Periodic solutions for second order Hamiltonian system with a p-Laplacian*

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Abstract

In this paper, by using an improved inequality, we improve an existence theorem of periodic solutions for second order Hamiltonian system with a p-Laplacian. Moreover, an estimate of solutions is also given. Our results improve those in some known literatures.

1. Introduction

Consider the ordinary p-Laplacian system

$$\begin{cases} \frac{d}{dt}\Phi_p(\dot{x}(t)) + \nabla F(t, x(t)) = 0, & \text{a.e. } t \in [0, T], \\ x(0) = x(T), \dot{x}(0) = \dot{x}(T). \end{cases} \tag{1.1}$$

where

$$\Phi_p(u) = |u|^{p-2}u = \left(\sum_{i=1}^{N} u_i^2\right)^{\frac{p-2}{2}} \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix},$$

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$T > 0$, $p > 1, q > 1, 1/p + 1/q = 1$, and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$, $(t, x) \to F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable and convex in x for almost every $t \in [0, T]$.

When $p = 2$, there are many existence results of periodic solutions for system (1.1) (see [1-6] and references therein). However, when $p > 1$, there are few papers to study these problems. In [7] and [8], the authors considered system (1.1) by using the dual least action principle and a generalized Mountain pass Lemma, respectively, and they obtained some existence results of solutions for system (1.1). In [9], we also considered system (1.1) by using the generalized Saddle point Theorem and obtained that system (1.1) has multiple solutions. Especially, in [7], Tian and Ge obtained the following results:

Theorem A *Suppose F satisfies the following conditions:*
(A_1) there exists $l \in L^{2\max\{q,p-1\}}(0, T; \mathbb{R}^N)$ such that for all $y \in \mathbb{R}^N$ and a.e. $t \in [0, T]$,

$$F(t, y) \geq \left(l(t), |y|^{\frac{p-2}{2}} y\right);$$

(A_2) there are constants $\alpha \in (0, T^{-p/q})$, $\alpha^{q-1} \in (0, T^{-q/p})$, $p > 1$, $\gamma \in L^{\max\{q,p-1\}}(0, T; \mathbb{R}^N)$ such that for $y \in \mathbb{R}^N$, and a.e. $t \in [0, T]$,

$$F(t, y) \leq \frac{\alpha^2}{p}|y|^p + \gamma(t);$$

(A_3) $\int_0^T F(t, y)dt \to +\infty$, as $|y| \to \infty$, $y \in \mathbb{R}^N$.
Then, system (1.1) has at least one solution.

In our paper, by using the improved inequality, we improve the condition (A_2) and also obtain an estimate of periodic solution for system (1.1).

2. Preliminaries

In the following, we use $| \cdot |$ to denote the Euclidean norm in $\mathbb{R}^N$. Let

$$\begin{aligned} W_T^{1,p} &= \{u : [0, T] \to \mathbb{R}^N |\ u(t) \text{ is absolutely continuous on } [0, T], \\ &\quad u(0) = u(T) \text{ and } \dot{u} \in L^p(0, T; \mathbb{R}^N)\}. \end{aligned}$$

Then, it follows from [2] that $W_T^{1,p}$ is a Banach space with the norm defined by

$$\|u\|_{W_T^{1,p}} = \left[\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt\right]^{1/p}, \quad u \in W_T^{1,p}.$$

It follows from [2] that $W_T^{1,p}$ is also reflexive and uniformly convex Banach space.

Let

$$X = \{v = (v_1, v_2) : v_1 \in W_T^{1,q}(0, T; \mathbb{R}^N), v_2 \in W_T^{1,p}(0, T; \mathbb{R}^N)\}$$

with the norm $\|v\| = \|v_1\|_{W_T^{1,q}} + \|v_2\|_{W_T^{1,p}}$. It is clear that X is a reflexive Banach space.

Let

$$\tilde{W}_T^{1,p} = \left\{ u \in W_T^{1,p} \;\middle|\; \int_0^T u(t)dt = 0 \right\}.$$

It is easy to know that $\tilde{W}_T^{1,p}$ is a subset of $W_T^{1,p}$ and $W_T^{1,p} = \mathbb{R}^N \oplus \tilde{W}_T^{1,p}$. Then $\tilde{X}$ stands for

$$\tilde{X} = \{v = (v_1, v_2) : v_1 \in \tilde{W}_T^{1,q}(0,T;\mathbb{R}^N), v_2 \in \tilde{W}_T^{1,p}(0,T;\mathbb{R}^N)\},$$

and $(W_T^{1,p})^*$ stands for the conjugate space of $W_T^{1,p}$. Then

$$X^* = \left\{ f = (f_1, f_2) : f_1 \in \left(W_T^{1,q}\right)^*, f_2 \in \left(W_T^{1,p}\right)^* \right\}$$

is the conjugate space of X. Furthermore, we define

$$Y = \{u = (u_1, u_2) : u_1 \in W_T^{1,p}(0,T;\mathbb{R}^N), u_2 \in W_T^{1,q}(0,T;\mathbb{R}^N)\}.$$

For $h \in L^1([0,T];\mathbb{R}^N)$, the mean value is defined by $\bar{h} = 1/T \int_0^T h(t)dt$. Besides this, $\|\cdot\|_\infty$, $\|\cdot\|_{L^k}$ and $\|\cdot\|_{W_T^{1,k}}$ stand for the norm in $C^0([0,T])$, $L^k([0,T])$ and $W_T^{1,k}$, respectively.

$\Gamma_0(\mathbb{R}^N)$ denotes the set of all convex lower semi-continuous (l.s.c.) functions $F : \mathbb{R}^N \to (-\infty, +\infty]$ whose effective domain $D(F) = \{u \in \mathbb{R}^N : F(u) < +\infty\}$ is nonempty. Let $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$, $(t,u) \to H(t,u)$ be a smooth Hamiltonian such that for each $t \in [0,T]$, $H(t,\cdot) \in \Gamma_0(\mathbb{R}^{2N})$ is strictly convex and $H(t,u)/|u| \to +\infty$, if $|u| \to \infty$. The Fenchel transform $H^*(t,\cdot)$ of $H(t,\cdot)$ is defined by

$$H^*(t,v) = \sup_{u \in \mathbb{R}^{2N}} \{(v,u) - H(t,u)\}$$

or

$$\begin{aligned} &H^*(t,v) = (v,u) - H(t,u) \\ &v = \nabla H(t,u), \quad \text{or} \quad u = \nabla H^*(t,v). \end{aligned} \tag{2.1}$$

If for $u = (u_1, u_2), u_1, u_2 \in \mathbb{R}^N$, $H(t,u)$ can be split into parts $H(t,u) = H_1(t,u_1) + H_2(t,u_2)$, then by (2.1), $H^*(t,v) = H_1^*(t,v_1) + H_2^*(t,v_2), v = (v_1,v_2), v_1, v_2 \in \mathbb{R}^N$. We denote by J the symplectic matrix. Then $J^2 = -I$ and $(Ju,v) = -(u,Jv)$ for all $u,v \in \mathbb{R}^{2N}$. It is clear that $(J\dot{v},v) = (\dot{v}_2,v_1) - (\dot{v}_1,v_2)$, where $v = (v_1,v_2)$, $v_i \in C(0,T;\mathbb{R}^N), i = 1,2$. The above knowledge and statement come from [2,7] and the references therein.

Let $x(t) = u_1(t)$, $\Phi_p(\dot{x}(t)) = \alpha u_2(t)$. Then system (1.1) is equivalent to the non-autonomous system

$$\begin{cases} \dot{u}_2(t) + \frac{1}{\alpha}\nabla F(t,u_1(t)) = 0, & \text{a.e. } t \in [0,T], \\ -\dot{u}_1(t) + \Phi_q(\alpha u_2(t)) = 0, & \\ u_i(0) = u_i(T), \;\; i = 1,2, & \end{cases} \tag{2.2}$$

that is

$$\begin{cases} J\dot{u}(t) + \nabla H(t,u(t)) = 0, & \text{a.e. } t \in [0,T], \\ u(0) = u(T), & \end{cases} \tag{2.3}$$

where $u=(u_1,u_2)$, $H(t,u)=H_1(t,u_1)+H_2(t,u_2)$,

$$H_1(t,u_1)=\frac{1}{\alpha}F(t,u_1),\quad H_2(t,u_2)=\frac{\alpha^{q-1}}{q}|u_2|^q, \tag{2.4}$$

where $H:[0,T]\times\mathbb{R}^{2N}\to\mathbb{R}$, $H_i:[0,T]\times\mathbb{R}^N\to\mathbb{R}, i=1,2$.

The dual action is defined on X by

$$\varphi(v)=\int_0^T\left[\frac{1}{2}(J\dot{v}(t),v(t))+H_1^*(t,\dot{v}_1(t))+H_2^*(t,\dot{v}_2(t))\right]dt,$$

where $v=(v_1,v_2)$, $H^*(t,\dot{v})=H_1^*(t,\dot{v}_1)+H_2^*(t,\dot{v}_2)$.

Lemma 2.1. (also see [9], Lemma 2.2) *Let $u\in\tilde{W}_T^{1,p}$. Then*

$$\|u\|_\infty\le\left(\frac{T}{q+1}\right)^{1/q}\left(\int_0^T|\dot{u}(s)|^p ds\right)^{1/p}, \tag{2.5}$$

and

$$\int_0^T|u(s)|^p ds\le\frac{T^p\Theta(p,q)}{(q+1)^{p/q}}\int_0^T|\dot{u}(s)|^p ds, \tag{2.6}$$

where

$$\Theta(p,q)=\int_0^1\left[s^{q+1}+(1-s)^{q+1}\right]^{p/q}ds.$$

Proof. Fix $t\in[0,T]$. For every $\tau\in[0,T]$, we have

$$u(t)=u(\tau)+\int_\tau^t\dot{u}(s)ds. \tag{2.7}$$

Set

$$\phi(s)=\begin{cases}s, & 0\le s\le t,\\ T-s, & t\le s\le T.\end{cases}$$

Integrating (2.7) over $[0,T]$ and using the Hölder's inequality, we obtain

$$\begin{aligned}
T|u(t)| &= \left|\int_0^T u(\tau)d\tau+\int_0^T\int_\tau^t\dot{u}(s)dsd\tau\right|\\
&\le \int_0^t\int_\tau^t|\dot{u}(s)|dsd\tau+\int_t^T\int_t^\tau|\dot{u}(s)|dsd\tau\\
&= \int_0^t s|\dot{u}(s)|ds+\int_t^T(T-s)|\dot{u}(s)|ds\\
&= \int_0^T\phi(s)|\dot{u}(s)|ds\\
&\le \left(\int_0^T[\phi(s)]^q ds\right)^{1/q}\left(\int_0^T|\dot{u}(s)|^p ds\right)^{1/p}\\
&= \frac{1}{(q+1)^{1/q}}\left[t^{q+1}+(T-t)^{q+1}\right]^{1/q}\left(\int_0^T|\dot{u}(s)|^p ds\right)^{1/p}.
\end{aligned} \tag{2.8}$$

Since $t^{q+1} + (T-t)^{q+1} \leq T^{q+1}$ for $t \in [0,T]$, it follows from (2.8) that (2.5) holds. On the other hand, from (2.8), we have

$$\begin{aligned}
T^p \int_0^T |u(t)|^p dt &\leq \frac{1}{(q+1)^{p/q}} \left(\int_0^T |\dot{u}(s)|^p ds \right) \int_0^T \left[t^{q+1} + (T-t)^{q+1} \right]^{p/q} dt \\
&\leq \frac{T^{1+p(q+1)/q}}{(q+1)^{p/q}} \left(\int_0^T |\dot{u}(s)|^p ds \right) \int_0^1 \left[s^{q+1} + (1-s)^{q+1} \right]^{p/q} ds \\
&= \frac{T^{2p}\Theta(p,q)}{(q+1)^{p/q}} \int_0^T |\dot{u}(s)|^p ds.
\end{aligned}$$

It follows that (2.6) holds. The proof is complete.

Remark 2.1. Obviously, our Lemma 2.1 improve Proposition 1.1 in [2] which shows that

$$\|u\|_\infty \leq T^{1/q} \|\dot{u}\|_{L^p}, \quad \|u\|_{L^p}^p \leq T^p \|\dot{u}\|_{L^p}^p. \tag{2.9}$$

Lemma 2.2. For every $v = (v_1, v_2) \in X$,

$$\int_0^T (J\dot{v}(t), v(t)) dt \geq -\frac{C}{p} \|\dot{v}_2\|_{L^p}^p - \frac{C}{q} \|\dot{v}_1\|_{L^q}^q; \tag{2.10}$$

for every $u = (u_1, u_2) \in Y$,

$$\int_0^T (J\dot{u}(t), u(t)) dt \geq -\frac{C}{q} \|\dot{u}_2\|_{L^q}^q - \frac{C}{p} \|\dot{u}_1\|_{L^p}^p, \tag{2.11}$$

where

$$C = \frac{T}{(q+1)^{1/q}} + \frac{T}{(p+1)^{1/p}}.$$

Proof. Let $v = \bar{v} + \tilde{v}$, where $\bar{v} = 1/T \int_0^T v(s) ds$. Then by Lemma 2.1, Hölder's inequality and Young's inequality, for $v \in X$, we have

$$\begin{aligned}
\int_0^T (J\dot{v}(t), v(t)) dt &= \int_0^T (J\dot{v}(t), \tilde{v}(t)) dt \\
&= \int_0^T [(\dot{v}_2(t), \tilde{v}_1(t)) - (\dot{v}_1(t), \tilde{v}_2(t))] dt \\
&\geq -\|\tilde{v}_1\|_\infty \int_0^T |\dot{v}_2(t)| dt - \|\tilde{v}_2\|_\infty \int_0^T |\dot{v}_1(t)| dt \\
&\geq -\frac{T}{(p+1)^{1/p}} \|\dot{v}_1\|_{L^q} \|\dot{v}_2\|_{L^p} - \frac{T}{(q+1)^{1/q}} \|\dot{v}_2\|_{L^p} \|\dot{v}_1\|_{L^q} \\
&= -C \|\dot{v}_2\|_{L^p} \|\dot{v}_1\|_{L^q} \\
&\geq -\frac{C}{p} \|\dot{v}_2\|_{L^p}^p - \frac{C}{q} \|\dot{v}_1\|_{L^q}^q.
\end{aligned}$$

Similarly to the above process, the result (2.11) holds for $u = (u_1, u_2) \in Y$.

Remark 2.2. Obviously, our Lemma 2.2 improve Lemma 3.3 in [7].

Lemma 2.3. [2, Proposition 1.4] Let $G \in C^1(\mathbb{R}^N, \mathbb{R})$ be a convex function. Then, for all $x, y \in \mathbb{R}^N$, we have

$$G(x) \geq G(y) + (\nabla G(y), x - y).$$

3. Main results and Proofs

Theorem 3.1 *Suppose F satisfies (A_1), (A_3) and the following condition:*
$(A_2)'$ there are constants $\alpha \in (0, (C/2)^{-p/q})$, $\alpha^{q-1} \in (0, (C/2)^{-q/p})$, $\gamma \in L^{\max\{q,p-1\}}$ $(0, T; \mathbb{R}^N)$ such that for all $y \in \mathbb{R}^N$, and a.e. $t \in [0, T]$,

$$F(t, y) \leq \frac{\alpha^2}{p}|y|^p + \gamma(t),$$

where

$$C = \frac{T}{(q+1)^{1/q}} + \frac{T}{(p+1)^{1/p}}.$$

Then, system (2.3) has at least one solution $u \in Y$ such that

$$v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = -J\left[u(t) - \frac{1}{T}\int_0^T u(s)ds\right] = \begin{pmatrix} -u_2(t) + \frac{1}{T}\int_0^T u_2(s)ds \\ u_1(t) - \frac{1}{T}\int_0^T u_1(s)ds \end{pmatrix}$$

minimizes the dual action

$$\varphi : X \to (-\infty, +\infty], \quad v \to \int_0^T \left[\frac{1}{2}(J\dot{v}(t), v(t)) + H^*(t, \dot{v}(t))\right] dt,$$

that is to say, system (1.1) has at least one solution $x \in W_T^{1,p}$.

Proof. The proof is same as in [7]. We only need to replace Lemma 3.3 in [7] with Lemma 2.2 and replace (2.9) with (2.5) in the process of proof.

Next, we consider the estimate of solutions for system (1.1).

Theorem 3.2 Assume that there exist $\alpha \in \left(0, \min\left\{C^{-1}, C^{-p/q}\right\}\right)$, $\beta \geq 0$, $\gamma \geq 0$ and $\delta > 0$ such that

$$\delta|y| - \beta \leq F(t, y) \leq \frac{\alpha^2}{p}|y|^p + \gamma \tag{3.1}$$

for all $t \in [0, T]$ and $y \in \mathbb{R}^N$. Then each solution x of system (1.1) satisfies

$$\int_0^T |x(t)|dt \leq \frac{(\gamma+\beta)T}{\delta} + \frac{T\alpha^q B^{1/p} D^{1/q}}{\delta(q+1)^{1/q}}, \tag{3.2}$$

$$\int_0^T |\dot{x}(t)|^p dt \leq \frac{pT(\gamma+\beta)}{1 - C\alpha}, \tag{3.3}$$

where

$$B = \frac{pT(\gamma+\beta)}{\alpha^q - C\alpha^{q+1}}, \quad D = \frac{qT(\gamma+\beta)}{\alpha^{1-q/p} - C\alpha}.$$

Proof. By (3.1), for $u = (u_1, u_2) \in \mathbb{R}^N \times \mathbb{R}^N$, we have

$$\begin{aligned} & \frac{\delta}{\alpha}|u_1| - \frac{\beta}{\alpha} + \frac{\alpha^{q-1}}{q}|u_2|^q \\ \leq\ & H(t, u) = \frac{1}{\alpha}F(t, u_1) + \frac{\alpha^{q-1}}{q}|u_2|^q \\ \leq\ & \frac{\alpha}{p}|u_1|^p + \frac{\gamma}{\alpha} + \frac{\alpha^{q-1}}{q}|u_2|^q. \end{aligned} \tag{3.4}$$

Then, we have

$$(u,v) - H(t,u) \geq (u,v) - \frac{\alpha}{p}|u_1|^p - \frac{\gamma}{\alpha} - \frac{\alpha^{q-1}}{q}|u_2|^q.$$

Since

$$\begin{aligned}
& (u,v) - \frac{\alpha}{p}|u_1|^p - \frac{\gamma}{\alpha} - \frac{\alpha^{q-1}}{q}|u_2|^q \\
= \;& (u_1,v_1) + (u_2,v_2) - \frac{\alpha}{p}|u_1|^p - \frac{\gamma}{\alpha} - \frac{\alpha^{q-1}}{q}|u_2|^q \\
\leq \;& |u_1||v_1| - \frac{\alpha}{p}|u_1|^p - \frac{\gamma}{\alpha} + |u_2||v_2| - \frac{\alpha^{q-1}}{q}|u_2|^q \\
\leq \;& \sup_{u_1\in\mathbb{R}^N}\left\{|u_1||v_1| - \frac{\alpha}{p}|u_1|^p - \frac{\gamma}{\alpha}\right\} + \sup_{u_2\in\mathbb{R}^N}\left\{|u_2||v_2| - \frac{\alpha^{q-1}}{q}|u_2|^q\right\} \\
= \;& \alpha^{-q/p}\frac{|v_1|^q}{q} - \frac{\gamma}{\alpha} + \frac{1}{p\alpha}|v_2|^p.
\end{aligned}$$

Hence,

$$H^*(t,v) \geq \alpha^{-q/p}\frac{|v_1|^q}{q} - \frac{\gamma}{\alpha} + \frac{1}{p\alpha}|v_2|^p. \tag{3.5}$$

By (2.1) and (3.4), we get

$$H^*(t,v) = (u,v) - H(t,u) \leq (u,v) + \frac{\beta}{\alpha}. \tag{3.6}$$

Then

$$\alpha^{-q/p}\frac{|v_1|^q}{q} - \frac{\gamma}{\alpha} + \frac{1}{p\alpha}|v_2|^p \leq (u,v) + \frac{\beta}{\alpha}. \tag{3.7}$$

Note that

$$v = \nabla H(t,u) = \begin{pmatrix} \nabla H_1(t,u_1) \\ \nabla H_2(t,u_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha}\nabla F(t,u_1) \\ \alpha^{q-1}|u_2|^{q-2}u_2 \end{pmatrix}.$$

Then by (2.1) and (3.7), we have

$$\alpha^{-q/p}\frac{\left|\frac{1}{\alpha}\nabla F(t,u_1)\right|^q}{q} - \frac{\gamma}{\alpha} + \frac{1}{p\alpha}\left|\alpha^{q-1}|u_2|^{q-2}u_2\right|^p \leq (u,\nabla H(t,u)) + \frac{\beta}{\alpha},$$

that is

$$\frac{\alpha^{-q/p-q}}{q}|\nabla F(t,u_1)|^q - \frac{\gamma}{\alpha} + \frac{\alpha^{q-1}}{p}|u_2|^q \leq (u,\nabla H(t,u)) + \frac{\beta}{\alpha}.$$

For each solution $u = (u_1,u_2)$ of system (2.3), it is easy to know that u_1 is the solution of (1.1). By (2.2) and (2.3), we know $\nabla F(t,u_1(t)) = -\alpha\dot{u}_2(t)$ and $\nabla H(t,u(t)) = -J\dot{u}(t)$. Hence

$$\frac{\alpha^{-q/p}}{q}|\dot{u}_2(t)|^q - \frac{\gamma}{\alpha} + \frac{\alpha^{q-1}}{p}|u_2(t)|^q \leq (u(t),-J\dot{u}(t)) + \frac{\beta}{\alpha}.$$

Integrating the above inequality over $[0,T]$ and using Lemma 2.2 and (2.2), we obtain

$$\begin{aligned}
\frac{\alpha^{-q/p}}{q}\|\dot{u}_2\|_{L^q}^q - \frac{\gamma T}{\alpha} + \frac{\alpha^{q-1}}{p}\|u_2\|_{L^q}^q &\leq -\int_0^T (u(t), J\dot{u}(t))dt + \frac{\beta T}{\alpha} \\
&\leq \frac{C}{q}\|\dot{u}_2\|_{L^q}^q + \frac{C}{p}\|\dot{u}_1\|_{L^p}^p + \frac{\beta T}{\alpha} \\
&= \frac{C}{q}\|\dot{u}_2\|_{L^q}^q + \frac{C}{p}\|\Phi_q(\alpha u_2)\|_{L^p}^p + \frac{\beta T}{\alpha} \\
&= \frac{C}{q}\|\dot{u}_2\|_{L^q}^q + \frac{C\alpha^q}{p}\|u_2\|_{L^q}^q + \frac{\beta T}{\alpha}.
\end{aligned}$$

So

$$\left(\frac{\alpha^{-q/p}}{q} - \frac{C}{q}\right)\|\dot{u}_2\|_{L^q}^q + \left(\frac{\alpha^{q-1}}{p} - \frac{C\alpha^q}{p}\right)\|u_2\|_{L^q}^q \leq \frac{T(\beta+\gamma)}{\alpha}.$$

Since $\alpha \in \left(0, \min\left\{C^{-1}, C^{-p/q}\right\}\right)$, we have

$$\|u_2\|_{L^q}^q \leq \frac{pT(\gamma+\beta)}{\alpha^q - C\alpha^{q+1}} = B, \quad \|\dot{u}_2\|_{L^q}^q \leq \frac{qT(\gamma+\beta)}{\alpha^{1-q/p} - C\alpha} = D.$$

Hence,

$$\|\dot{u}_1\|_{L^p}^p = \|\Phi_q(\alpha u_2)\|_{L^p}^p = \alpha^q\|u_2\|_{L^q}^q \leq B\alpha^q. \tag{3.8}$$

It follows that (3.3) holds. Since F is continuously differentiable and convex in x, then by Lemma 2.3, (3.1), (2.2), (2.5), Hölder's inequality and (3.8), we have

$$\begin{aligned}
\delta\int_0^T |u_1(t)|dt - \beta T &\leq \int_0^T F(t, u_1(t))dt \\
&\leq \int_0^T [F(t,0) + (\nabla F(t, u_1(t)), u_1(t))]dt \\
&\leq \gamma T - \int_0^T (\alpha\dot{u}_2(t), u_1(t))dt \\
&\leq \gamma T + \alpha\|\tilde{u}_1\|_\infty \int_0^T |\dot{u}_2(t)|dt \\
&\leq \gamma T + \alpha T^{1/p}\|\tilde{u}_1\|_\infty \left(\int_0^T |\dot{u}_2(t)|^q dt\right)^{1/q} \\
&\leq \gamma T + \alpha\frac{T}{(q+1)^{1/q}}\|\dot{u}_1\|_{L^p}\|\dot{u}_2\|_{L^q} \\
&\leq \gamma T + \frac{T\alpha^q B^{1/p} D^{1/q}}{(q+1)^{1/q}}.
\end{aligned}$$

So, we get

$$\int_0^T |u_1(t)|dt \leq \frac{(\gamma+\beta)T}{\delta} + \frac{T\alpha^q B^{1/p} D^{1/q}}{\delta(q+1)^{1/q}}.$$

It follows that (3.2) holds. The proof is complete.

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