Hopf algebra actions on differential graded algebras and applications

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Abstract

Let H be a finite dimensional semisimple Hopf algebra, A a differential graded (dg for short) H-module algebra. Then the smash product algebra $A\#H$ is a dg algebra. For any dg $A\#H$-module M, there is a quasi-isomorphism of dg algebras: $\mathrm{RHom}_A(M,M)\#H \longrightarrow \mathrm{RHom}_{A\#H}(M\otimes H, M\otimes H)$. This result is applied to d-Koszul algebras, Calabi-Yau algebras and AS-Gorenstein dg algebras.

1 Introduction

Let G be a finite group, R be a G-group algebra. For modules M and N over the skew group algebra $R * G$, there are "natural" G-actions on the extension groups $\mathrm{Ext}^*_R(M,N)$ and the Hochschild cohomologies of R (see [19, 20]). Naturally, the group actions on extension groups or Hochschild cohomologies can be generalized to Hopf algebra actions [7, 23]. It seems not easy to find an explicit relation between the extensions of modules over $R * G$ and that of modules over R. Let R be a positively graded algebra and G be a finite group of grading preserving automorphisms of R. In [19], the author established an interesting link between the Yoneda algebra of the trivial module over $R * G$ and that of the trivial module over R. In [20], the authors discussed the Hochschild cohomology algebra of skew group algebras. In both papers, the results were proved through discussing the structures of extension groups directly.

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Since the extension groups of modules and Hochschild cohomologies can be computed through differential graded (dg, for short) modules and dg algebras, we try to deal with the group actions on extensions in the framework of dg settings, and we find that this is an efficient way to do these things. So we first need to discuss properties of the group (more generally, Hopf algebra) actions on dg algebras. Let H be a finite dimensional semisimple Hopf algebra, and A be a dg H-module algebra. Our main result (Theorem 2.8) says that there is a quasi-isomorphism of dg algebras: $\mathrm{RHom}_A(M,M)\#H \longrightarrow \mathrm{RHom}_{A\#H}(M\otimes H, M\otimes H)$ for any dg $A\#H$-module M. This quasi-isomorphism yields the isomorphism in [19, Theorem 10] and generalizes of [24, Theorem 2.3] to the level of derived functors. We apply the main result to d-Koszul algebras, Calabi-Yau algebras and AS-Gorenstein algebras. We show that the smash product algebra of a d-Koszul algebra and finite dimensional semisimple Hopf algebra is also a d-Koszul algebra, and the Galois covering algebras of a d-Koszul Calabi-Yau algebras are also Calabi-Yau.

Throughout, k is an algebraically closed field of characteristic zero and all algebras are k-algebras; unadorned $\otimes$ means $\otimes_k$ and Hom means Hom_k. By a dg algebra we mean a cochain dg algebra, that is, a graded algebra $A = \bigoplus_{n\in\mathbb{Z}} A^n$ with a differential d of degree 1, such that for all homogeneous elements $a, b \in A$ we have $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ where $|a|$ denotes the degree of a. An associative algebra R may be regarded as a dg algebra concentrated in degree zero with zero differentials, and then a complex of R-modules may be regarded as a dg R-module. A (left) dg A-module is a graded A-module $M = \oplus_{n\in\mathbb{Z}}M^n$ with a differential d of degree 1 such that for all homogeneous elements $a \in A$ and $m \in M$ we have $d(am) = d(a)m + (-1)^{|a|}ad(m)$. Similarly, we have right dg modules. In this paper, by a dg module we always mean a left dg module.

Let A be a dg algebra, M and N dg A-modules. We write $\mathrm{Hom}_A(M,N) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}^i_A(M,N)$, where $\mathrm{Hom}^i_A(M,N)$ is the set of all graded A-module maps of degree i. Then $\mathrm{Hom}_A(M,N)$ is a complex with the canonical differential d which acts on a homogeneous element $f \in \mathrm{Hom}_A(M,N)$ by $d(f) = d_N \circ f - (-1)^{|f|} f \circ d_M$. A dg A-module P is said to be *homotopically projective* (or K-projective) if $\mathrm{Hom}_A(P,-)$ preserves the quasi-trivial dg modules, and a dg A-module I is said to be *homotopically injective* (or K-injective) if $\mathrm{Hom}_A(-,I)$ preserves the quasi-trivial dg modules (see [16, Chapter 8]).

For more properties of dg algebras and modules, we refer to the references [1, 8, 14, 17].

2 Hopf algebra actions on dg algebras

Let A be a dg algebra, H a Hopf algebra. We call A a *dg H-module algebra* if

(i) A is a graded H-module algebra, that is; for $a \in A^i$, $b \in A^j$ and $h \in H$, $h\cdot a \in A^i$ and $h\cdot(ab) = (h_{(1)}\cdot a)(h_{(2)}\cdot b)$, and

(ii) the differential d of A is compatible with the H-module action, that is, $d(h\cdot a) = h\cdot d(a)$.

If A is a left dg H-module algebra, then the cohomology algebra $H(A)$ is a graded H-module algebra, and the smash product $A\#H$ is also a dg algebra with the differential $\delta = d \otimes id$.

Proposition 2.1. *If A is a dg H-module algebra, then $H(A\#H) \cong H(A)\#H$ as graded algebras.*

Since A is a dg subalgebra of $A\#H$, any dg $A\#H$-modules M and N can be viewed as dg A-modules. Also M and N are complexes of H-modules. If the antipode S of H is bijective, there is a natural H-module structure on the complex $\mathrm{Hom}_A(M,N)$. Explicitly, for $f \in \mathrm{Hom}_A(M,N)$, and $h \in H$, the H-module action is defined by

$$(h \rightharpoonup f)(m) = h_{(2)} f(S^{-1}h_{(1)}m), \qquad \text{for } m \in M,\ h \in H. \tag{1}$$

Given an H-module X, we write $X^H = \{x \in X | hx = \varepsilon(h)x, \text{ for all } h \in H\}$ for the invariant submodule. If M is a complex of H-modules, then M^H is a subcomplex of M.

Lemma 2.2. *Let H be a Hopf algebra with a bijective antipode.*

(i) *If A is a dg H-module algebra, M and N are dg $A\#H$-modules, then*

$$\mathrm{Hom}_{A\#H}(M,N) \cong \mathrm{Hom}_A(M,N)^H$$

as complexes of vector spaces

(ii) *If H is semisimple, then $(\ )^H$ preserves exact sequences.*

Proof. The assertion (i) follows directly from the definition. For the assertion (ii), just observe that $(\ )^H \cong \mathrm{Hom}_H(k,-)$ is exact. ■

Let M be a dg A-module. Then $\mathrm{Hom}_A(M,M)$ is a dg algebra. For convenience, the multiplication of $\mathrm{Hom}_A(M,M)$ is defined as follows: for homogeneous elements $f,g \in \mathrm{Hom}_A(M,M)$, $f * g = (-1)^{|f||g|} g \circ f$.

From now on, H will always be a Hopf algebra with a bijective antipode.

Lemma 2.3. *Let A be a dg H-module algebra, M a dg $A\#H$-module. Then $B = \mathrm{Hom}_A(M,M)$ is a dg H-module algebra.*

Proof. Straightforward. ■

Let M be a dg $A\#H$-module, W a dg H-modules. Then $M \otimes W$ is a dg $A\#H$-module through the action defined by

$$(a\#h)(m \otimes w) = (a\#h_{(1)})m \otimes h_{(2)}w. \tag{2}$$

Lemma 2.4. *Let M and N be dg $A\#H$-modules, W a dg H-modules. Then the Hom-Tensor adjoint isomorphism*

$$\varphi : \mathrm{Hom}(W, \mathrm{Hom}_A(M,N)) \longrightarrow \mathrm{Hom}_A(M \otimes W, N)$$

is an H-module morphism.

Proof. For homogeneous elements $f \in \mathrm{Hom}(W, \mathrm{Hom}_A(M,N))$ and $m \in M$, $w \in W$, recall that $\varphi(f)(m \otimes w) = (-1)^{|m||w|} f(w)(m)$. Hence for $h \in H$, we have

$$\begin{aligned}\varphi(h \rightharpoonup f)(m \otimes w) &= (-1)^{|m||w|}(h \rightharpoonup f)(w)(m) \\ &= (-1)^{|m||w|}[h_{(1)} \rightharpoonup (f(S^{-1}(h_{(1)})w))](m) \\ &= (-1)^{|m||w|} h_{(3)}(f(S^{-1}(h_{(1)})w))(S^{-1}(h_{(2)})m),\end{aligned}$$

$$\begin{aligned}(h \rightharpoonup \varphi(f))(m \otimes w) &= h_{(2)}\varphi(f)(S^{-1}(h_{(1)})(m \otimes w)) \\ &= h_{(3)}\varphi(f)(S^{-1}(h_{(2)})m \otimes S^{-1}(h_{(1)})w)) \\ &= (-1)^{|m||w|} h_{(3)}(f(S^{-1}(h_{(1)})w))(S^{-1}(h_{(2)})m).\end{aligned}$$

Therefore φ is an H-module morphism. ∎

Proposition 2.5. *Let H be a finite dimensional semisimple Hopf algebra, A a dg H-module algebra. Then a dg $A\#H$-module P is K-projective if and only if P is K-projective as a dg A-module.*

Proof. Assume that P is a K-projective A-module. We only need to show that the functor $\mathrm{Hom}_{A\#H}(P,-)$ preserves the quasi-trivial dg modules. By Lemma 2.2, we have

$$\mathrm{Hom}_{A\#H}(P,-) = (\ )^H \circ \mathrm{Hom}_A(P,-).$$

Since H is semisimple, by Lemma 2.3 the functor $(\ )^H$ preserves exact sequences. Hence $(\ )^H \circ \mathrm{Hom}_A(P,-)$ preserves quasi-trivial dg modules.

Conversely, suppose that P is a K-projective $A\#H$-module. Since P is homotopically equivalent to a semifree dg $A\#H$-module (see [1, 8]), we may assume that P is semifree. Let

$$0 \subseteq P(0) \subseteq P(1) \subseteq \cdots \subseteq P(n) \subseteq P(n+1) \subseteq \cdots$$

be a semifree filtration of the dg $A\#H$-module P. Then $P(n+1)/P(n)$ is a free dg $A\#H$-module (i.e., it is a direct sum of shifts of $A\#H$). Moreover, $A\#H$ is also a free dg A-module. Hence, the above filtration is also a semifree filtration of the dg A-module P. So P is K-projective as a dg A-module. ∎

Proposition 2.6. *Let A and H be as above. A dg $A\#H$-module I is K-injective if and only if it is K-injective as a dg A-module.*

Proof. Assume that I is a K-injective dg $A\#H$-module. We have to show that the functor $\mathrm{Hom}_A(-,I)$ preserves quasi-trivial dg A-modules. As a dg A-module, we have $I \cong \mathrm{Hom}_{A\#H}(A\#H, I)$. Hence we get

$$\mathrm{Hom}_A(-,I) \cong \mathrm{Hom}_A(-, \mathrm{Hom}_{A\#H}(A\#H, I)) \cong \mathrm{Hom}_{A\#H}(A\#H \otimes_A -, I).$$

Since $A\#H$ is a free right dg A-module, $A\#H \otimes_A -$ preserves quasi-trivial dg A-modules. Therefore $\mathrm{Hom}_A(-,I)$ preserves the quasi-trivial dg modules.

The other direction follows from the following isomorphism:

$$\mathrm{Hom}_{A\#H}(-,I) \cong (\ )^H \circ \mathrm{Hom}_A(-,I).$$

∎

Lemma 2.7. *Let P and Q be dg $A\#H$-modules. If H is finite dimensional, then there is a natural isomorphism of complexes of vector spaces:*

$$\mathrm{Hom}_{A\#H}(P \otimes H, Q \otimes H) \cong \mathrm{Hom}_A(P, Q) \otimes H.$$

Proof. By Lemma 2.2, we have

$$\mathrm{Hom}_{A\#H}(P \otimes H, Q \otimes H) \cong \mathrm{Hom}_A(P \otimes H, Q \otimes H)^H.$$

On the other hand, we have an isomorphism of complexes of H-modules

$$\mathrm{Hom}_A(P \otimes H, Q \otimes H) \cong \mathrm{Hom}(H, \mathrm{Hom}_A(P, Q \otimes H)).$$

Hence

$$\begin{aligned} \mathrm{Hom}_{A\#H}(P \otimes H, Q \otimes H) &\cong \mathrm{Hom}(H, \mathrm{Hom}_A(P, Q \otimes H))^H \\ &\cong \mathrm{Hom}_H(H, \mathrm{Hom}_A(P, Q \otimes H)) \\ &\cong \mathrm{Hom}_A(P, Q \otimes H) \\ &\cong \mathrm{Hom}_A(P, Q) \otimes H. \end{aligned}$$ ■

We may write out explicitly the isomorphism in the lemma above as

$$\theta : \mathrm{Hom}_A(P, Q) \otimes H \longrightarrow \mathrm{Hom}_{A\#H}(P \otimes H, Q \otimes H),$$

acting on elements as

$$\theta(f \otimes h)(p \otimes g) = g_{(2)} f(S^{-1}(g_{(1)})p) \otimes g_{(3)} h,$$

where $f \in \mathrm{Hom}_A(P, Q)$, $p \in P$ and $g, h \in H$.

Let M be a dg $A\#H$-module. Let P be a K-projective resolution of the dg $A\#H$-module M. From Proposition 2.5, it follows that P is also K-projective as a dg A-module. Then $\mathrm{RHom}_A(M, M) = \mathrm{Hom}_A(P, P)$, and hence is a dg algebra. By Lemma 2.3, $\mathrm{RHom}_A(M, M)$ is a dg H-module algebra. Of course the dg H-module algebra structure of $\mathrm{RHom}_A(M, M)$ depends on the choice of the K-projective resolution of M. However the dg algebra structures on $\mathrm{RHom}_A(M, M)$ induced from different K-projective resolutions are quasi-isomorphic to each other as A_∞-algebras. This does not matter since such dg algebras have the same homological properties. Also the H-module structures are compatible with the associated quasi-isomorphisms.

Theorem 2.8. *Let H be a finite dimensional semisimple Hopf algebra, A a dg H-module algebra. If M is a dg $A\#H$-module, then there is a quasi-isomorphism of dg algebras:*

$$\mathrm{RHom}_A(M, M)\#H \longrightarrow \mathrm{RHom}_{A\#H}(M \otimes H, M \otimes H).$$

Proof. Let P be a K-projective resolution of the dg $A\#H$-module M. Then dg $A\#H$-module $P \otimes H$ is quasi-isomorphic to the dg $A\#H$-module $M \otimes H$. By Proposition 2.5, $P \otimes H$ is a K-projective dg $A\#H$-module. Hence we have $\mathrm{RHom}_{A\#H}(M, M) = \mathrm{Hom}_{A\#H}(P \otimes H, P \otimes H)$ and $\mathrm{RHom}_A(M, M) = \mathrm{Hom}_A(P, P)$. By Lemma 2.7, we have a quasi-isomorphism of complexes:

$$\theta : \mathrm{Hom}_A(P, P) \otimes H \longrightarrow \mathrm{Hom}_{A\#H}(P \otimes H, P \otimes H),$$

and for $f \in \mathrm{Hom}_A(P,P)$, $h,g \in H$ and $p \in P$,

$$\theta(f \otimes h)(p \otimes g) = g_{(2)} f(S^{-1}(g_{(1)})p) \otimes g_{(3)} h.$$

We claim that θ is a morphism of dg algebras.

For homogeneous elements $f, f' \in \mathrm{Hom}_A(P,P)$, $h, h', g \in H$ and $p \in P$, we have

$$\begin{aligned}
& \theta((f'\#h')(f\#h))(p \otimes g) \\
= \; & \theta(f' * (h'_{(1)} \rightharpoonup f)\#h'_{(2)}h)(p \otimes g) \\
= \; & g_{(2)}[f' * (h'_{(1)} \rightharpoonup f)](S^{-1}(g_{(1)})p) \otimes g_{(3)}h'_{(2)}h \\
= \; & (-1)^{|f||g|} g_{(2)}(h'_{(1)} \rightharpoonup f)\left(f'(S^{-1}(g_{(1)})p\right) \otimes g_{(3)}h'_{(2)}h,
\end{aligned}$$

and

$$\begin{aligned}
& [\theta(f'\#h') * \theta(f\#h)](p \otimes g) \\
= \; & (-1)^{|f||g|}\theta(f\#h) \circ \theta(f'\#h')(p \otimes g) \\
= \; & (-1)^{|f||g|}\theta(f\#h)(g_{(2)}f'(S^{-1}(g_{(1)})p) \otimes g_{(3)}h') \\
= \; & (-1)^{|f||g|} g_{(4)}h'_{(2)} f[S^{-1}(g_{(3)}h'_{(1)})g_{(2)}f'(S^{-1}(g_{(1)})p)] \otimes g_{(5)}h'_{(3)}h \\
= \; & (-1)^{|f||g|} g_{(2)}(h'_{(1)} \rightharpoonup f)\left(f'(S^{-1}(g_{(1)})p\right) \otimes g_{(3)}h'_{(2)}h.
\end{aligned}$$

Hence θ is compatible with the multiplications, and by Lemma 2.7 it is a quasi-isomorphism. ■

Let M and N be dg $A\#H$-modules. Since $\mathrm{RHom}_A(M,N)$ is a complex of H-modules, the extension group $\mathrm{Ext}^*_A(M,N) = \oplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_A(M,N)$ is a graded H-module, and $\mathrm{Ext}^*_A(M,M)$ is a graded H-module algebra.

Corollary 2.9. *Let A and H be as above, M and N be dg $A\#H$-modules.*

(i) $\mathrm{Ext}^*_{A\#H}(M,N) \cong \mathrm{Ext}^*_A(M,N)^H$;

(ii) $\mathrm{Ext}^*_{A\#H}(M,M) \cong \mathrm{Ext}^*_A(M,M)^H$ *as graded algebras;*

(iii) $\mathrm{Ext}^*_{A\#H}(M \otimes H, M \otimes H) \cong \mathrm{Ext}^*_A(M,M)\#H$ *as graded algebras.*

Proof. (i) Let P and Q be K-projective resolutions of the dg $A\#H$-modules M and N respectively. Then $\mathrm{RHom}_{A\#H}(M,N) = \mathrm{Hom}_{A\#H}(P,Q) \cong \mathrm{Hom}_A(P,Q)^H$. Hence

$$\begin{aligned}
\mathrm{Ext}^*_{A\#H}(M,N) &= \mathrm{H}^*(\mathrm{RHom}_{A\#H}(M,N)) \cong \mathrm{H}^*(\mathrm{Hom}_A(P,Q)^H) \\
&\cong (\mathrm{H}^*\,\mathrm{Hom}_A(P,Q))^H \cong \mathrm{Ext}^*_A(M,N)^H.
\end{aligned}$$

The assertion (ii) is directly from (i). Then assertion (iii) is a direct consequence of Theorem 2.8. ■

Group actions on extension groups have been discussed by several authors (see [19, 20]) by the use of the traditional homological tools. In Corollary 2.9, when $H = kG$ is a group algebra of finite group G, then assertion (ii) becomes Proposition 2.6 in [20], and assertion (iii) becomes Theorem 10 in [19]. Moreover, assertion (iii) is a generalization of [24, Theorem 2.3] to the level of derived functors.

3 Applications

Throughout this section, H is a finite dimensional semisimple Hopf algebra.

3.1 Hopf algebra actions on d-Koszul algebras

Let $R = \oplus_{n\geq 0} R_n$ be a positively graded algebra such that R_0 is semisimple, $\dim R_i < \infty$ for all $i \geq 0$ and $R_i R_j = R_{i+j}$. Recall that R is called a homogeneous algebra if $R \cong T_{R_0}(R_1)/I$, where I is an ideal generated by elements in $\underbrace{R_1 \otimes_{R_0} \cdots \otimes_{R_0} R_1}_{d\ factors}$ $(d \geq 2)$. R is called a *connected* graded algebra if $R_0 \cong k$. A homogeneous algebra R is called a *d-Koszul* algebra if the trivial module R_0 has a graded projective resolution

$$\cdots \longrightarrow P^{-n} \longrightarrow P^{-n+1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow R_0 \longrightarrow 0,$$

such that the graded module P^{-n} is generated in degree $\frac{n}{2}d$ if n is even and $\frac{n-1}{2}d + 1$ if n is odd for all $n \geq 0$. When $d = 2$, then a d-Koszul algebra is usually called a *Koszul* algebra which was introduced by Priddy in [21]. The concept of d-Koszul algebra was introduced by Berger in [2], where a d-Koszul algebra is called a generalized Koszul algebra. Many interesting algebras are proved to be d-Koszul algebras. For example, 3-dimensional graded Calabi-Yau algebras [6] are d-Koszul algebras.

For simplicity, write $E^i(R) = \mathrm{Ext}^i_R(R_0, R_0)$ and $E(R) = \oplus_{i\geq 0} \mathrm{Ext}^i_R(R_0, R_0)$. Endowed with the Yoneda product, $E(R)$ is a graded algebra. We call $E(R)$ sometimes the *Yoneda Ext-algebra* of R. For the d-Koszul algebras, we have the following properties.

Theorem 3.1. *[3, 5, 11, 12] Let R be as above.*

(i) R is a Koszul algebra if and only if $E(R)$ is generated by $E^0(R)$ and $E^1(R)$.

(ii) If R is a homogeneous algebra, then R is a d-Koszul algebra ($d \geq 3$) if and only if $E(R)$ is generated by $E^0(R)$, $E^1(R)$ and $E^2(R)$.

Applying the main result from the last section, we are able to show that the d-Koszulness of a graded algebra can be lifted to a smash product of the graded algebra.

Theorem 3.2. *Let R be a homogeneous algebra. Assume that there is an H-action on R so that R is a graded H-module algebra. Then $R\#H$ is a d-Koszul algebra if and only if R is a d-Koszul algebra. Moreover, $E(R\#H) \cong E(R)\#H$.*

Proof. Since R is an H-module algebra, R is a graded $R\#H$-module. In particular, R_0 is an $R\#H$-module. Let

$$P^\bullet := \qquad \cdots \longrightarrow P^{-n} \longrightarrow P^{-n+1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow R_0 \longrightarrow 0$$

be a graded projective resolution of the $R\#H$-module R_0. Write $B = R\#H$. Then $B_0 = R_0\#H$. As a left graded B-module, $B_0 \cong R_0 \otimes H$, where the left B-module structure of $R_0 \otimes H$ is defined by the equation (2). Therefore

$$\cdots \longrightarrow P^{-n} \otimes H \longrightarrow P^{-n+1} \otimes H \longrightarrow \cdots \longrightarrow P^0 \otimes H \longrightarrow B_0 \longrightarrow 0$$

is a graded projective resolution of the B-module B_0. We have the following isomorphisms of graded algebras

$$\begin{aligned} E(R\#H) &= \oplus_{i\geq 0} H^i \operatorname{Hom}_{R\#H}(P^\bullet \otimes H, P^\bullet \otimes H) \\ &\cong \oplus_{i\geq 0} H^i(\operatorname{Hom}_R(P^\bullet, P^\bullet)\#H) \\ &\cong E(R)\#H. \end{aligned}$$

It is clear that $E(R\#H)$ is generated by $E^0(R\#H)$, $E^1(R\#H)$ and $E^2(R\#H)$ if and only if $E(R)$ is generated by $E^0(R)$, $E^1(R)$ and $E^2(R)$. Now the proof follows directly from Theorem 3.1. ∎

If R is Koszul and H is the group algebra of a finite group G, then the theorem above implies [19, Theorem 14].

3.2 Calabi-Yau algebras

Let R be a positively graded algebra, $R^e = R \otimes R^{op}$ be the enveloping algebra. R is called a *graded Calabi-Yau algebra* of dimension p (in the sense of Ginzburg [9]) if (i) R is homologically smooth, that is, as an R^e-module R has a projective resolution of finite length given by finitely generated modules; (ii) there is a graded R-R-bimodule isomorphism [4]

$$\operatorname{Ext}^i_{R^e}(R, R^e) \cong \begin{cases} 0, & i \neq p, \\ R(l), & i = p, \end{cases} \tag{3}$$

where l is an integer, and $R(l)$ is the shift of R.

Let E be a finite dimensional graded algebra. We say that E is *graded symmetric* if there is an integer n and a homogeneous nondegenerate bilinear form $\langle -, - \rangle : E \times E \longrightarrow k(n)$ such that $\langle xy, z\rangle = \langle x, yz\rangle$ and $\langle x, y\rangle = (-1)^{|x||y|}\langle y, x\rangle$ for all homogeneous elements $x, y, z \in E$.

Proposition 3.3. *Let Q be a finite quiver. A d-Koszul algebra $R = kQ/I$ is a Calabi-Yau algebra if and only if $E(R)$ is a graded symmetric algebra.*

Proof. Assume that R is a Calabi-Yau algebra of dimension p. By [15, Lemma 4.1] the triangulated category $D^b(R)$ is a Calabi-Yau category, where $D^b(R)$ is the triangulated subcategory of the derived category of R consisting of complexes whose cohomology has finite total dimension. Then the Yoneda Ext-algebra $E(R) = \oplus_{n\geq 0} \operatorname{Ext}^n_R(R_0, R_0) = \oplus_{0\leq n\leq p} \operatorname{Hom}_{D^b(R)}(R_0, R_0[n])$ is graded symmetric (see [15, Sect. 2.6], or the appendix of [6]).

Conversely, assume $E(R)$ is graded symmetric. Suppose that the global dimension of R is p. By [3, Theorem 1.2] or [18, Theorem 12.5], R is an AS-Gorenstein algebra. By [4, Proposition 4.5] R is a Calabi-Yau algebra of dimension p if and only if $\varepsilon^{p+1} \circ \phi = id$, where ε is the isomorphism of R defined by $\varepsilon(r) = (-1)^{|r|}r$ for a homogeneous element $r \in R$, and ϕ is the isomorphism of R such that $\phi|_{R_1}$ is the dual map of the restriction map of the Nakayama automorphism of $E(R)$ to $E^1(R)$. If $d \geq 3$, then $\mathrm{gldim}(R) = p$ must be odd. In this case, $E(R)$ is exactly a symmetric algebra. Hence the Nakayama automorphism of $E(R)$ is the identity. Therefore $\phi = id$. Since p is odd, $\varepsilon^{p+1} = id$. Hence $\varepsilon^{p+1} \circ \phi = id$. That is, R is a Calabi-Yau algebra. If $d = 2$, then $\mathrm{gldim}(R) = p$ can be any positive integer. Thus if p is odd, the proof is the same as above. However, if p is even, then the Nakayama automorphism ν of $E(R)$ satisfies $\nu(x) = (-1)^{|x|}x$ for homogeneous elements $x \in E(R)$. Hence $\phi = \varepsilon$ and $\varepsilon^{p+1} \circ \phi = id$. That is, R is a Calabi-Yau algebra. ∎

Let G be a finite group. Suppose that R is an $\mathbb{N} \times G$-graded algebra such that $R_{0,e} = k$ and $R_{0,g} = 0$ for $g \neq e$, and $R_i = \oplus_{g \in G} R_{i,g}$ is finite dimensional for all $i \geq 0$. Let M and N be finite generated $\mathbb{N} \times G$-graded R-modules. Then $\mathrm{Hom}_R(M, N)$ is an $\mathbb{N} \times G$-graded vector space. On the other hand, the $\mathbb{N} \times G$-graded algebra R has a natural kG^*-module structure so that it is an ($\mathbb{N}$-)graded kG^*-module algebra. Similarly, the graded $\mathbb{N} \times G$-graded R-modules M and N can be regarded as graded $R\#kG^*$-modules. Hence $\mathrm{Hom}_R(M, N)$ is a graded kG^*-module with the module structure given by the equation (1). However, the $\mathbb{N} \times G$-grading on $\mathrm{Hom}_R(M, N)$ also induces a graded kG^*-module structure. It is not hard to check that the two kG^*-modules described as above coincide. Similarly, if $P^\bullet$ and $Q^\bullet$ are complexes of graded $\mathbb{N} \times G$-graded R-modules, then $\mathrm{Hom}_R(P^\bullet, Q^\bullet)$ is a complex of $\mathbb{N} \times G$-graded vector spaces.

We say that an $\mathbb{N} \times G$-graded algebra R is a *Calabi-Yau algebra of dimension p* if the isomorphism in (3) also respects the G-grading. That is, the isomorphism $\mathrm{Ext}^p_{R^e}(R, R^e) \cong R(l)$ is also an isomorphism of G-graded spaces.

Corollary 3.4. *Let R be as above. Then R is a d-Koszul Calabi-Yau algebra of dimension p, then so is $R\#kG^*$.*

Proof. If R is a d-Koszul algebra, by Theorem 3.2 $R\#kG^*$ is a d-Koszul algebra. By Theorem 2.8, $\mathrm{RHom}_{R\#kG^*}(k \otimes kG^*, k \otimes kG^*) \cong \mathrm{RHom}_R(k, k)\#kG^*$. Since R is a Calabi-Yau algebra of dimension p, the global dimension of R is p. Let

$$P^\bullet := \qquad 0 \longrightarrow P^{-p} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow k \longrightarrow 0$$

be a minimal $\mathbb{N} \times G$-graded projective resolution of the trivial module k. By the Koszulness of R, the projective module P^i is finitely generated for all $i \leq 0$. Now

$$\mathrm{RHom}_R(k, k)\#kG^* = \mathrm{Hom}_R(P^\bullet, P^\bullet)\#kG^*,$$

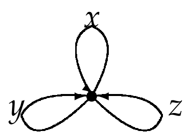
and hence

$$E(R\#kG^*) \cong H^*(\mathrm{Hom}_R(P^\bullet, P^\bullet))\#kG^* \cong E(R)\#kG^*.$$

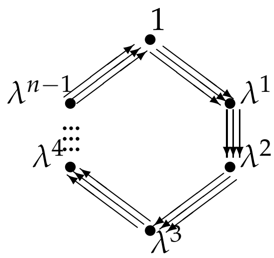
Since the Yoneda Ext-algebra $E(R)$ is the cohomology algebra of the dg algebra $\mathrm{Hom}_R(P^\bullet, P^\bullet)$, which certainly respects the $\mathbb{N} \times G$-gradings of the projective modules, $E(R)$ is an $\mathbb{N} \times G$-graded algebra. Since R is Calabi-Yau and the isomorphism $\mathrm{Ext}^p_{R^e}(R, R^e) \cong R(l)$ is also an isomorphism of G-graded vector spaces, one can check easily that $E(R)$ is in fact an $\mathbb{N} \times G$-graded symmetric algebra. Then $E(R)\#kG^*$ must be a graded symmetric algebra. It follows that $E(R\#kG^*)$ is a graded symmetric algebra. Therefore $R\#kG^*$ is a Calabi-Yau algebra by Proposition 3.3. Note that the global dimensions of R and $R\#kG^*$ are the same. Thus $R\#kG^*$ is also a Calabi-Yau algebra of dimension p since the global dimension of $R\#kG^*$ coincides with the Calabi-Yau dimension. ■

The above corollary is also a direct consequence of [7, Theorem 17] since in this case $\mathrm{Ext}^p_R(R, R^e)$ is isomorphic to $R(l)$ both as an R-R-bimodule and as a left kG^*-modules. The method of [7] is meant to compute the Hochschild cohomology of the smash product algebra by utilizing the spectral sequence obtained in [23]. However, when it comes to d-Koszul case we only need to compute the Yoneda Ext-algebra of the smash product $R\#kG^*$.

Example 3.5. Let $R = k[x, y, z]$ be the polynomial algebra. With the natural grading, R is a $\mathbb{N}$-graded algebra. It is well known that R is a Koszul Calabi-Yau algebra of dimension 3. Let λ be a primitive nth root of unit, and $G = \{\lambda^i | i \in \mathbb{Z}\}$ be the group generated by λ. Set $R_{i,g} = 0$ if $g \neq \lambda^i$ and $R_{i,\lambda^i} = R_i$. Then R is an $\mathbb{N} \times G$-graded algebra. By Theorem 3.2, $R\#kG^*$ is a Koszul algebra. In fact, $R\#kG^*$ is a Galois covering [10, 20] of the polynomial algebra $k[x, y, z]$. Explicitly, let Q be the quiver



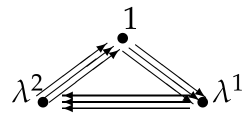
with relations $\rho = \{xy - yx, xz - zx, zy - yz\}$. Let Q' be the following quiver:



For each $0 \le i \le n-1$, the arrows leaving from the vertex λ^i are labeled as x_i, y_i and z_i respectively. The relations of the quiver Q' is $\rho' = \{x_{i+1}y_i - y_{i+1}x_i, x_{i+1}z_i - z_{i+1}x_i, y_{i+1}z_i - z_{i+1}y_i | 0 \le i \le n-1\}$, where $x_n = x_0$, $y_n = y_0$ and $z_n = z_0$. Define a map $F : (Q', \rho') \longrightarrow (Q, \rho)$ of graphs with relations by sending all the vertices to the unique vertex of Q and sending arrows x_i to x, y_i to y and z_i to z for all i. Then F is a regular covering in the sense of [10]. Let $S = kQ'/(\rho')$ be the quotient algebra of the path algebra kQ' by modulo the two-side ideal generated by the relations ρ'. Then it is direct to check that $R\#kG^* \cong S$ as $\mathbb{N}$-graded algebras.

In general, $R\#kG^*$ is not a Calabi-Yau algebra. This is because the kG^*-action on R is not compatible with the Calabi-Yau property of R. In fact, R is Calabi-Yau of dimension 3 as an $\mathbb{N} \times G$-graded algebra if and only if λ is a third primitive root of the unit. Now if $G = \{1, \lambda, \lambda^2\}$, then, by Corollary 3.4, $R\#kG^*$ is a Calabi-

Yau algebra of dimension 3. The associated quiver Q' is as follows:



By [6, 22], $R\#kG^*$ must be defined by a superpotential $W' \in kQ'/[KQ', kQ']$ so that $R\#kG^* \cong kQ'/(\partial_a W' | a \in Q'_1)$. In fact, the defining superpotential of $k[x, y, z]$ is $W = xyz - yxz$, and the defining superpotential of $R\#kG^*$ is the "lifting" of W, that is, $W' = f_1 + f_2 + f_3$, where $f_1 = x_2y_1z_0 - y_2x_1z_0$, $f_2 = z_2x_1y_0 - x_2z_1y_0$, and $f_3 = y_2z_1x_0 - z_2y_1x_0$.

3.3 Koszul dg algebras and AS-Gorenstein dg algebras

The concept of Koszul dg algebra was introduced in [13]. It was shown that there were some duality properties between a Koszul dg algebra and its Yoneda Ext-algebra.

Definition 3.6. *[13, 18]* Let $A = \oplus_{n \geq 0} A^n$ be a dg algebra such that A^0 is semisimple and the differential vanishes on A^0.

(i) A is called a *Koszul dg algebra* if $\mathrm{Ext}^i_A(A_0, A_0) = 0$ for $i \neq 0$.

(ii) A is called a *AS-Gorenstein dg algebra* (AS stands for Artin-Schelter) if there is an integer n such that $\mathrm{RHom}_A(A^0, A) \cong A^0[n]$ as right dg A-modules.

Proposition 3.7. *Let A be a dg H-module algebra. Then A is a Koszul dg algebra if and only if $A\#H$ is a Koszul dg algebra.*

Proof. Since A^0 is semisimple, $A^0\#H$ is also semisimple. By Corollary 2.9,

$$\mathrm{Ext}^*_{A\#H}(A^0\#H, A^0\#H) \cong \mathrm{Ext}^*_A(A^0, A^0)\#H$$

as graded algebras. Hence $\mathrm{Ext}^i_{A\#H}(A^0\#H, A^0\#H) = 0$ for $i \neq 0$ if and only if $\mathrm{Ext}^i_A(A, A) = 0$ for all $i \neq 0$. The proof then follows directly from Definition 3.6. ∎

Proposition 3.8. *Let A be a dg H-module algebra such that $A^0 = k$. Then A is an AS-Gorenstein dg algebra if and only if $A\#H$ is an AS-Gorenstein dg algebra.*

Proof. By Theorem 2.8, $\mathrm{RHom}_{A\#H}(A^0\#H, A\#H) \cong \mathrm{RHom}_A(A^0, A)\#H$. Moreover, from the proof of Theorem 2.8, one sees that the isomorphism is compatible with the right $A\#H$-module structures. If A is AS-Gorenstein, then $\mathrm{RHom}_{A\#H}(A^0\#H, A\#H) \cong A^0\#H[n]$. Hence $A\#H$ is an AS-Gorenstein algebra. Conversely, if $A\#H$ is AS-Gorenstein, then $\mathrm{RHom}_A(A^0, A)\#H \cong A^0\#H[n] \cong H[n]$. It follows that $\mathrm{Ext}^i_A(A^0, A) = 0$ for $i \neq n$, and $\mathrm{Ext}^n_A(A^0, A)$ must be of dimension 1. By suitable truncations of the right dg A-module $\mathrm{RHom}_A(A^0, A)$, we get $\mathrm{RHom}_A(A^0, A) \cong k[n]$. Therefore A is AS-Gorenstein. ∎

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