Configuration of groups and paradoxical decompositions

A. Rejali A. Yousofzadeh

Abstract

Let G be a non-amenable discrete group. We construct the paradoxical decomposition of G, by using the configuration equations under some conditions. This gives a positive answer to a question proposed by G. A. Willis. We find an upper bound for the Tarski number of a given group, by using configurations.

1 introduction and preliminaries

Let G be a discrete group. The equivalence between non-amenability and existence of paradoxical decompositions is a result due to A. Tarski and it is called the Tarski alternative theorem (see [2], [3], [5], [6] ,[9]) and [10]).

In [1] it is proved that if G is finitely generated and admits a paradoxical decomposition, then the system of configuration equations introduced by Rosenblatt and Willis admits no non-negative solutions. In the present paper, we introduce a paradoxical condition for Rosenblatt-Willis configuration equation systems and prove the following slightly weaker converse to the Abdollali-Rejali-Willis above mentioned result: Let G be a finitely generated group and assume that the Rosenblatt- Willis configuration equation system (associated with an ordered finite generating subset and a finite partition of G) admits no non-negative solutions and satisfies the paradoxical condition. Then G admits a paradoxical decomposition (and therefore is non-amenable).

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The proof is constructive and we deduce an estimate from above of the Tarski number of such a non-amenable group in terms of the cardinality of the configuration set associated with the given finite generating subset and the given finite partition of G.

The first section of this paper is designed to introduce the notation and the concepts involved in the statements of the main theorems. The process of constructing the paradoxical decomposition is given in section 2. At first we need some preliminaries.

Definition 1.1. [9] A discrete group is amenable if it admits a $G-$invariant finitely additive probability measure, that is a map $\mu : \mathcal{P}(G) \to [0,1]$ defined on the set $\mathcal{P}(G)$ of all subsets of G such that

$$\mu(A \coprod B) = \mu(A) + \mu(B) \qquad for\ \ disjoint\ subsets\ A\ and\ B\ of\ G,$$

$$\mu(gA) = \mu(A) \qquad for\ g \in G\ and\ A \subseteq G,$$

$$\mu(G) = 1,$$

where $\coprod$ is disjoint union and $gA = \{ga : \ a \in A\}$ for $g \in G$ and $A \subseteq G$.

Proposition 1.2. Let G be a group. Then the following statements are equivalent

1) There exist a partition $\{A_1, \dots, A_n, B_1, \dots B_m\}$ of G and $g_1, \dots, g_n$ and $h_1, \dots,$ h_m in G such that $\{g_jA_j\}_{j=1}^n$ and $\{h_jB_j\}_{j=1}^m$ form partitions of G.

2) There exist pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m$ of G and elements $g_1, \dots, g_n$ and $h_1, \dots, h_m$ in G such that $\{g_jA_j\}_{j=1}^n$ and $\{h_jB_j\}_{j=1}^m$ form partitions of G.

3) There exist pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m$ of G and elements $g_1, \dots, g_n$ and $h_1, \dots, h_m$ in G such that

$$G = \bigcup_{j=1}^{n} g_jA_j = \bigcup_{j=1}^{m} h_jB_j$$

(not necessarily pairwise disjoint).

Proof. Clearly (1) implies (2) and (2) implies (3). Suppose (2) holds. Let $A = \bigcup_{j=1}^n A_j$ and $B = \bigcup_{j=1}^m B_j$. Note that $A \cap B = \emptyset$. Define $f : G \to B$ by $f(x) = b_x$, where $x = h_jb_x$ for some unique $b_x \in B_j$.

Put $D := \bigcup_{k=0}^\infty f^k(A)$ and $T = (G \setminus A) \setminus f(D)$. Note that $A \cap f(D) = \emptyset$ and $A \cup f(D) = D$.

Let $D_j := B_j \cap h_j^{-1}D$. Then

$$T \cup (\bigcup_{j=1}^{m} D_j) = G \setminus A$$

and

$$G = \bigcup_{i=1}^{n} g_iA_i = eT \cup (\bigcup_{j=1}^{m} h_jD_j),$$

where e denotes the neutral element of the group G. Hence (1) holds.

Suppose (3) holds. Define inductively $A'_1 = A_1$, $A'_k = A_k \setminus g_k^{-1}(\bigcup_{i=1}^{k-1} g_i A'_i)$, $B'_1 = B_1$ and $B'_k = B_k \setminus h_k^{-1}(\bigcup_{i=1}^{k-1} h_i B'_i)$. It is easy to show that $\{g_i A'_i,\ i = 1, \ldots, n\}$ and $\{h_j B'_j,\ j = 1, \ldots, m\}$ form partitions of G, so (2) holds. ■

We say that G admits a complete paradoxical decomposition, if condition (1) of Proposition (1.2) holds.

The number $\tau = n + m$ for n and m in the previous definition is called the Tarski number of that paradoxical decomposition; the minimum of all such numbers over all the possible paradoxical decompositions of G, is called the Tarski number of G and denoted by $\tau(G)$. In the case that there is no paradoxical decomposition, we set $\tau(G) = \infty$ (see [4]).

Let G be a finitely generated group. Let $\mathfrak{g} = (g_1, \ldots, g_n)$ be an ordered generating set for G and $\mathcal{E} = \{E_1, \ldots, E_m\}$ be a finite partition of G. A configuration corresponding to this generating sequence and partition is an $(n+1)-$tuple $C = (k_0, k_1, \ldots, k_n)$, where $k_i \in \{1, \ldots, m\}$ for each i, such that there is $x \in G$ with $x \in E_{k_0}$ and $g_i x \in E_{k_i}$ for each $i \in \{1, \ldots, n\}$. The set of configurations corresponding to the generating sequence $\mathfrak{g}$ and partition $\mathcal{E}$ of G, will be denoted by $Con(\mathfrak{g}, \mathcal{E})$, which is clearly a finite set.

A configuration is thus an $(n+1)-$tuple of positive integers. The configuration set $Con(\mathfrak{g}, \mathcal{E})$ records how the generators in $\mathfrak{g}$ multiply between sets in the partition $\mathcal{E}$.

The configuration $C = (k_1, \ldots, k_n)$ may be described equivalently as a labeled tree. The tree has one vertex of degree n, labeled by k_0. Emanating from this vertex are edges labeled $1, \ldots, n$ and the other vertex of the $i - th$ edge is labeled k_i. When the generators are distinct, this tree is a subgraph of the Cayley graph of the finitely generated group $G = \langle g_1, g_2, \ldots, g_n \rangle$, the edges labels indicate which generator gives rise to the edge and the vertex label show which set of the partition $\mathcal{E}$ the vertex belongs to. From this perspective the configuration set $Con(\mathfrak{g}, \mathcal{E})$ is a set of rooted trees having height 1. This finite set carries information about G.

J. M. Rosenblatt and G. A. Willis [8] considered the amenability of groups based on finite generating sets and finite partitions of G. For completeness we summarize the main ideas.

The statement of the result involves the notion of the system of configuration equations corresponding to the configuration set $Con(\mathfrak{g}, \mathcal{E})$. There are $|Con(\mathfrak{g}, \mathcal{E})|$ variables in the system of equations. They are denoted by f_C, where $C \in Con(\mathfrak{g}, \mathcal{E})$. These are $|\mathfrak{g}||\mathcal{E}| = mn$ equations in the system.

Let C be a configuration in $Con(\mathfrak{g}, \mathcal{E})$. Call $x_0 \in G$ a base point of C if there is a sequence of elements $x_1, \ldots, x_n$ such that $x_i = g_i x_0$, for each $i \in \{1, 2, \ldots, n\}$ and $x_i \in E_{k_i}$, for each $i \in \{0, 1, \ldots, n\}$. In this case $x_1, \ldots, x_n$ are called branch points of C. Define

$$x_0(C) = \{x \in G :\ \ x \textit{ is a base point of } C\}.$$

Thus $\{x_0(C) :\ C \in Con(\mathfrak{g}, \mathcal{E})\}$ is a refinement of $\mathcal{E}$. Moreover "C is a configuration in $Con(\mathfrak{g}, \mathcal{E})$" means that $x_0(C)$ is not empty.

Let $x_j(C) = g_j x_0(C)$, for $1 \leq j \leq n$ and $C \in Con(\mathfrak{g}, \mathcal{E})$. The configuration equations corresponding to the configuration set $Con(\mathfrak{g}, \mathcal{E})$ are the equations:

$$\sum_{C:\, x_j(C)\subseteq E_i} f_C = \sum_{C:\, x_0(C)\subseteq E_i} f_C,$$

where $1 \leq i \leq m$ and $1 \leq j \leq n$. This system of equations will be denoted by $Eq(\mathfrak{g}, \mathcal{E})$. A nonnegative solution to $Eq(\mathfrak{g}, \mathcal{E})$ will be called normalized if $\sum_{C\in Con(\mathfrak{g},\mathcal{E})} f_C = 1$.

Proposition 1.3. *[8]* There is a normalized solution of every possible instance of the configuration equations of a group G if and only if G is amenable.

The link between amenability and normalized solutions of the configuration equations was seen via a refinement of the partition $\mathcal{E}$, in [1]. Let M be an invariant mean for G and set $f_C = M(\chi_{x_0(C)})$, for $C \in Con(\mathfrak{g}, \mathcal{E})$. Then $\{f_C : C \in Con(\mathfrak{g}, \mathcal{E})\}$ is a normalized solution of the configuration equation $Eq(\mathfrak{g}, \mathcal{E})$.

Note that if $\{f_C\}$ is a nonnegative nonzero solution of $Eq(\mathfrak{g}, \mathcal{E})$ and $F = \sum_{C\in Con(\mathfrak{g},\mathcal{E})} f_C$, then $\{\frac{f_C}{F}\}$ is a normalized solution of $Eq(\mathfrak{g}, \mathcal{E})$.

Using the above notation, we have

$$\coprod_{C\in Con(\mathfrak{g},\mathcal{E})} x_0(C) = G. \tag{1.1}$$

In the following examples, the corresponding configuration equations to a configuration set is obtained. For convenience we write f_i for f_{C_i}.

Example 1.4. Let $G = (\mathbb{Z}, +)$, $\mathfrak{g} = (1)$ and E_1 be the set of all negative integers and E_2 be the set of all non-negative integers. Then

$$Con(\mathfrak{g}, \mathcal{E}) = \{(1,1), (1,2), (2,2)\}.$$

In this case $m = 2$, $n = 1$ and there are 3 configurations whose corresponding equations are:

$$f_1 + f_2 = f_1 \text{ and } f_3 = f_2 + f_3.$$

Clearly $(f_1, f_2, f_3) = (1/2, 0, 1/2)$ is a normalized solution of $Eq(\mathfrak{g}, \mathcal{E})$.

Example 1.5. Let $G = \mathbb{F}_2$ be the free group on generators a and b, so that each element may be uniquely represented as a reduced word $w = w_1 w_2 \cdots w_k$ where $w_i = a, b, a^{-1}$ or b^{-1}. Let $\mathfrak{g} = (a, b)$ and $E_1 = \{w \in F_2 | \; w_1 = a\}$, $E_2 = \{w \in F_2 | \; w_1 = b\}$ and $E_3 = \{w \in F_2 | \; w = e \text{ or } w_1 = a^{-1} \text{ or } b^{-1}\}$, so that $\mathcal{E} = \{E_1, E_2, E_3\}$ is a partition of F_2. Then $Con(\mathfrak{g}, \mathcal{E}) = \{(1,1,2), (2,1,2), (3,1,2), (3,3,2), (3,2,2), (3,1,3), (3,1,1)\}$. In this case $m = 3$, $n = 2$ and there are 7 configurations whose corresponding equations are:

$$\begin{aligned}
f_1 &= f_1 + f_2 + f_3 + f_6 + f_7,\\
f_1 &= f_7,\\
f_2 &= f_5,\\
f_2 &= f_1 + f_2 + f_3 + f_4 + f_5,\\
f_3 + f_4 + f_5 + f_6 + f_7 &= f_4,\\
f_3 + f_4 + f_5 + f_6 + f_7 &= f_6.
\end{aligned}$$

Clearly $Eq(\mathfrak{g}, \mathcal{E})$ has no nonnegative nonzero solution.

2 Paradoxical decomposition and free groups

It is interesting to find the relation between the paradoxical decomposition and a system of configuration equations with no normalized solution. Considering Proposition(1.2), the following is immediate.

Proposition 2.1. Let $Con(\mathfrak{g}, \mathcal{E}) = \{C_1, \ldots C_l\}$ and suppose there exist positive integers n, m such that $n + m \leq l$ and elements $h_1, \ldots, h_n, k_1, \ldots, k_m \in G$ such that

$$G = \bigcup_{i=1}^{n} h_i x_0(C_i) = \bigcup_{j=1}^{m} k_j x_0(C_{n+j}).$$

Then G admits a paradoxical decomposition and $\tau(G) \leq |Con(\mathfrak{g}, \mathcal{E})|$.

In the following example we use configurations to construct a paradoxical decomposition. This generalizes the construction of paradoxical decomposition of free group $\mathbb{F}_2$ by using configuration set in [8], for $\mathbb{F}_n$.

Let n be a positive integer. We say that the group G admits a $G = nG$ paradoxical decomposition if there exist a partition

$$\{E_{11}, E_{12}, \ldots, E_{1m_1}; \ldots; E_{n1}, E_{n2}, \ldots, E_{nm_n}\}$$

and a generating sequence

$$\{g_{11}, g_{12}, \ldots, g_{1m_1}; \ldots; g_{n1}, g_{n2}, \ldots, g_{nm_n}\}$$

of G such that for each i, $1 \leq i \leq n$,

$$G = \bigcup_{k=1}^{m_i} g_{ik} E_{ik}.$$

Example 2.2. Let G be a finitely generated free group. Then there exist a partition $\mathcal{E}$ and a generating set $\mathfrak{g}$ of G such that the configuration equation $Eq(\mathfrak{g}, \mathcal{E})$ has no nonnegative nonzero solution. Furthermore there is a $G = nG$ paradoxical decomposition of G, for some positive integer n.

Proof. Let $\{g_1, \ldots, g_n\}$ be a free base of $\mathbb{F}_n$. Each non identity element of G is a reduced word $w = w_1^{\varepsilon_1} w_2^{\varepsilon_2} \cdots w_k^{\varepsilon_k}$, where $w_i \in \{g_1, \ldots g_n\}$ and $\varepsilon_i \in \{1, -1\}$. Put

$E_i = \{w : \ w_1 = g_i\} \qquad for \ 1 \leq i \leq n$ and

$E_{n+1} = \{w : \ w_1 \in \{g_1^{-1}, ..., g_n^{-1}\}\} \cup \{e\}$.

Then the configuration equations corresponding to the mentioned (free) generators and partition have no nonnegative nonzero solution and in addition they provide a paradoxical decomposition for G.

By the definition of configurations, putting $\mathfrak{g} = (g_1, \ldots, g_n)$ and $\mathcal{E} = \{E_1, \ldots, E_{n+1}\}$, we have $Con(\mathfrak{g}, \mathcal{E}) = \{C_1, \ldots C_{n^2+n+1}\}$, where

$$\begin{cases} C_1 = (\boxed{1}, 1, 2, ..., n) \\ C_2 = (\boxed{2}, 1, 2, ..., n) \\ \vdots \\ C_n = (\boxed{n}, 1, 2, ..., n) \end{cases}$$

$$C_{n+1} = (n+1, 1, 2, ..., n)$$

$$\begin{cases} C_{n+2} = (n+1, \boxed{\text{n+1}}, 2, 3, ..., n) \\ C_{n+3} = (n+1, \boxed{2}, 2, 3, ..., n) \\ \vdots \\ C_{2n+1} = (n+1, \boxed{\text{n}}, 2, 3, ..., n) \end{cases}$$

$$\begin{cases} C_{2n+2} = (n+1, 1, \boxed{\text{n+1}}, 3, 4, ..., n) \\ C_{2n+3} = (n+1, 1, \boxed{1}, 3, 4, ..., n) \\ C_{2n+4} = (n+1, 1, \boxed{3}, 3, 4, ..., n) \\ \vdots \\ C_{3n+1} = (n+1, 1, \boxed{\text{n}}, 3, 4, ..., n) \end{cases}$$

$$\begin{cases} C_{3n+2} = (n+1, 1, 2, \boxed{\text{n+1}}, 4, 5, ..., n) \\ C_{3n+3} = (n+1, 1, 2, \boxed{1}, 4, 5, ..., n) \\ C_{4n+4} = (n+1, 1, 2, \boxed{2}, 4, 5, ..., n) \\ C_{4n+5} = (n+1, 1, 2, \boxed{4}, 4, 5, ..., n) \\ \vdots \\ C_{4n+1} = (n+1, 1, 2, \boxed{\text{n}}, 4, 5, ..., n) \end{cases}$$

$$\vdots$$

$$\begin{cases} C_{n^2+2} = (n+1, 1, 2, ..., n-1, \boxed{\text{n+1}}) \\ C_{n^2+3} = (n+1, 1, 2, ..., n-1, \boxed{1}) \\ C_{n^2+4} = (n+1, 1, 2, ..., n-1, \boxed{2}) \\ \vdots \\ C_{n^2+n+1} = (n+1, 1, 2, ..., n-1, \boxed{\text{n-1}}). \end{cases}$$

Above, the set of boxed numbers in each n first column together with one of the numbers boxed in the $(n+1)st$ column is $\{1, \ldots, n+1\}$. For examples, for $1 \leq k \leq n^2+n+1$, write $C_k = (C_{k,0}, C_{k,1}, \ldots, C_{k,n})$. Hence we have

$$\{C_{1,0}, C_{2,0}, \ldots, C_{n,0}, C_{n^2+2,n+1}\} = \{1, \ldots, n+1\}$$

and

$$\{C_{n+2,1}, C_{n+3,1}, \ldots, C_{2n+1,1}, C_{n^2+3,n+1}\} = \{1, \ldots, n+1\}.$$

This means that we can make G as union of $\{E_1, \ldots, E_{n+1}\}$, n times and independently. Therefore we have

$$\begin{aligned} G &= x_0(C_1) \cup ... \cup x_0(C_n) \cup g_n x_0(C_{n^2+2}) \\ G &= g_1[x_0(C_{n+2}) \cup ... \cup x_0(C_{2n+1})] \cup g_n x_0(C_{n^2+3}) \\ G &= g_k[x_0(C_{kn+2}) \cup ... \cup x_0(C_{kn+n+1})] \cup g_n x_0(C_{n^2+2+k}), \end{aligned} \tag{2.1}$$

for $2 \leq k \leq n-1$.

It is a $G = nG$ paradoxical decomposition (see the statements before Example 2.2). In fact two first lines of (2.1) form a paradoxical decomposition. But this decomposition is not complete. We can use the process mentioned in Proposition 1.2 to complete this decomposition. But we prefer to do this in a way which depends on the structure of this special example as follow.

Note that $x_0(C_{n+2}) \cup x_0(C_{n+1}) \subseteq E_{n+1}$ and $x_1(C_{n+2}) = E_{n+1}$. Now let $A = \bigcup_{m=1}^{\infty} g_1^{-m} x_0(C_{n+1})$ and $B = x_0(C_{n+2}) \backslash A$. By an easy induction we have $A \subseteq x_0(C_{n+2})$. This choosing of A shows that $g_1 A = A \cup x_0(C_{n+1})$ and therefore the paradoxical decomposition of G can be constructed as below:

$G = x_0(C_1) \cup ... \cup x_0(C_n) \cup g_n x_0(C_{n^2+2})$,
$G = g_1 B \cup A \cup x_0(C_{n+1}) \cup g_1[x_0(C_{n+3} \cup ... \cup x_0(C_{2n+1}] \cup g_n x_0(C_{n^2+3})$,
$G = g_k[x_0(C_{kn+2}) \cup ... \cup x_0(C_{kn+n+1})] \cup g_n x_0(C_{n^2+2+k})$,
for $2 \leq k \leq n-1$.

Note that this decomposition is not complete yet. We can change it to a complete paradoxical decomposition. Clearly

$$\begin{cases} G = x_0(C_1) \cup ... \cup x_0(C_n) \cup g_n x_0(C_{n^2+2}) \\ G = g_1[x_0(C_{n+2}) \cup ... \cup x_0(C_{2n+1})] \cup g_n x_0(C_{n^2+3}) \end{cases} \tag{2.2}$$

forms an uncomplete paradoxical decomposition. We now complete this decomposition. The set

$$M := x_0(C_{n+1}) \bigcup \Big(\bigcup_{i=n+3}^{4n+2} x_0(C_i) \Big) \bigcup \Big(\bigcup_{i=n^2+4}^{n^2+n+1} x_0(C_i) \Big)$$

is not used in (2.2).
Put

$$S = \bigcup_{m=1}^{\infty} g_1^{-m} M \quad \text{and} \quad T = x_0(C_{n+2}) \setminus S.$$

Then $g_1 S = M \cup S$ and therefore $x_1(C_{n+2}) = g_1 x_0(C_{n+2}) = g_1 T \cup M \cup S$ and finally we have by (2.2)

$$G = g_1 T \cup M \cup S \cup g_1 x_0(C_{n+3}) \cup ... \cup g_1 x_0(C_{2n+1}) \cup g_n x_0(C_{n^2+3}).$$

This together with

$$G = x_0(C_1) \cup ... \cup x_0(C_n) \cup g_n x_0(C_{n^2+2})$$

forms a complete paradoxical decomposition for G.

We now show that the configuration equation $Eq(\mathfrak{g}, \mathcal{E})$ has no nonzero nonnegative solution.

Note that by the definition of configuration equations we have:

$$\sum_{C:\, x_1(C) \subseteq E_1} f_C = \sum_{C:\, x_0(C) \subseteq E_1} f_C$$

and

$$\sum_{C:\, x_2(C) \subseteq E_2} f_C = \sum_{C:\, x_0(C) \subseteq E_2} f_C.$$

In other words,

$$f_{C_1} = f_{C_1} + f_{C_2} + \cdots + f_{C_{n+1}} + f_{C_{2n+2}} + \cdots + f_{C_{n^2+n+1}},$$
$$f_{C_2} = f_{C_1} + f_{C_2} + \cdots + f_{C_{2n+1}} + f_{C_{3n+2}} + \cdots + f_{C_{n^2+n+1}},$$

which implies $f_{C_1} = \cdots = f_{C_{n^2+n+1}} = 0$. ■

3 Paradoxical decomposition and configurations

In [1], the authors proved that if G admits a paradoxical decomposition, then the corresponding system of configuration equations has no normalized solution. In the following, for completeness we give a proof for it.

Let $\{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m; g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_m\}$ be a complete paradoxical decomposition of a finitely generated group G. Put $\mathfrak{g} = (g_1^{-1}, \ldots, g_n^{-1}, h_1^{-1}, \ldots, h_m^{-1}, t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are elements of G such that $G = \langle \mathfrak{g} \rangle$. Put also $\mathcal{E} = \{A_1, \ldots, A_n, B_1, \ldots B_m\}$. Since by assumption $\{g_1 A_1, \ldots, g_n A_n\}$ is a partition for G, for each $C \in Con(\mathfrak{g}, \mathcal{E})$ there is exactly one $j \in \{1, \ldots, n\}$ such that $x_j(C) \subseteq A_j$. Thus

$$\sum_{C \in Con(\mathfrak{g}, \mathcal{E})} f_C = \sum_{j=1}^{n} \sum_{x_j(C) \subseteq A_j} f_C.$$

Therefore we have by Definition (1.5),

$$\sum_{C \in Con(\mathfrak{g}, \mathcal{E})} f_C = \sum_{j=1}^{n} \sum_{x_0(C) \subseteq A_j} f_C. \tag{3.1}$$

Similarly

$$\sum_{C \in Con(\mathfrak{g}, \mathcal{E})} f_C = \sum_{k=1}^{m} \sum_{x_0(C) \subseteq B_k} f_C. \tag{3.2}$$

But we know by (1.1), that $\coprod_{C \in Con(\mathfrak{g}, \mathcal{E})} x_0(C) = G$. Thus by adding equations (3.1) and (3.2), we have

$$2 \sum_{C \in Con(\mathfrak{g}, \mathcal{E})} f_C \leq \sum_{C \in Con(\mathfrak{g}, \mathcal{E})} f_C.$$

This shows that $Eq(\mathfrak{g}, \mathcal{E})$ has no nonzero nonnegative solution.

In the following we give a positive answer to the converse of this fact, under some suitable conditions (see Definition 2.4).

The existence of a paradoxical decomposition, non amenability and the existence of no normalized solution for some configuration equation of a group G are equivalent [8]. In the following, we want to present a constructive way for finding a paradoxical decomposition by using a system of configuration equations with no normalized solution. We present this under some conditions. The general case is still open.

Before proving the main Theorem, we introduce some notation which is used throughout this paper.

Notation. Classify the members of $Con(\mathfrak{g},\mathcal{E})$ as below:

$$A_i^j = \{C \in Con(\mathfrak{g},\mathcal{E}) : \ x_j(C) \subseteq E_i\} \qquad (1 \leq i \leq m, 1 \leq j \leq n). \tag{3.3}$$

It is clear that for each $j \in \{0,...,n\}$, $\{A_1^j,...,A_m^j\}$ is a partition of $Con(\mathfrak{g},\mathcal{E})$.

Lemma 3.1. For each $i \in \{1,\ldots,m\}$ and $j,j' \in \{0,\ldots,n\}$,

$$g_{j'}^{-1}g_j\left(\bigcup_{C\in A_i^j} x_0(C)\right) = \bigcup_{C\in A_i^{j'}} x_0(C).$$

Proof. By definition

$$g_{j'}\left(\bigcup_{C\in A_i^{j'}} x_0(C)\right) = \bigcup_{C\in A_i^{j'}} x_{j'}(C) = E_i = \bigcup_{C\in A_i^{j}} x_j(C) = g_j\left(\bigcup_{C\in A_i^j} x_0(C)\right).$$

Therefore

$$g_{j'}^{-1}g_j\left(\bigcup_{C\in A_i^j} x_0(C)\right) = \bigcup_{C\in A_i^{j'}} x_0(C). \qquad \blacksquare$$

By [5, Proposition 2.4], G is amenable if and only if there is a normalized solution for every possible instance of the configuration equations. Let the corresponding system whose solution is normalized, have the form $AX = 0$, where A is an $(nm) \times |Con(\mathfrak{g},\mathcal{E})|$ matrix whose entries are 0, 1 or -1, X is the vector $[f_C]$, where C runs over $Con(\mathfrak{g},\mathcal{E})$. Suppose $Con(\mathfrak{g},\mathcal{E}) = \{C_1,\ldots,C_l\}$ and $A = [a_{st}]$. If $1 \leq i \leq m$, then $a_{st} = 1$ [resp. $a_{st} = -1$] if and only if $x_s(C_t) \subseteq E_i$ and $x_0(C_t) \nsubseteq E_i$ [resp. $x_s(C_t) \subseteq E_i$ and $x_0(C_t) \nsubseteq E_i$] for $(i-1)n+1 \leq s \leq in$; otherwise $a_{st} = 0$. It is easy to see that a matrix equation $AX = 0$ has a nonzero solution if and only if $rank(A)$ is less than the number of columns of A. Therefore that matrix has no nonzero solutions if and only if $rank(A) = |Con(\mathfrak{g},\mathcal{E})|$.

We now state the key lemma of this paper.

Lemma 3.2. If the system $Eq(\mathfrak{g},\mathcal{E})$ has no nonnegative nonzero solutions and $AX = 0$ is the corresponding system, then by row operations, A can be changed into an equivalent matrix B with nonnegative entries and no zero column.

Proof. Let A be the coefficient matrix of the configuration equations system and let $rank(A) = r$. Then A is equivalent to the following matrix (by a new indexing of the members of $Con(\mathfrak{g},\mathcal{E})$, if necessary)

$$A' = \begin{pmatrix} I & D \\ 0 & 0 \end{pmatrix},$$

where I is the $r \times r$ identity matrix. Note that if the system has no nonzero solutions, then D may be omitted. But if it has some nonzero solution which are not nonnegative, then D is an $r \times (|Con(\mathfrak{g}, \mathcal{E})| - r)$ matrix with no zero columns. We claim that in all of D's columns, if there are negative entries, then there are also positive entries. Indeed if all of the entries of the k^{th} column were negative or zero for example $(\alpha_{ik})_{i=1}^{mn}$, where each $\alpha_{ik} \leq 0$, then $f_{C_k} = 1, f_{C_1} = -\alpha_{1k}$, $f_{C_2} = -\alpha_{2k}, ..., f_{C_r} = -\alpha_{rk}$ and $f_{C_j} = 0$, for $r < j \neq k$ would be a nonnegative nonzero solution of the system, which is a contradiction. Therefore by suitable adding of rows, the obtained matrix B say, involves just nonnegative entries.

Suppose that j-th column of B is zero. Then ($f_{C_i} = 0$, for $i \neq j$ and $f_{C_j} = 1$) is a nonzero nonnegative solution for the system, which is a contradiction. ∎

Remark 3.3. Let $|Con(\mathfrak{g}, \mathcal{E})| = l$ and $(a_1, \ldots, a_l)$ be a row of matrix B such that $a_{i_1}, \ldots, a_{i_k}$ are positive for some $1 \leq i_1 < \cdots < i_k \leq l$, otherwise zero. Then by the above argument this row is made by adding and subtracting of some rows of the matrix A. In other words, there are $t_1, \ldots, t_s \in \{1, \ldots, m\}$ (not necessarily distinct) such that

$$a_{i_1} f_{C_{i_1}} + \cdots + a_{i_k} f_{C_{i_k}} = \sum_{r=1}^{s} \Big(\sum_{C \in A_{t_r}^{j_{t_r}}} f_C - \sum_{C \in A_{t_r}^{k_{t_r}}} f_C \Big),$$

for some $j_{t_r}, k_{t_r} \in \{0, \ldots, n\}$. (The case $k = j$ is possible.) In the next theorem we give a way for finding the paradoxical decomposition of a group using configuration equation with no nonnegative nonzero solution, under the following condition.

Definition 3.4. Let A and B be as above and L_i^j be the coefficient vector of the equation

$$\sum_{x_j(C) \subseteq E_i} f_C - \sum_{x_0(C) \subseteq E_i} f_C = 0.$$

We say that $Eq(\mathfrak{g}, \mathcal{E})$ satisfies the *paradoxical condition* if each row of B is of the form $\sum_{i=1}^{m} R_i$, where $R_i \in \{L_i^j, -L_i^j, L_i^j - L_i^k : \ 1 \leq j, k \leq n\}$ and

$$A = \begin{pmatrix} L_1^1 \\ \vdots \\ L_1^n \\ \vdots \\ L_m^1 \\ \vdots \\ L_m^n \end{pmatrix}.$$

In other words, each row of B is of the form $\sum_{i=1}^{m} (L_i^{j_i} - L_i^{k_i})$, for some $j_i, k_i \in \{0, \ldots, n\}$.

Example 3.5. Let G be a 2-generated group, $\mathfrak{g} = (g_1, g_2)$ and $\mathcal{E} = \{E_1, E_2, E_3\}$. Let also $Con(\mathfrak{g}, \mathcal{E})$ consists of $C_1 = (1,1,2), C_2 = (1,2,3), C_3 = (2,3,1), C_4 = (2,3,2), C_5 = (3,3,1)$. Then $Eq(\mathfrak{g}, \mathcal{E})$ has a nonzero solution $f_{C_1} = f_{C_3} = -f_{C_4} = 1, f_{C_2} = f_{C_5} = 0$. But it does not have any nonnegative nonzero solution and satisfies the paradoxical condition. Indeed,

$$\begin{array}{lll} A_1^0 = \{C_1, C_2\} & A_1^1 = \{C_1\} & A_1^2 = \{C_3, C_5\} \\ A_2^0 = \{C_3, C_4\} & A_2^1 = \{C_2\} & A_2^2 = \{C_1, C_4\} \\ A_3^0 = \{C_5\} & A_3^1 = \{C_3, C_4, C_5\} & A_3^2 = \{C_2\} \end{array}$$

It is easily checked that $Eq(\mathfrak{g}, \mathcal{E})$ satisfies the paradoxical condition. In fact,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -L_1^1 \\ L_3^1 \\ L_2^2 + L_3^1 \\ -L_1^1 - L_3^2 \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 3.6. *(Main Theorem)* Let G be a finitely generated group. Let $\mathfrak{g}$ be an ordered finite generating set for G and $\mathcal{E}$ a finite partition of G. Suppose that the associated system of configuration equations $Eq(\mathfrak{g}, \mathcal{E})$ admits no nonzero non-negative solution and satisfies the paradoxical condition. Then G admits a paradoxical decomposition.

Proof. Let $|Con(\mathfrak{g}, \mathcal{E})| = l$ and $B = (R_s)$. Suppose $R_s = (b_{s1}, \ldots, b_{sl})$ is a nonzero row of B for some $1 \leq s \leq r$, where $r = rank(B)$. On the other hand, by the paradoxical condition, $R_s = \sum_{i=1}^m (L_i^{j'_i} - L_i^{j_i})$, for some $j_i, j'_i \in \{0, \ldots, n\}$. Therefore by Remark(3.3)

$$\sum_{u=1}^{l} b_{su} f_{C_u} = \sum_{i=1}^{m} \sum_{C \in A_i^{j'_i}} f_C - \sum_{i=1}^{m} \sum_{C \in A_i^{j_i}} f_C$$

and hence

$$\sum_{i=1}^{m} \sum_{C \in A_i^{j'_i}} f_C = \sum_{i=1}^{m} \sum_{C \in A_i^{j_i}} f_C + \sum_{u=1}^{l} b_{su} f_{C_u}. \tag{3.4}$$

Each b_{su} is nonnegative and so by (3.4), each $C \in A_i^{j_i}$, appears in left hand side of (3.4), p times, say. So $C \in A_{k_{C,i_1}}^{j'_{k_{C,i_1}}} \bigcap \cdots \bigcap A_{k_{C,i_p}}^{j'_{k_{C,i_p}}}$. Similarly for each $b_{su} \neq 0$, and each $u \in \{1, \ldots, l\}$ each C_u appears in left hand side of (3.4), t times, say. So there are distinct $k_{u_1}, \ldots, k_{u_t}$ such that $C \in A_{k_{u_1}}^{j'_{k_{u_1}}} \bigcap \cdots \bigcap A_{k_{u_t}}^{j'_{k_{u_t}}}$. Clearly $k_{C,i_1}, \ldots, k_{C,i_p}, k_{u_1} \ldots, k_{u_t}$ are pairwise distinct.

Denote

$$\begin{cases} I_s = \{(C,i): \quad C \in A_i^{j_i}, \quad 1 \le i \le m\}, & \\ A_{(C,i)} = g_{j'_{k_{C,i}}} x_0(C) & for\ (C,i) \in I_s, \\ h_{(C,i)} = g_{j_i} g^{-1}_{j'_{k_{C,i}}} & for\ (C,i) \in I_s, \\ B_u = g_{j'_{k_u}} x_0(C_u) & if \quad b_{su} \neq 0, \\ g_u = g^{-1}_{j'_{k_u}} & if \quad b_{su} \neq 0. \end{cases} \tag{3.5}$$

Then we have

$$\bigcup_{\alpha \in I_s} h_\alpha A_\alpha = \bigcup_{i=1}^{m} \bigcup_{C \in A_i^{j_i}} g_{j_i} g^{-1}_{j'_{k_{C,i}}} g_{j'_{k_{C,i}}} x_0(C) = \bigcup_{i=1}^{m} \bigcup_{C \in A_i^{j_i}} g_{j_i} x_0(C) = G \tag{3.6}$$

and the last two equations of (3.5) imply that:

$$x_0(C_u) = g_u B_u \qquad \text{if } b_{su} \neq 0.$$

Furthermore, by Lemma (3.1)

$$\coprod_{C \in A_i^{j'_i}} x_0(C) = g^{-1}_{j'_i} \coprod_{C \in A_i^0} x_0(C), \qquad (1 \le i \le m).$$

So for each i, $\{g_{j'_i} x_0(C): \ C \in A_i^{j'_i}\}$ is a partition of $\coprod_{C \in A_i^0} x_0(C)$. But it is clear that $\{A_1^0, \ldots, A_m^0\}$ is a partition of $Con(\mathfrak{g}, \mathcal{E})$. Therefore $\mathcal{A} = \{g_{j'_i} x_0(C) : \ C \in A_i^{j'_i}, \ 1 \le i \le m\}$ forms a partition for G.

It is easily checked that for each i and j, $|A_i^j| \le l - m + 1$. Therefore $|I_s| \le m(l - m + 1)$.

Let $rank(B) = r$. We prove by induction, for $1 \le s \le r$, that there are a finite index set J_s and disjoint subsets F_α and B_u^i with $b_{iu} \neq 0$ of G and $g_\alpha, h_u^i \in G$ for $\alpha \in J_s$ and $1 \le i \le s$ such that

$$G = \bigcup_{\alpha \in J_s} g_\alpha F_\alpha \text{ and } \ x_0(C_u) = h_u^i B_u^i, \ \text{ whenever } b_{iu} \neq 0.$$

It is correct for $s = 1$, by the above argument. Suppose it is true for s. Since $G = \bigcup_{\alpha \in J_s} g_\alpha F_\alpha$, by (3.6) again, there are disjoint subsets D_β, $\beta \in J_{s+1}$ and B_u, with $b_{(s+1)u \neq 0}$ of G, for some finite index set J_{s+1}, such that

$$\bigcup_{\beta \in J_{s+1}} g_\beta D_\beta = G = \bigcup_{\alpha \in \alpha} g_\alpha F_\alpha \ \text{ and } \ h_u B_u = x_0(C_u) \ \text{ for } \ b_{(s+1)u \neq 0}.$$

Hence there exist some finite index set J' and disjoint subsets D'_η, B'_u of $\bigcup_{\alpha \in J_s} A_\alpha$ and members g'_η, h'_u such that $G = \bigcup_{\eta \in J'} g'_\eta D'_\eta$ and $x_0(C_u) = h'_u B'_u$ which are as desired.

In particular for $s = r$,

$$\{D'_\eta : \ \eta \in J_r\} \coprod \{B_{ur} : \ b^r_u \neq 0\} \coprod \bigcup_{i=1}^{r-1} \{B^i_u : \ b_{iu} \neq 0\} \subseteq G$$

and

$$G = \bigcup_{\eta \in J_r} g'_\eta D'_\eta.$$

In addition,

$$x_0(C_u) = h^i_u B^i_u, \quad (1 \leq i \leq r, \ b_{iu} \neq 0) \tag{3.7}$$

But by Lemma (3.2), for each $1 \leq u \leq l$ there is $1 \leq i_u \leq r$ such that $b_{i_u u} \neq 0$. Hence by using (3.7)

$$G = \bigcup_{u=1}^{l} x_0(C_u) = \bigcup_{u=1}^{l} h^{i_u}_u B^{i_u}_u.$$

This completes the proof. ■

It follows by an easy induction that $|J_r| \leq \delta^r$, where $\delta = m(l - m + 1)$. Since $m, r \leq l$, we have the following Corollary.

Corollary 3.7. Let $Con(\mathfrak{g}, \mathcal{E})$ be a configuration set of group G satisfying the paradoxical condition and $|Con(\mathfrak{g}, \mathcal{E})| = l$. Then $\tau(G) \leq l + \delta^r$, where $\delta = l^2$. Hence $\tau(G) \leq l + l^{2l}$.

Remark 3.8. By using the previous theorem, we can find the paradoxical decomposition of the group G in Example (3.5).

As we have seen in that example, $Eq(\mathfrak{g}, \mathcal{E})$ satisfies the paradoxical condition. One can show (by using the process in the proof of Theorem,) that

$$\{A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, B_4, B_5, C\}$$

is a partition of G, where $A_1 = g_1^{-2} x_0(C_1), A_2 = g_1^{-2} x_0(C_2), A_3 = g_1(x_0(C_3),$ $A_4 = g_2 x_0(C_4), A_5 = g_2^{-1} x_0(C_5), B_1 = g_2 x_0(C_1), B_2 = g_1^{-1} x_0(C_2), B_3 =$ $g_1^2(x_0(C_3), B_4 = g_1 x_0(C_4), B_5 = g_1^2(x_0(C_5)$ and $C = g_1^2 x_0(C_4)$.
Furthermore

$$G = g_1^2 A_1 \bigcup g_1^2 A_2 \bigcup g_1^{-1} A_3 \bigcup g_2^{-1} A_4 \bigcup g_2 A_5$$

and

$$G = g_2^{-1} B_1 \bigcup g_1 B_2 \bigcup g_1^{-2} B_3 \bigcup g_1^{-1} B_4 \bigcup g_1^{-2} B_5$$

give a paradoxical decomposition of G.

Definition 3.9. Let G be a group and $Con(\mathfrak{g}, \mathcal{E})$ be a configuration set of G. Let also sets A^s_r be as in (3.3) and suppose that

(i) there exist $1 \leq i \leq m$ and $0 \leq j_i, k_i \leq n$ such that $A^{j_i}_i \subsetneq A^{k_i}_i$,

(ii) for each $1 \leq s \leq m$ there exists $0 \leq t_s \leq n$ such that $A^{t_s}_s \subseteq A^{j_i}_i \setminus A^{k_i}_i$.

Then $Eq(\mathfrak{g}, \mathcal{E})$ is said to satisfy the strong paradoxical condition.

Example 3.10. Let $G = \mathbb{F}_n$ the free group of rank n and $(\mathfrak{g}, \mathcal{E})$ be as in Example (2.2). Then $Eq(\mathfrak{g}, \mathcal{E})$ satisfies the strong paradoxical condition. [For $A_1^0 \subsetneqq A_1^1$, $A_s^0 \subseteq A_1^1 \setminus A_1^0$ for $2 \leq s \leq n$ and $A_{n+1}^n, A_1^n \subseteq A_1^1 \setminus A_1^0$.]

Lemma 3.11. If $Eq(\mathfrak{g}, \mathcal{E})$ satisfies the strong paradoxical condition, this system also satisfies the paradoxical condition.

Proof. Let $Eq(\mathfrak{g}, \mathcal{E})$ satisfy the strong paradoxical condition. It is easily checked that for each j, $\sum_{i=1}^m L_i^j = 0$, in particular,

$$\sum_{s \neq i} L_s^{t_i} = -L_i^{t_i}.$$

Let B be as in Lemma 3.2. Then

$$B = \begin{pmatrix} -L_1^{t_1} + L_i^{k_i} - L_i^{j_i} \\ \vdots \\ -L_{i-1}^{t_{i-1}} + L_i^{k_i} - L_i^{j_i} \\ -L_{i+1}^{t_{i+1}} + L_i^{k_i} - L_i^{j_i} \\ \vdots \\ -L_m^{t_m} + L_i^{k_i} - L_i^{j_i} \\ \sum_{s \neq i} L_s^{t_i} + L_i^{k_i} - L_i^{j_i} \\ 0 \end{pmatrix},$$

which follows that $Eq(\mathfrak{g}, \mathcal{E})$ satisfies the paradoxical condition. ∎

Example (3.5) shows that $Eq(\mathfrak{g}, \mathcal{E})$ satisfies the paradoxical condition but it does not satisfy the strong paradoxical condition.

Corollary 3.12. If $Eq(\mathfrak{g}, \mathcal{E})$ satisfies the strong paradoxical condition, then G has a paradoxical decomposition.

In the following theorem we give a similar and independent proof for this corollary, such that the paradoxical decomposition is complete.

Theorem 3.13. Let G be a group, $\mathfrak{g} = (g_1, \ldots, g_n)$ be a generating sequence of G, $\mathcal{E} = \{E_1, \ldots, E_m\}$ be a finite partition of G and $Eq(\mathfrak{g}, \mathcal{E})$ satisfies the strong paradoxical condition and sets A_i^j be as in (3.3). Then G has a complete paradoxical decomposition.

Proof. At first we find the paradoxical decomposition and then complete it. Set

$\mathcal{I} := \bigcup\{x_0(C) : \quad C \in A_i^{j_i}\}$,

$\mathcal{J} := \bigcup\{x_0(C) : \quad C \in A_i^{k_i} \setminus A_i^{j_i}\}$,

$g := g_{j_i}^{-1} g_{k_i}$,

$B_s := \mathcal{J} \setminus \bigcup\{x_0(C) : \; C \in A_s^{t_s}\}$, for $1 \leq s \leq m$, Note that with this notation and by Lemma(3.1), Definition (2.9)(ii) means for $1 \leq s \leq m$,

$$\mathcal{J} = (g_{t_s}^{-1} \bigcup_{C \in A_s^0} x_0(C)) \bigcup B_s \tag{3.8}$$

and

$$\mathcal{I} = g \bigcup_{C \in A_i^{k_i}} x_0(C). \tag{3.9}$$

On the other hand it follows by Definition (2.9)(i) that,

$$\mathcal{I} = g(\mathcal{I} \bigcup \mathcal{J}). \tag{3.10}$$

Now (3.9) implies

$$\mathcal{I} = g(g \bigcup_{C \in A_i^{k_i}} x_0(C) \bigcup \mathcal{J}) = g^2 \mathcal{I} \bigcup g\mathcal{J}.$$

By induction and (3.10) we have for each $r \in \mathbb{N}$,

$$\mathcal{I} = g^r \mathcal{I} \bigcup g^{r-1} \mathcal{J} \bigcup \cdots \bigcup g^2 \mathcal{J} \bigcup g\mathcal{J}. \tag{3.11}$$

Therefore by (3.11) and (3.8) and for $r = m+1$,

$$\begin{aligned} \mathcal{I} = g^{m+1}\mathcal{I} \bigcup g^m (g_{t_1}^{-1} \coprod_{C \in A_1^0} x_0(C) \bigcup B_1) \bigcup g^{m-1} (g_{t_2}^{-1} \coprod_{C \in A_2^0} x_0(C) \bigcup B_2) \\ \bigcup \cdots \bigcup g(g_{t_m}^{-1} \coprod_{C \in A_m^0} x_0(C) \bigcup B_m). \end{aligned} \tag{3.12}$$

Put $A := g^{m+1}\mathcal{I}$ and $A_s := (g^{m+1-s} g_{t_s}^{-1} \coprod_{C \in A_s^0} x_0(C))$, for $1 \leq s \leq m$. Thus

$$\{A\} \bigcup \{A_s : \quad 1 \leq s \leq m\} \bigcup \{x_0(C) : \quad C \notin A_i^{j_i}\} \bigcup \{g^m B_1, \ldots, gB_m\}$$

is a partition of G. In addition by (1.1) and (3.12)

$$G = \coprod_{s=1}^{m} g_{t_s} g^{m+1-s} A_s. \tag{3.13}$$

The fact that

$$\begin{aligned} G &= \mathcal{I} \cup \bigcup \{x_0(C) : \quad C \in Con(\mathfrak{g}, \mathcal{E}) \setminus A_i^{j_i}\} \\ &= g^{-m-1} A \cup \bigcup \{e.x_0(C) : \quad C \in Con(\mathfrak{g}, \mathcal{E}) \setminus A_i^{j_i}\}, \end{aligned}$$

where e is the identity element of G, with (3.13) shows the existence of a paradoxical decomposition for G. This decomposition may be uncomplete. We complete it as below.

Set $D_s = \bigcup_{k=m+1-s}^{\infty} g^k B_s$. Then $D_s \subseteq \mathcal{I}$ and $g^{-1} D_s = D_s \bigcup g^{m-s} B_s$. Thus $A = g^{m+1}\mathcal{I} = g^{m+2}[(D_1 \bigcup g^{m-1} B_1) \bigcup \cdots \bigcup (D_m \bigcup B_m) \bigcup (\mathcal{I} \setminus \bigcup_{s=1}^{m} D_s)]$.

By substituting this in the above, the paradoxical decomposition will be complete. ∎

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Department of Mathematics, University of Isfahan,
Isfahan 81746-73441, Iran
email:rejali@sci.ui.ac.ir, a.yousofzade@sci.ui.ac.ir