n and every $t \in [0, kT]$ by (3.9).

Set $\delta = (\text{meas } E)/2$. It follows from (3.5) and Lemma 2.11 that there exists a subset E_δ of E with $\text{meas}(E \backslash E_\delta) < \delta$ such that

$$|x|^{-q^+\alpha} F(t, x) \to +\infty \ \text{ as } |x| \to +\infty$$

uniformly for all $t \in E_\delta$, which implies that

$$\text{meas} E_\delta = \text{meas} E - \text{meas}(E \backslash E_\delta) > \delta > 0 \tag{3.11}$$

and for every $N > 0$, there exists $M \geq 1$ such that

$$|x|^{-q^+\alpha} F(t, x) \geq N \tag{3.12}$$

for all $|x| \geq M$ and all $t \in E_\delta$. By (3.9) and (3.10), we have

$$|u_n(t)| \geq M \tag{3.13}$$

for large n and every $t \in [0, kT]$. It follows (1.8), (3.8), (3.11)-(3.13) that

$$\begin{aligned}
\varphi_k(u_n) &\leq (C_5|\bar{u}_n|^\alpha + C_6)^{q^+} - \int_{[0,kT]\backslash E_\delta} \gamma(t)dt - \int_{E_\delta} N|u_n(t)|^{q^+\alpha} dt \\
&\leq (C_5|\bar{u}_n|^\alpha + C_6)^{q^+} - \int_{[0,kT]\backslash E_\delta} \gamma(t)dt - 2^{-q^+\alpha}|\bar{u}_n|^{q^+\alpha}\delta N
\end{aligned}$$

for large n. Hence, we have

$$\limsup_{n\to\infty} |\bar{u}_n|^{-q^+\alpha} \varphi_k(u_n) \leq C_5^{q^+} - 2^{-q^+\alpha}\delta N.$$

By the arbitrariness of $N > 0$, we have

$$\limsup_{n\to\infty} |\bar{u}_n|^{-q^+\alpha} \varphi_k(u_n) = -\infty,$$

which contradicts the boundedness of $\varphi_k(u_n)$. Hence $(|\bar{u}_n|)$ is bounded, and $\{\|u_n\|\}$ is bounded by (3.2) and (3.8).

The sequence $\{u_n\}$ has a subsequence, also denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \text{ weakly in } W_{kT}^{1,p(t)} \quad \text{and} \quad u_n \to u \text{ strongly in } C\big([0,kT];\mathbb{R}^N\big) \tag{3.14}$$

and $\|u_n\|_\infty \leq C_7$ is bounded by Lemma 2.6, where C_7 is a positive constant.

We conclude that

$$\left|\int_0^{kT} \Big(\nabla F(t,u_n(t)),u_n(t)-u(t)\Big)dt\right| \leq ka_0\|u_n-u\|_\infty \int_0^T b(t)dt \\ \to 0, \tag{3.15}$$

by assumption (A) and (3.14), where $a_0 = \max\limits_{0\leq s\leq C_7} a(s)$.

By Lemma 2.12, we have

$$\langle \varphi_k'(u_n),u_n-u\rangle = \int_0^{kT} \Big[\Big((|\dot{u}_n(t)|^{p(t)-2}\dot{u}_n(t),\dot{u}_n(t)-\dot{u}(t)\Big)- \\ \Big(\nabla F(t,u_n(t)),u_n(t)-u(t)\Big)\Big]dt,$$

and $\langle \varphi_k'(u_n),u_n-u\rangle \to 0$ by the assumption of $\varphi_k'(u_n)\to 0$ and the boundedness of $\{\|u_n\|\}$.

Then it follows from (2.6), (2.7) and (3.15) that

$$\begin{aligned}\langle J_k'(u_n),u_n-u\rangle &= \int_0^{kT} \Big(|\dot{u}_n(t)|^{p(t)-2}\dot{u}_n(t),\dot{u}_n(t)-\dot{u}(t)\Big)dt \\ &= \langle \varphi_k'(u_n),u_n-u\rangle + \int_0^{kT} \Big(\nabla F(t,u_n(t)),u_n(t)-u(t)\Big)dt \\ &\to 0\end{aligned} \tag{3.16}$$

Moreover, since $J_k'(u)$ is a bounded linear function, we get $\langle J_k'(u),u_n-u\rangle \to 0$, which combined with (3.16) implies that

$$\lim_{n\to\infty} \langle J_k'(u_n)-J_k'(u),u_n-u\rangle = 0 \tag{3.17}$$

It follows from (3.14), (3.17) and Lemma 2.13 that $\{u_n\}$ admits a convergent subsequence, so we conclude that φ_k satisfies (PS) condition.

We prove that φ_k satisfies the other conditions of the saddle point theorem. Set

$$e_k(t) = k(\cos k^{-1}\omega t)x_0$$

for all $t\in\mathbb{R}$ and some $|x_0|=1$, where $\omega = 2\pi/T$. Then we have

$$\dot{e}_k(t) = -\omega(\sin k^{-1}\omega t)x_0$$

Hence we have

$$\begin{aligned}
\varphi_k(x+e_k) &\le \frac{1}{p^-}\int_0^{kT} |\omega(\sin k^{-1}\omega t)|^{p(t)}dt - \int_0^{kT} F(t,x+e_k)dt \\
&\le \frac{kT}{p^-}(\omega^{p^+}+1) - \int_{[0,kT]\setminus E_\delta} \gamma(t)dt - N\int_{E_\delta} |x+k(\cos k^{-1}\omega t)x_0|^{q^+\alpha}dt \\
&\le \frac{kT}{p^-}(\omega^{p^+}+1) - \int_{[0,kT]\setminus E_\delta} \gamma(t)dt - NM^{q^+\alpha}\mathrm{meas}E_\delta \\
&\le \frac{kT}{p^-}(\omega^{p^+}+1) - \int_{[0,kT]\setminus E_\delta} \gamma(t)dt - N\mathrm{meas}E_\delta
\end{aligned}$$

for all $|x| \ge M+k$, which implies that

$$\varphi_k(x+e_k) \to -\infty \text{ as } |x| \to \infty \tag{3.18}$$

by the arbitrariness of N.

Let $\tilde{W}_{kT}^{1,p(t)}$ be the subspace of $W_{kT}^{1,p(t)}$ given by

$$\tilde{W}_{kT}^{1,p(t)} = \left\{ u \in W_{kT}^{1,p(t)};\ \int_0^{kT} u(t)dt = 0 \right\},$$

then we have

$$\varphi(u) \to +\infty \tag{3.19}$$

as $\|u\| \to \infty$ in $\tilde{W}_{kT}^{1,p(t)}$. In fact it follows from Lemma 2.7 that

$$\begin{aligned}
&\left|\int_0^{kT} [F(t,u(t)) - F(t,0)]dt\right| \\
&\qquad = \left|\int_0^{kT}\int_0^1 (\nabla F(t,su(t)),u(t))dsdt\right| \\
&\qquad \le \int_0^{kT}\int_0^1 f(t)|su(t)|^\alpha |u(t)|dsdt + \int_0^{kT}\int_0^1 g(t)|u(t)|dsdt \\
&\qquad \le \|u\|_\infty^{\alpha+1}\int_0^{kT} f(t)dt + \|u\|_\infty \int_0^{kT} g(t)dt \\
&\qquad \le C_8\Big(\int_0^{kT} |\dot{u}(t)|^{p(t)}dt\Big)^{(\alpha+1)/p^-} + C_9\Big(\int_0^{kT} |\dot{u}(t)|^{p(t)}dt\Big)^{1/p^-} + C_{10}
\end{aligned}$$

for all $u \in \tilde{W}_{kT}^{1,p(t)}$ and some positive constants C_8, C_9 and C_{10}.

Hence we have

$$\begin{aligned}
\varphi_k(u) &= \int_0^{kT} \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt - \int_0^{kT} [F(t,u(t)) - F(t,0)]dt - \int_0^{kT} F(t,0)dt \\
&\ge \frac{1}{p^+}\int_0^{kT} |\dot{u}(t)|^{p(t)}dt - C_8\Big(\int_0^{kT} |\dot{u}(t)|^{p(t)}dt\Big)^{(\alpha+1)/p^-} \\
&\quad - C_9\Big(\int_0^{kT} |\dot{u}(t)|^{p(t)}dt\Big)^{1/p^-} - \int_0^{kT} F(t,0)dt - C_{10}
\end{aligned}$$

for all $u \in \tilde{W}_{kT}^{1,p(t)}$, which implies (3.19) by (3.3).

By (3.18), (3.19) and the saddle point theorem (see Theorem 4.6 in [14]), there exists a critical point $u_k \in W_{kT}^{1,p(t)}$ for φ_k such that

$$-\infty < \inf_{\tilde{W}_{kT}^{1,p(t)}} \varphi_k \leq \varphi_k(u_k) \leq \sup_{\mathbb{R}^N + e_k} \varphi_k \tag{3.20}$$

Arguing in a similar way in [22], we can prove that $\|u_k\|_\infty \to \infty$ as $k \to \infty$.

4 Example

In this section, we give one example to illustrate our results.

Example 4.1. In system (1.1), let $p(t) = \cos \omega t + 7$, and let

$$F(t,x) = |x|^4 + \sin \omega t$$

where ω denotes the positive constant $2\pi/T$. Then

$$|\nabla F(t,x)| \leq 4|x|^3 \quad \text{and} \quad |x|^{-18/5} F(t,x) \to +\infty$$

for every $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

These show that all conditions of Theorem 3.2 are satisfied, where

$$\alpha = 3, \;\; p^- = 6, \;\; q^+ = \frac{6}{5}.$$

By Theorem 3.2, system (1.1) has kT-periodic solution $u_k \in W_{kT}^{1,p(t)}$ for every positive integer k such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$, but it is obvious that the results in [1-2, 9-12, 16-17, 21-23, 25-26] can't be applied to our example.

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