On the Fourth Power Mean of a Sum Analogues to Character Sums Over Short Intervals*

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Abstract

The main purpose of this paper is to study the mean value properties of a sum analogues to character sums by using the estimate for character sums and elementary methods, and finally give two asymptotic formulae for its fourth power mean over short intervals.

1 Introduction

This paper is concerned with character sums of the type

$$\sum_{1\le n\le N}(-1)^n\chi(n),$$

where χ denotes a Dirichlet character modulo q, $q \ge 2$ is an integer, which is called as a sum analogues to character sums. As for the classical character sums

$$\sum_{N+1\le n\le N+H}\chi(n),$$

where N and H are integers with $H \ge 1$. Burgess [1] [2] showed that

$$\sum_{N+1\le n\le N+H}\chi(n) \ll_{\epsilon,t} H^{1-\frac{1}{t}}q^{\frac{t+1}{4t^2}+\epsilon} \tag{1}$$

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holds if q is cubefree or $t = 1, 2, 3$. When $t = 1$, (1) is a slightly weakened version of the famous Pólya-Vinogradov inequality [3]. For the higher moment, Montgomery and Vanghan [4] established the upper bound for any positive integer k,

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \max_{1 \leq y \leq q} \left| \sum_{n \leq y} \chi(n) \right|^{2k} \ll \phi(q) \cdot q^k,$$

where $\phi(q)$ is the Euler function, and χ_0 is the principal character modulo q. Cochrane and Zheng [5] proved

$$\frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{N+1 \leq n \leq N+H} \chi(n) \right|^{2k} \ll_{\epsilon,k} p^{k-1+\epsilon} + N^k p^{\epsilon}$$

with p a prime. In recent years, Xu and Zhang [6] studied the asymptotic properties for the $2k-$th power mean value of character sums and got

$$\sum_{\chi(-1)=1}^{*} \left| \sum_{a < \frac{q}{4}} \chi(a) \right|^{2k} = \frac{J(q)q^k}{16} \left(\frac{\pi}{8}\right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right) \prod_{p \nmid q} A(0, k, p, 2) + O(q^{k+\epsilon})$$

and

$$\sum_{\chi(-1)=-1}^{*} \left| \sum_{a < \frac{q}{4}} \chi(a) \right|^{2k} =$$
$$C(k)q^k J(q) \zeta^{2k-1}(2) \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid 2q} A(0, k, p, 2) + O(q^{k+\epsilon}),$$

for any odd integer $q \geq 5$, where $\sum^*$ denotes the summation over all primitive characters, $J(q)$ denotes the number of primitive characters mod q, $C(k)$ is a computable constant depending on k, and

$$A(m, k, p, s) = \sum_{i=0}^{2k-2} \frac{1}{p^{is}} \sum_{j=0}^{i} (-1)^j C_{2k-1}^{j} C_{k+m+i-j-1}^{i-j} C_{k+i-j-1}^{i-j}, \ \ C_m^n = \frac{m!}{n!(m-n)!}.$$

Later, χ was generalized to the non-principal. In [7], by using the properties of Dedekind sum and Cochrane sum, a sharp asymptotic formula was given as follows

$$\sum_{\chi \neq \chi_0} \left| \sum_{a < \frac{q}{4}} \chi(a) \right|^4 = \frac{21\phi^4(q)}{256q} \prod_{p^\alpha \| q} \frac{\frac{(p+1)^3}{p(p^2+1)} - \frac{1}{p^{3\alpha-1}}}{1 + \frac{1}{p} + \frac{1}{p^2}} + O(q^{2+\epsilon}),$$

where $\prod_{p^\alpha \| q}$ denotes the product over all prime divisors p of q with $p^\alpha | q$ and $p^{\alpha+1} \nmid q$.

It is interesting to consider more mean value of character sums. Due to the results above, a natural question arises: If there exists an asymptotic formula for higher moment of the sum analogues to character sums over short intervals. In this paper, we shall use the estimate for character sums and elementary methods to give a fourth power mean formula for the sum analogues to character sums. Furthermore, a relationship between the sum analogues to character sums and general Kloosterman sum are also studied.

2 Mean value of the sum analogues to character sums

Theorem 1 Let $q \geq 3$ be an odd integer, $k > 0$ be any fixed real number. Then for any real number N with $1 < N < \sqrt{q}$, we have the asymptotic formula

$$\sum_{\chi \bmod q} \left| \sum_{n \leq N} (-1)^n n^k \chi(n) \right|^4 = \frac{80 \ln N - 50 \ln 2}{(2k+1)^3 \pi^2} \cdot \frac{\phi^4(q)}{q^3} \cdot N^{4k+2} \prod_{p|q} \frac{p}{p+1} + O\left(\phi(q) \cdot N^{4k+2} \cdot 2^{\omega(q)}\right),$$

where $\omega(q)$ denotes the number of all different prime divisors of q.

Taking $k = 0$, we may immediately deduce the following:

Corollary Let $q \geq 3$ be an odd integer. Then for any real number N with $1 < N < \sqrt{q}$, we have the asymptotic formula

$$\sum_{\chi \bmod q} \left| \sum_{n \leq N} (-1)^n \chi(n) \right|^4 = \frac{80 \ln N - 50 \ln 2}{\pi^2} \cdot \frac{\phi^4(q)}{q^3} \cdot N^2 \prod_{p|q} \frac{p}{p+1} + O\left(\phi(q) \cdot N^2 \cdot 2^{\omega(q)}\right).$$

Note The method we use can be generalized to study $2s$−th$(s \geq 2)$ power mean of the sums, but the constant will be very complicate, so we do not give a general conclusion here.

To prove Theorem 1, we need a lemma:

Lemma 1 Let $q \geq 3$ be a fixed integer. Then for any real number $N > 1$, we have the asymptotic formula

$$\sum_{\substack{1 \leq n \leq N \\ (n,\, q)=1}} \frac{\phi(n)}{n^2} = \frac{6}{\pi^2} \prod_{p|q} \frac{p}{p+1} \left(\ln N + \gamma + \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} + \sum_{p|q} \frac{p \cdot \ln p}{p^2 - 1} \right) + O\left(\frac{2^{\omega(q)} \ln N}{\sqrt{N}} \right),$$

where γ is the Euler constant, $\Lambda(n)$ is the Mangoldt function, and $\sum_{p|q}$ denotes the summation over all different prime divisors of q.

Proof. Note that $\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$, where $\mu(d)$ is the Möbius function. From Theorem 4.2 of [3] we have

$$\sum_{\substack{1\le n\le N\\(n,\,q)=1}} \frac{\phi(n)}{n^2} = \sum_{\substack{md\le N\\(md,\,q)=1}} \frac{\mu(d)}{d^2\cdot m} = \sum_{\substack{d\le\sqrt{N}\\(d,\,q)=1}} \frac{\mu(d)}{d^2} \sum_{\substack{m\le\frac{N}{d}\\(m,\,q)=1}} \frac{1}{m}$$

$$+ \sum_{\substack{m\le\sqrt{N}\\(m,\,q)=1}} \frac{1}{m} \sum_{\substack{d\le\frac{N}{m}\\(d,\,q)=1}} \frac{\mu(d)}{d^2} - \left(\sum_{\substack{d\le\sqrt{N}\\(d,\,q)=1}} \frac{\mu(d)}{d^2} \right) \cdot \left(\sum_{\substack{m\le\sqrt{N}\\(m,\,q)=1}} \frac{1}{m} \right). \quad (2)$$

Note that the asymptotic formulae

$$\sum_{\substack{m\le N\\(m,\,q)=1}} \frac{1}{m} = \frac{\phi(q)}{q}\left(\ln N + \gamma + \sum_{p|q} \frac{\ln p}{p-1}\right) + O\left(\frac{2^{\omega(q)}}{N}\right), \quad (3)$$

$$\sum_{\substack{d\le N\\(d,\,q)=1}} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} \prod_{p|q} \left(1-\frac{1}{p^2}\right)^{-1} + O\left(\frac{1}{N}\right) \quad (4)$$

and

$$\sum_{\substack{d\le N\\(d,\,q)=1}} \frac{\mu(d)\ln d}{d^2} = \sum_{\substack{d=1\\(d,\,q)=1}}^{\infty} \frac{\mu(d)}{d^2}\left(-\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} + \sum_{p|q}\frac{\ln p}{p^2-1}\right)$$

$$+O\left(\frac{2^{\omega(q)}\ln N}{\sqrt{N}}\right). \quad (5)$$

Combining (2), (3), (4) and (5) we may immediately get

$$\sum_{\substack{1\le n\le N\\(n,\,q)=1}} \frac{\phi(n)}{n^2} = \sum_{\substack{d\le\sqrt{N}\\(d,\,q)=1}} \frac{\mu(d)}{d^2}\left[\frac{\phi(q)}{q}\left(\ln\frac{N}{d} + \gamma + \sum_{p|q}\frac{\ln p}{p-1}\right)\right]$$

$$+O\left(\frac{2^{\omega(q)}}{\sqrt{N}}\right) + O\left(\sum_{1\le n\le\sqrt{N}} \frac{1}{d}\cdot\frac{2^{\omega(q)}}{N}\right)$$

$$= \frac{6}{\pi^2}\prod_{p|q}\left(1-\frac{1}{p+1}\right)\left(\ln N + \gamma + \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^2} + \sum_{p|q}\frac{\ln p}{p-1} - \sum_{p|q}\frac{\ln p}{p^2-1}\right)$$

$$+O\left(\frac{2^{\omega(q)}\ln q}{\sqrt{N}}\right)$$

$$= \frac{6}{\pi^2}\prod_{p|q}\frac{p}{p+1}\left(\ln N + \gamma + \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^2} + \sum_{p|q}\frac{p\cdot\ln p}{p^2-1}\right) + O\left(\frac{2^{\omega(q)}\ln q}{\sqrt{N}}\right).$$

This proves Lemma 1.

Now we shall complete the proof of Theorem 1.

For an odd integer $q \geq 3$ we have

$$\sum_{\chi \bmod q} \left| \sum_{n \leq N} (-1)^n n^k \chi(n) \right|^4 = \sum_{\chi \bmod q} \left| \sum_{\substack{n \leq N \\ 2|n}} n^k \chi(n) - \sum_{\substack{n \leq N \\ 2\nmid n}} n^k \chi(n) \right|^4$$

$$= \sum_{\chi \bmod q} \left| 2^{k+1}\chi(2) \sum_{n \leq \frac{N}{2}} n^k \chi(n) - \sum_{n \leq N} n^k \chi(n) \right|^4$$

$$= \sum_{\chi \bmod q} \left| \sum_{n \leq N} n^k \chi(n) \right|^4 + 2^{4(k+1)} \sum_{\chi \bmod q} \left| \sum_{n \leq \frac{N}{2}} n^k \chi(n) \right|^4$$

$$-2 \cdot 2^{3k+4} \sum_{\chi \bmod q} \overline{\chi}(2) \left(\sum_{n \leq \frac{N}{2}} n^k \chi(n) \right) \cdot \left(\sum_{n \leq \frac{N}{2}} n^k \overline{\chi}(n) \right)^2 \cdot \left(\sum_{n \leq N} n^k \chi(n) \right)$$

$$+2 \cdot 2^{2(k+1)} \sum_{\chi \bmod q} \overline{\chi}(4) \left(\sum_{n \leq \frac{N}{2}} n^k \overline{\chi}(n) \right)^2 \cdot \left(\sum_{n \leq N} n^k \chi(n) \right)^2$$

$$-2 \cdot 2^{k+2} \sum_{\chi \bmod q} \overline{\chi}(2) \left(\sum_{n \leq N} n^k \chi(n) \right)^2 \cdot \left(\sum_{n \leq \frac{N}{2}} n^k \overline{\chi}(n) \right) \cdot \left(\sum_{n \leq N} n^k \overline{\chi}(n) \right)$$

$$+2^{2(k+2)} \sum_{\chi \bmod q} \left| \sum_{n \leq \frac{N}{2}} n^k \chi(n) \right|^2 \left| \sum_{n \leq N} n^k \chi(n) \right|^2 . \tag{6}$$

If $q > 2$ is even, the terms in (6) involving $\overline{\chi}(2)$ and $\overline{\chi}(4)$ will vanish.

We estimate the fourth term in (6) firstly. For any real number N with $1 \leq N \leq \sqrt{q}$, with the orthogonality relations for character modulo q we have

$$\sum_{\chi \bmod q} \overline{\chi}(4) \left(\sum_{n \leq \frac{N}{2}} n^k \overline{\chi}(n) \right)^2 \cdot \left(\sum_{n \leq N} n^k \chi(n) \right)^2$$
$$= \sum_{m \leq \frac{N}{2}} \sum_{n \leq N} \sum_{u \leq N} \sum_{v \leq \frac{N}{2}} \sum_{\chi \bmod q} (mnuv)^k \chi(\overline{4mv}nu)$$
$$= \phi(q) \sum_{\substack{m \leq \frac{N}{2} \\ 4mv = nu}}{}' \sum_{n \leq N}{}' \sum_{u \leq N}{}' \sum_{v \leq \frac{N}{2}}{}' (mnuv)^k, \tag{7}$$

where $\sum_{m \leq N}{}'$ denotes the summation over all m such that $1 \leq m \leq N$ with $(m, q) = 1$.

For $4mv = nu$, let $(m,n) = d$, then $m = m_1 d$, $n = n_1 d$ with $(m_1, n_1) = 1$. Then $4m_1 v = n_1 u$.

If $2 \nmid n_1$, let $u = 4m_1 u_1$, then $v = n_1 u_1$, we deduce that

$$\begin{aligned}
&\sum_{m\le\frac{N}{2}}{}' \sum_{\substack{n\le N\\ 4mv=nu}}{}' \sum_{u\le N}{}' \sum_{v\le\frac{N}{2}}{}' (mnuv)^k \\
\Omega: \quad &= 4^k \sum_{\substack{m\le\frac{N}{2}\\ (2m,\,n)=1}}{}' m^{2k} \sum_{n\le N}{}' n^{2k} \sum_{d\le\min\{\frac{N}{2m},\,\frac{N}{n}\}}{}' d^{2k} \sum_{u\le\min\{\frac{N}{4m},\,\frac{N}{2n}\}}{}' u^{2k}.
\end{aligned} \tag{8}$$

For convenience we split the sums over m or n into the following cases: (i) $m = n = 1$; (ii) $1 < m \le \frac{N}{2}, 1 \le n < 2m$; (iii) $1 < n \le N, 1 \le m < \frac{n}{2}$. Note that

$$\sum_{n\le N}{}' n^k = \frac{1}{k+1}\cdot\frac{\phi(q)}{q}\cdot N^{k+1} + O\left(N^k\cdot 2^{\omega(q)}\right), \tag{9}$$

so the case (i),

$$\begin{aligned}
\Omega &= 4^k \sum_{d\le\frac{N}{2}}{}' d^{2k} \sum_{u\le\frac{N}{4}}{}' u^{2k} \\
&= \frac{1}{(2k+1)^2\cdot 2^{4k+3}}\cdot\frac{\phi^2(q)}{q^2}\cdot N^{4k+2} + O\left(N^{4k+1}\cdot 2^{\omega(q)}\right).
\end{aligned}$$

For the case (ii), by Lemma 1

$$\begin{aligned}
\Omega &= 4^k \sum_{\substack{1<m\le\frac{N}{2}\\ (2m,\,n)=1}}{}' m^{2k} \sum_{1\le n<2m}{}' n^{2k} \sum_{d\le\frac{N}{2m}}{}' d^{2k} \sum_{u\le\frac{N}{4m}}{}' u^{2k} \\
&= \frac{1}{(2k+1)^2\cdot 2^{4k+3}} \sum_{\substack{1<m\le\frac{N}{2}\\ (2m,\,n)=1}}{}' m^{2k} \sum_{1\le n<2m}{}' n^{2k}\cdot\frac{\phi^2(q)}{q^2}\cdot\frac{N^{4k+2}}{m^{4k+2}} \\
&\quad +O\left(\sum_{\substack{1<m\le\frac{N}{2}\\ (2m,\,n)=1}}{}' m^{2k} \sum_{1\le n<2m}{}' n^{2k}\cdot\frac{N^{4k+1}2^{\omega}(q)}{m^{4k+1}} \right) \\
&= \frac{1}{(2k+1)^2\cdot 2^{4k+3}}\cdot\frac{\phi^2(q)}{q^2}\cdot N^{4k+2} \\
&\quad \sum_{1<m\le\frac{N}{2}}{}' \frac{1}{m^{2k+2}}\left(\frac{1}{2k+1}\cdot\frac{\phi(2mq)}{2mq}\cdot(2m)^{2k+1}\right) + O\left(N^{4k+2}\cdot 2^{\omega(q)}\right)
\end{aligned}$$

$$= \frac{1}{(2k+1)^3 \cdot 2^{2k+3}} \cdot \frac{\phi^3(q)}{q^3} \cdot N^{4k+2} \sideset{}{'}\sum_{1<m\le\frac{N}{2}} \frac{\phi(2m)}{m^2} + O\left(N^{4k+2} \cdot 2^{\omega(q)}\right)$$

$$= \frac{1}{(2k+1)^3 \cdot 2^{2k+3}} \cdot \frac{\phi^3(q)}{q^3} \cdot N^{4k+2} \left(2 \sideset{}{'}\sum_{1<m\le\frac{N}{2}} \frac{\phi(m)}{m^2} - \sideset{}{'}\sum_{\substack{1<m\le\frac{N}{2} \\ (2,\, m)=1}} \frac{\phi(m)}{m^2} \right)$$

$$+O\left(N^{4k+2} \cdot 2^{\omega(q)}\right)$$

$$= \frac{1}{(2k+1)^3 \cdot 2^{2k}\pi^2} \cdot \frac{\phi^3(q)}{q^3} \cdot N^{4k+2} \ln\frac{N}{2} \prod_{p|q} \frac{p}{p+1} + O\left(N^{4k+2} \cdot 2^{\omega(q)}\right).$$

Similarly, the case (iii) when $1 < n \le N, 1 \le m < \frac{n}{2}$

$$\Omega = 4^k \sideset{}{'}\sum_{1<n\le N} n^{2k} \sideset{}{'}\sum_{\substack{1\le m<\frac{n}{2} \\ (2m,\, n)=1}} m^{2k} \sideset{}{'}\sum_{d\le\frac{N}{n}} d^{2k} \sideset{}{'}\sum_{u\le\frac{N}{2n}} u^{2k}$$

$$= \frac{1}{(2k+1)^3 \cdot 2^{2k+2}} \cdot \frac{\phi^3(q)}{q^3} \cdot N^{4k+2} \sideset{}{'}\sum_{\substack{1<n\le N \\ (2,\, n)=1}} \frac{\phi(n)}{n^2} + O\left(N^{4k+2} \cdot 2^{\omega(q)}\right)$$

$$= \frac{1}{(2k+1)^3 \cdot 2^{2k}\pi^2} \cdot \frac{\phi^3(q)}{q^3} \cdot N^{4k+2} \ln N \prod_{p|q} \frac{p}{p+1} + O\left(N^{4k+2} \cdot 2^{\omega(q)}\right).$$

Therefore, combining the cases, we have if $2 \nmid n_1$

$$\sideset{}{'}\sum_{m\le\frac{N}{2}} \sideset{}{'}\sum_{\substack{n\le N \\ 4mv=nu}} \sideset{}{'}\sum_{u\le N} \sideset{}{'}\sum_{v\le\frac{N}{2}} (mnuv)^k$$

$$= \frac{1}{(2k+1)^3 \cdot 2^{2k}\pi^2} \cdot \frac{\phi^3(q)}{q^3} \cdot N^{4k+2} \ln\frac{N^2}{2} \prod_{p|q} \frac{p}{p+1} + O\left(N^{4k+2} \cdot 2^{\omega(q)}\right) \quad (10)$$

If $2||n_1$, that is $n_1 = 2n_2$ with $2 \nmid n_2$, let $u = 2m_1u_1$, then $v = n_2u_1$, we get

$$\phi(q) \sideset{}{'}\sum_{m\le\frac{N}{2}} \sideset{}{'}\sum_{\substack{n\le N \\ 4mv=nu}} \sideset{}{'}\sum_{u\le N} \sideset{}{'}\sum_{v\le\frac{N}{2}} (mnuv)^k$$

$$= 4^k\phi(q) \sideset{}{'}\sum_{m\le\frac{N}{2}} m^{2k} \sideset{}{'}\sum_{\substack{n\le\frac{N}{2} \\ (2,\, n)=1 \\ (m,\, 2n)=1}} n^{2k} \sideset{}{'}\sum_{d\le\min\{\frac{N}{2m},\, \frac{N}{2n}\}} d^{2k} \sideset{}{'}\sum_{u\le\min\{\frac{N}{2m},\, \frac{N}{2n}\}} u^{2k}$$

$$= 4^k\phi(q) \sideset{}{'}\sum_{d\le\frac{N}{2}} d^{2k} \sideset{}{'}\sum_{u\le\frac{N}{2}} u^{2k} + 2 \cdot 4^k\phi(q) \sideset{}{'}\sum_{1<m\le\frac{N}{2}} m^{2k} \sideset{}{'}\sum_{\substack{1\le n<m \\ (2,\, n)=1 \\ (m,\, 2n)=1}} n^{2k} \sideset{}{'}\sum_{d\le\frac{N}{2m}} d^{2k} \sideset{}{'}\sum_{u\le\frac{N}{2m}} u^{2k}$$

$$= \frac{1}{(2k+1)^3 \cdot 2^{2k}\pi^2} \cdot \frac{\phi^4(q)}{q^3} \cdot N^{4k+2} \ln\frac{N}{2} \prod_{p|q} \frac{p}{p+1} + O\left(\phi(q) \cdot N^{4k+2} \cdot 2^{\omega(q)}\right). \quad (11)$$

If $4|n_1$, that is $n_1 = 4n_3$, let $u = m_1u_1$, then $v = n_3u_1$, we also get

$$
\begin{aligned}
&\phi(q) \sideset{}{'}\sum_{m\le \frac{N}{2}} \sideset{}{'}\sum_{\substack{n\le N\\ 4mv=nu}} \sideset{}{'}\sum_{u\le N} \sideset{}{'}\sum_{v\le \frac{N}{2}} (mnuv)^k \\
&= 4^k\phi(q) \sideset{}{'}\sum_{m\le \frac{N}{2}} m^{2k} \sideset{}{'}\sum_{\substack{n\le \frac{N}{4}\\ (m,\,4n)=1}} n^{2k} \sideset{}{'}\sum_{d\le \min\{\frac{N}{2m},\, \frac{N}{4n}\}} d^{2k} \sideset{}{'}\sum_{u\le \min\{\frac{N}{m},\, \frac{N}{2n}\}} u^{2k} \\
&= 4^k\phi(q) \sideset{}{'}\sum_{d\le \frac{N}{4}} d^{2k} \sideset{}{'}\sum_{u\le \frac{N}{2}} u^{2k} + 4^k\phi(q) \sideset{}{'}\sum_{\substack{1<m\le \frac{N}{2}\\ (2,\,m)=1}} m^{2k} \sideset{}{'}\sum_{\substack{1\le n<m\\ (m,\,n)=1}} n^{2k} \sideset{}{'}\sum_{d\le \frac{N}{2m}} d^{2k} \sideset{}{'}\sum_{u\le \frac{N}{m}} u^{2k} \\
&\quad +4^k\phi(q) \sideset{}{'}\sum_{1<n\le \frac{N}{4}} n^{2k} \sideset{}{'}\sum_{\substack{1\le m<2n\\ (m,\,4n)=1}} n^{2k} \sideset{}{'}\sum_{d\le \frac{N}{4n}} d^{2k} \sideset{}{'}\sum_{u\le \frac{N}{2n}} u^{2k} \\
&= \frac{1}{(2k+1)^3\cdot 2^{2k}\pi^2}\cdot \frac{\phi^4(q)}{q^3}\cdot N^{4k+2}\ln\frac{N^2}{8}\prod_{p|q}\frac{p}{p+1} \\
&\qquad\qquad +O\left(\phi(q)\cdot N^{4k+2}\cdot 2^{\omega(q)}\right). \qquad (12)
\end{aligned}
$$

Combining (10), (11) and (12), we may immediately obtain

$$
\begin{aligned}
&\sum_{\chi \bmod q} \overline{\chi}(4) \left(\sum_{n\le N/2} n^k\overline{\chi}(n)\right)^2 \cdot \left(\sum_{n\le N} n^k\chi(n)\right)^2 \\
&= \frac{1}{(2k+1)^3\cdot 2^{2k}\pi^2}\cdot \frac{\phi^4(q)}{q^3}\cdot N^{4k+2}\ln\frac{N^5}{32}\prod_{p|q}\frac{p}{p+1} + O\left(\phi(q)\cdot N^{4k+2}\cdot 2^{\omega(q)}\right).
\end{aligned}
$$

Similar methods can be used to estimate the other terms in (6), we deduce that

$$
\begin{aligned}
\sum_{\chi \bmod q} \left|\sum_{n\le N} n^k\chi(n)\right|^4 &= \frac{12}{(2k+1)^3\pi^2}\cdot \frac{\phi^4(q)}{q^3}\cdot N^{4k+2}\ln N\prod_{p|q}\frac{p}{p+1} \\
&\quad +O\left(\phi(q)\cdot N^{4k+2}\cdot 2^{\omega(q)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\chi \bmod q} \left|\sum_{n\le N/2} n^k\chi(n)\right|^4 &= \frac{3}{(2k+1)^3\cdot 2^{4k}\pi^2}\cdot \frac{\phi^4(q)}{q^3}\cdot N^{4k+2}\ln\frac{N}{2}\prod_{p|q}\frac{p}{p+1} \\
&\quad +O\left(\phi(q)\cdot N^{4k+2}\cdot 2^{\omega(q)}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\chi \bmod q} \overline{\chi}(2) \left(\sum_{n\le N/2} n^k\chi(n)\right) \cdot \left(\sum_{n\le N/2} n^k\overline{\chi}(n)\right)^2 \cdot \left(\sum_{n\le N} n^k\chi(n)\right) \\
&= \frac{1}{(2k+1)^3\cdot 2^{3k}\pi^2}\cdot \frac{\phi^4(q)}{q^3}\cdot N^{4k+2}\ln\frac{N^4}{32}\prod_{p|q}\frac{p}{p+1} + O\left(\phi(q)\cdot N^{4k+2}\cdot 2^{\omega(q)}\right),
\end{aligned}
$$

$$\sum_{\chi \bmod q} \overline{\chi}(2) \left(\sum_{n \le N} n^k \chi(n) \right)^2 \cdot \left(\sum_{n \le N/2} n^k \overline{\chi}(n) \right) \cdot \left(\sum_{n \le N} n^k \overline{\chi}(n) \right)$$
$$= \frac{1}{(2k+1)^3 \cdot 2^k \pi^2} \cdot \frac{\phi^4(q)}{q^3} \cdot N^{4k+2} \ln \frac{N^8}{64} \prod_{p|q} \frac{p}{p+1} + O\left(\phi(q) \cdot N^{4k+2} \cdot 2^{\omega(q)} \right)$$

and

$$\sum_{\chi \bmod q} \left| \sum_{n \le N/2} n^k \chi(n) \right|^2 \left| \sum_{n \le N} n^k \chi(n) \right|^2$$
$$= \frac{6}{(2k+1)^3 \cdot 2^{2k} \pi^2} \cdot \frac{\phi^4(q)}{q^3} \cdot N^{4k+2} \ln \frac{N}{2} \prod_{p|q} \frac{p}{p+1} + O\left(\phi(q) \cdot N^{4k+2} \cdot 2^{\omega(q)} \right).$$

Finally, we obtain

$$\sum_{\chi \bmod q} \left| \sum_{n \le N} (-1)^n n^k \chi(n) \right|^4$$
$$= \frac{80 \ln N - 50 \ln 2}{(2k+1)^3 \pi^2} \cdot \frac{\phi^4(q)}{q^3} \cdot N^{4k+2} \prod_{p|q} \frac{p}{p+1} + O\left(\phi(q) \cdot N^{4k+2} \cdot 2^{\omega(q)} \right).$$

This completes the proof of Theorem 1.

3 A hybrid mean value with general Kloosterman sum

Let $q \ge 3$ be a positive integer. For any integers r and s, the classical Kloosterman sums $K(r,s;q)$ are defined as follows:

$$K(r,s;q) = \sum_{a \le q}' \mathrm{e}\left(\frac{ra + s\bar{a}}{q} \right),$$

where $a\bar{a} \equiv 1 (\bmod q)$ and $\mathrm{e}(y) = \mathrm{e}^{2\pi i y}$. Perhaps, the most famous property of $K(r,s;q)$ is the estimate (see [8])

$$|K(r,s;q)| \le d(q) q^{\frac{1}{2}} (r,s,q)^{\frac{1}{2}}, \tag{13}$$

where $d(q)$ is the divisor function, and (r,s,q) denotes the greatest common divisor of r, s and q.

The general Kloosterman sums $K(r,s,\chi;q)$ is defined as follows

$$K(r,s,\chi;q) = \sum_{a \le q} \chi(a) \mathrm{e}\left(\frac{ra + s\bar{a}}{q} \right).$$

This summation is very important, because the classical Kloosterman sums is a special case of the general Kloosterman sums when χ is a principal character modulo q. For an arbitrary composite number q, we do not know how large

$K(r,s,\chi;q)$ is. In fact, the value of $K(r,s,\chi;q)$ is quite irregular when q is not a prime. However, it is surprising that $K(r,s,\chi;q)$ enjoys many good distribution properties. Zhang [9] got an identity for the fourth power mean of the general Kloosterman sums

$$\sum_{\chi \bmod q} \sum_{m=1}^{q} |K(m,n,\chi;q)|^4 = \phi^2(q)q^2 d(q) \prod_{p^\alpha \| q} \left(1 - \frac{2}{\alpha+1} \cdot \frac{p^{\alpha-1}-1}{p^\alpha(p-1)} + \frac{\alpha - 4p^{\alpha-1}}{(\alpha+1)p^\alpha}\right).$$

Zhang [10] proved that

$$\sum_{\chi \neq \chi_0}^{*} |K(m,n,\chi;q)|^2 \cdot \left|\sum_{x<\frac{q}{4}} \chi(x)\right|^4 = \frac{21}{256} J(q)\phi(q)q^2 \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^{\frac{7}{2}+\epsilon}).$$

It is natural to consider the mean value property of the sum analogues to character sums with general Kloosterman sums. In this section, we try to give an asymptotic formula for them and prove the following theorem.

Theorem 2 Let $q \geq 3$ be an odd integer. Then for any real number N with $1 < N < q^{\frac{1}{6}}$ and integers r and s with $(rs,q) = 1$, we have the asymptotic formula

$$\begin{aligned} &\sum_{\chi \bmod q} |K(m,n,\chi;q)|^2 \cdot \left|\sum_{n\leq N} (-1)^n n^k \chi(n)\right|^4 \\ &= \frac{80 \ln N - 50 \ln 2}{(2k+1)^3 \pi^2} \cdot \frac{\phi^5(q)}{q^3} \cdot N^{4k+2} \prod_{p|q} \frac{p}{p+1} + O\left(\phi(q) \cdot N^{4k+2} \cdot 2^{\omega(q)}\right) \\ &\quad + O(d^2(q)\phi^4(q)q^{-\frac{5}{2}} \cdot N^{4k+5}). \end{aligned}$$

To prove Theorem 2, we need the following lemma

Lemma 2 Let $q \geq 3$ be an integer. Then for any real number N with $1 < N < q^{\frac{1}{6}}$ and integers r and s with $(rs,q) = 1$, we have

$$\begin{aligned} &\sum_{\chi \bmod q} \chi(a) \left[\sum_{a=2}^{q} \sideset{}{'}\sum_{b=1}^{q} e\left(\frac{rb(a-1) + s\bar{b}(\bar{a}-1)}{q}\right)\right] \cdot \left|\sum_{n\leq N} (-1)^n n^k \chi(n)\right|^4 \\ &\ll d^2(q)\phi^4(q)q^{-\frac{5}{2}} \cdot N^{4k+5}. \end{aligned}$$

Proof. With the orthogonality relations for characters, we deduce that

$$\sum_{\chi \bmod q} \chi(a) \left[\sum_{a=2}^{q} {\sum_{b=1}^{q}}' e\left(\frac{rb(a-1)+s\bar{b}(\bar{a}-1)}{q} \right) \right] \cdot \left| \sum_{n\le N} (-1)^n n^k \chi(n) \right|^4$$

$$\ll \sum_{a=2}^{q} {\sum_{b=1}^{q}}' e\left(\frac{rb(a-1)+s\bar{b}(\bar{a}-1)}{q} \right) \sum_{\chi \bmod q} \chi(a) \left| \sum_{n\le N} n^k \chi(n) \right|^4$$

$$= \phi(q) \sum_{a=2}^{q} {\sum_{b=1}^{q}}' e\left(\frac{rb(a-1)+s\bar{b}(\bar{a}-1)}{q} \right) \mathop{{\sum}' {\sum}' {\sum}' {\sum}'}_{\substack{m\le N\ n\le N\ u\le N\ v\le N \\ anv\equiv mu (\bmod q)}} (mnuv)^k$$

$$= \phi(q) \mathop{{\sum}' {\sum}' {\sum}' {\sum}'}_{\substack{m\le N\ n\le N\ u\le N\ v\le N \\ nv\neq mu}} (mnuv)^k {\sum_{b=1}^{q}}' e\left(\frac{rb(mu\overline{nv}-1)+s\bar{b}(\overline{mu}nv-1)}{q} \right). \quad (14)$$

Noting that for any integer c with $(c,\ q) = 1$

$$(c-1,\ q) = (c\bar{c}-c,\ q) = (c(\bar{c}-1),\ q) = (\bar{c}-1,\ q),$$

and

$$(r(c-1),\ s(\bar{c}-1),\ q) = (c-1,\ q)$$

when $(rs,\ q) = 1$. From (13) we get

$$\phi(q) \mathop{{\sum}' {\sum}' {\sum}' {\sum}'}_{\substack{m\le N\ n\le N\ u\le N\ v\le N \\ nv\neq mu}} (mnuv)^k {\sum_{b=1}^{q}}' e\left(\frac{rb(mu\overline{nv}-1)+s\bar{b}(\overline{mu}nv-1)}{q} \right)$$

$$\ll \phi(q)d(q)q^{\frac{1}{2}} \mathop{{\sum}' {\sum}' {\sum}' {\sum}'}_{\substack{m\le N\ n\le N\ u\le N\ v\le N \\ nv\equiv mu (\bmod d) \\ nv\neq mu}} (mnuv)^k (mu\overline{nv}-1,\ q)^{\frac{1}{2}}$$

$$= \phi(q)d(q)q^{\frac{1}{2}} \sum_{d|q} d^{\frac{1}{2}} \mathop{{\sum}' {\sum}' {\sum}' {\sum}'}_{\substack{m\le N\ n\le N\ u\le N\ v\le N \\ nv\equiv mu (\bmod d) \\ nv\neq mu}} (mnuv)^k. \quad (15)$$

Then we estimate the inner summations of (15)

$$\mathop{{\sum}' {\sum}' {\sum}' {\sum}'}_{\substack{m\le N\ n\le N\ u\le N\ v\le N \\ nv\equiv mu (\bmod d) \\ nv\neq mu}} (mnuv)^k = {\sum_{n\le N}}' n^k {\sum_{u\le N}}' {\sum_{v\le N}}' v^k \sum_{l=1}^{\left[\frac{uN-nv}{d}\right]} (nv+ld)^k$$

$$= \sum_{i=1}^{k} C_k^i {\sum_{n\le N}}' n^{k+i} {\sum_{u\le N}}' {\sum_{v\le N}}' v^{k+i} \sum_{l=1}^{\left[\frac{uN-nv}{d}\right]} (ld)^{k-i}$$

$$\le \sum_{i=1}^{k} C_k^i {\sum_{n\le N}}' n^{k+i} {\sum_{u\le N}}' {\sum_{v\le N}}' v^{k+i} d^{k-i} \left(\frac{uN}{d}\right)^{k-i+1} \ll \frac{1}{d} \frac{\phi^3(q)}{q^3} \cdot N^{4k+5}, \quad (16)$$

where we have used (9). Combining (15) and (16), we get Lemma 2.

Now we prove Theorem 2.

Let $q \geq 3$ be an odd integer. Then for any real number N with $1 < N < q^{\frac{1}{6}}$ and integers r and s with $(rs,\ q) = 1$, from the properties of residue systems we have

$$
\begin{aligned}
|K(r,s,\chi;q)|^2 &= \sum_{a=1}^{q} \chi(a) \mathrm{e}\left(\frac{ra+s\bar{a}}{q}\right) \overline{\chi}(b) \mathrm{e}\left(-\frac{rb+s\bar{b}}{q}\right) \\
&= \sum_{a=1}^{q} \sum_{b=1}^{q}{}' \chi(a) \mathrm{e}\left(\frac{rb(a-1)+s\bar{b}(\bar{a}-1)}{q}\right) \\
&= \phi(q) + \sum_{a=2}^{q} \sum_{b=1}^{q}{}' \chi(a) \mathrm{e}\left(\frac{rb(a-1)+s\bar{b}(\bar{a}-1)}{q}\right).
\end{aligned}
$$

With Theorem 1 and Lemma 2,

$$
\begin{aligned}
&\sum_{\chi \bmod q} |K(r,s,\chi;q)|^2 \left| \sum_{n \leq N} (-1)^n n^k \chi(n) \right|^4 \\
&= \phi(q) \left| \sum_{n \leq N} (-1)^n n^k \chi(n) \right|^4 + \sum_{a=2}^{q} \sum_{b=1}^{q}{}' \mathrm{e}\left(\frac{rb(a-1)+s\bar{b}(\bar{a}-1)}{q}\right) \\
&\quad \times \sum_{\chi \bmod q} \chi(a) \left| \sum_{n \leq N} (-1)^n n^k \chi(n) \right|^4 \\
&= \frac{80 \ln N - 50 \ln 2}{(2k+1)^3 \pi^2} \cdot \frac{\phi^5(q)}{q^3} \cdot N^{4k+2} \prod_{p|q} \frac{p}{p+1} + O\left(\phi(q) \cdot N^{4k+2} \cdot 2^{\omega(q)}\right) \\
&\quad + O(d^2(q) \phi^4(q) q^{-\frac{5}{2}} \cdot N^{4k+5}).
\end{aligned}
$$

This completes the proof of Theorem 2.

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