Uniqueness of meromorphic functions sharing four values IM and one set in an angular domain*

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Abstract

In this paper, we investigate the uniqueness problem of meromorphic functions sharing four distinct values IM and a finite set in an angular domain. One result which we obtained generalizes and extends the former results.

1 Introduction and main results

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}(=\mathbb{C}\bigcup\{\infty\})$ to denote the extended complex plane, and $X(\subset \mathbb{C})$ to denote an angular domain, a transcendental meromorphic function is meromorphic in the whole complex plane $\mathbb{C}$ and not rational. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f), m(r,f), N(r,f)$ and so on, that can be found, for instance, in [5,15].

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $X \subseteq \mathbb{C}$. Define

$$E_X(S,f) = \bigcup_{a\in S}\{z\in X | f_a(z) = 0, \quad \textit{counting} \quad \textit{multiplicities}\},$$

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$$\overline{E}_X(S,f)=\bigcup_{a\in S}\{z\in X|f_a(z)=0,\quad ignoring\quad multiplicities\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

Let f and g be two non-constant meromorphic functions in $\mathbb{C}$. If $E_X(S,f)=E_X(S,g)$, we say f and g share the set S CM(counting multiplicities) in X. If $\overline{E}_X(S,f)=\overline{E}_X(S,g)$, we say f and g share the set S IM(ignoring multiplicities) in X. In particular, when $S=\{a\}$, where $a\in\widehat{\mathbb{C}}$, we say f and g share the value a CM in X if $E_X(S,f)=E_X(S,g)$, and we say f and g share the value a IM in X if $\overline{E}_X(S,f)=\overline{E}_X(S,g)$. When $X=\mathbb{C}$, we give the simple notation as before,$E(S,f),\overline{E}(S,f)$ and so on(see [13]). In addition, if f and g share a IM in X such that the same zeros of $f-a$ and $g-a$ have different multiplicities, it is said that f and g share a DM in X.

R.Nevanlinna(see [9]) proved the following well-known theorems.

Theorem 1.1. (see [9]) *If f and g are two non-constant meromorphic functions that share five distinct values a_1,a_2,a_3,a_4,a_5 IM in $X=\mathbb{C}$, then $f(z)\equiv g(z)$.*

Theorem 1.2. (see [9]) *If f and g are two distinct non-constant meromorphic functions that share four distinct values a_1,a_2,a_3,a_4 CM in $X=\mathbb{C}$, then f is a Möbius transformation of g , two of the shared values, say a_1 and a_2 are Picard values, and the cross ratio $(a_1,a_2,a_3,a_4)=-1$.*

In 1993, M. Reinders [10] investigated the uniqueness problem of non-constant meromorphic functions sharing four distinct values IM and obtained the following result.

Theorem 1.3. (see [10]) *Let f and g be distinct non-constant meromorphic functions sharing four distinct values $a_j(j=1,2,3,4)$ IM. If there exists $a,b\in\widehat{\mathbb{C}}\setminus\{a_1,a_2,a_3,a_4\}$ such that $f(z)=a\Longrightarrow g(z)=b$, then either $f(z)$ and $g(z)$ are the function $f=L\circ\hat{f}\circ h$ and $g=L\circ\hat{g}\circ h$ or $f=T\circ g$, where L,T are Möbius transformation, h is a non-constant entire function and*

$$\hat{f}=\frac{e^z+1}{(e^z-1)^2},\quad \hat{g}=\frac{(e^z+1)^2}{8(e^z-1)}.$$

After their very work, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many investigations (see [13]). In [16], Zheng studied the uniqueness problem under the condition that five values are shared in some angular domain in $\mathbb{C}$. It is an interesting topic to investigate the uniqueness with shared values in the remaining part of the complex plane removing an unbounded closed set, see [1,2,6,7,8,11,12,16,17]. Zheng J.H. [15], Cao T.B. and Yi H.X. [2], Xu J.F. and Yi H.X. [12] continued to investigate the uniqueness of meromorphic functions sharing five values and four values, Lin W.C., Mori S. and Tohge K. [6] and Lin W.C., Mori S. and Yi H.X. [7] investigated the uniqueness of meromorphic and entire functions sharing sets in an angular domain. They obtained some important results. To state the next results, we require the following basic notations and definitions of meromorphic functions in an angular domain(see [5,16,17]).

Let f be a meromorphic function on the angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq \arg z \leq \beta\}$ and $0 < \beta - \alpha \leq 2\pi$. Define

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\{\log^+|f(te^{i\alpha})| + \log^+|f(te^{i\beta})|\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f) = \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$

$$C_{\alpha,\beta}(r,f) = 2\sum_{1<|b_\mu|<r}\left(\frac{1}{|b_\mu|^\omega} - \frac{|b_\mu|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_\mu - \alpha),$$

$$D_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f), \quad S_{\alpha,\beta}(r,f) = D_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f),$$

where $\omega = \frac{\pi}{\beta-\alpha}$ and $b_\mu = |b_\mu|e^{i\theta_\mu}(\mu = 1,2,\cdots)$ are the poles of f on $\overline{\Omega}(\alpha,\beta)$ counted according to their multiplicities. $S_{\alpha,\beta}(r,f)$ is called the Nevanlinna's angular characteristic, and $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\overline{\Omega}(\alpha,\beta)$, and $\overline{C}_{\alpha,\beta}(r,f)$ is the reduced function of $C_{\alpha,\beta}(r,f)$. Similarly, when $a \neq \infty$, we will use the notations $A_{\alpha,\beta}(r,\frac{1}{f-a})$, $B_{\alpha,\beta}(r,\frac{1}{f-a})$, $C_{\alpha,\beta}(r,\frac{1}{f-a})$, $S_{\alpha,\beta}(r,\frac{1}{f-a})$ and so on.

In 2008, Cao and Yi [1] investigated the problem of two transcendental analytic functions f,g sharing three values DM in an angular domain and obtained the following result.

Theorem 1.4. (see [1, Theorem 1.]) *There are no two distinct transcendental analytic functions f and g that share three distinct values a_1, a_2, a_3 DM in one angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$, provided that*

$$\lim_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log(rT(r,f))} = \infty, \quad (r \notin E).$$

Theorem 1.5. (see [2, Theorem 1.3.]) *Let f and g be two transcendental meromorphic functions. Given one angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leq 2\pi$, we assume that f and g share five distinct values $a_j(j = 1,2,3,4,5)$ IM in X. Then $f(z) \equiv g(z)$, provided that*

$$\lim_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log(rT(r,f))} = \infty, \quad (r \notin E).$$

We may denote Theorem 1.5 by $5IM$ theorem.

Zheng J.H. [17] raised the question: Does $2CM + 2IM = 4CM$ hold?

In 2009, one of the authors of this paper dealt with the above question and obtained the following result.

Theorem 1.6. (see [2, Corollary 1.1.]) *Let f and g be two distinct transcendental meromorphic functions. Given one angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leq 2\pi$, we assume that f and g share two distinct values $a_j(j = 1,2)$ CM and another two distinct values a_3, a_4 IM in X. Then a_1, a_2, a_3, a_4 are shared CM in X of f and g, provided that*

$$\lim_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log(rT(r,f))} = \infty, \quad (r \notin E).$$

For the case where three values are shared CM and one IM was investigated by Zheng [17]. Zheng [17] obtained one Theorem which we denoted by a simple notation $3CM + 1IM = 4CM$.

In 2009, one of the authors of this paper investigated that f, g share four values IM in one angular domain and obtained an analogous result as Theorem 1.2 in one angular domain.

Theorem 1.7. (see [2, Theorem 1.5.]) *Let f and g be two distinct transcendental meromorphic functions. Given one angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \le 2\pi$, we assume that f and g share four distinct values $a_j (j = 1, 2, 3, 4)$ CM in X, and that*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty, \quad (r \notin E).$$

Then f is a Möbius transformation of g , two of the shared values, say a_1 and a_2 are Picard values in X, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.

It is a natural question to ask: What had happened when f and g share four distinct values $a_j (j = 1, 2, 3, 4)$ IM and satisfy that $f(z) = a \Longrightarrow g(z) = b$ in one angular domain where $a, b \in \widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3, a_4\}$ for Theorem 1.3?

In this paper, we will deal with a more general form of the above question and obtain the following result.

Theorem 1.8. *Let f and g be two transcendental meromorphic functions. Given one angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \le 2\pi$, we assume that f and g share four distinct values $a_j (j = 1, 2, 3, 4)$ IM in X and $\overline{E}_X(S, f) \subset \overline{E}_X(S, g)$, where $S = \{b_1, \ldots, b_m\}, m \ge 1$ and $b_1, \ldots, b_m \in \widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3, a_4\}$, and that*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r))} = \infty, \quad (r \notin E), \tag{1}$$

where $T(r) = \max\{T(r, f), T(r, g)\}$. Then f and g share all values CM, thus it follows that either $f \equiv g$ or f is a Möbius transformation of g. Furthermore, if the number of the values in S is odd, then $f \equiv g$.

Remark 1.1. *The special case $m = 1$ of this Theorem immediately yields Theorem 1.5. In fact, when $m = 1$, set $S = \{a_5\}$. If f, g share a_5 IM, which implies $\overline{E}_X(S, f) \subset \overline{E}_X(S, g)$, then by Theorem 1.8, we can get $f \equiv g$.*

2 Some Lemmas

To prove our result, we require the following Lemmas.

Lemma 2.1. (see [4,14].) *Let f be a nonconstant meromorphic function on $\overline{\Omega}(\alpha, \beta)$. Then for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\left(r, \frac{1}{f - a}\right) = S_{\alpha,\beta}(r, f) + \varepsilon(r, a),$$

where $\varepsilon(r, a) = O(1)$ as $r \to \infty$.

Lemma 2.2. (see [4,P138].) *Let f be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. Given one angular domain on $\overline{\Omega}(\alpha,\beta)$. Then for any $1 \le r < R$, we have*

$$A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) \le K\left\{\left(\frac{R}{r}\right)^{\omega}\int_1^R \frac{\log^+ T(r,f)}{t^{1+\omega}}dt + \log^+\frac{r}{R-r} + \log\frac{R}{r} + 1\right\},$$

and

$$B_{\alpha,\beta}\left(r, \frac{f'}{f}\right) \le \frac{4\omega}{r^{\omega}} m\left(r, \frac{f'}{f}\right),$$

where $\omega = \frac{\pi}{\beta-\alpha}$ and K is a positive constant not depending on r and R.

Remark 2.1. *Nevanlinna conjectured that*

$$D_{\alpha,\beta}\left(r, \frac{f'}{f}\right) = A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f}\right) = o\left(S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right)\right) \tag{2}$$

when r tends to $+\infty$ outside an exceptional set of finite linear measure, and he proved that $D_{\alpha,\beta}\left(r, \frac{f'}{f}\right) = O(1)$ when the function f is meromorphic in $\mathbb{C}$ and has finite order. In 1974, Gol'dberg[3] constructed a counter-example to show that (2) is not valid (see [17]). However, it follows from Lemma 2.2, that

$$D_{\alpha,\beta}\left(r, \frac{f'}{f}\right) = A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f}\right) = R(r,f),$$

where $R(r,f) = O\{\log(rT(r,f))\}$ as $r \to \infty (r \notin E)$ and E is a set with finite linear measure.

Throughout the paper, we denote by $R(r,*)$ quantities satisfying

$$R(r,*) = O(\log(rT(r,*))), \quad r \notin E,$$

where E is a set with finite linear measure.

Remark 2.2. *From the definition of $A_{\alpha,\beta}(r,f), B_{\alpha,\beta}(r,f), C_{\alpha,\beta}(r,f), D_{\alpha,\beta}(r,f), S_{\alpha,\beta}(r,f)$ and Lemma 2.1 and Lemma 2.2, we can see that the properties of $C_{\alpha,\beta}(r,f), D_{\alpha,\beta}(r,f)$ and $S_{\alpha,\beta}(r,f)$ are the same as for the more familiar quantities $N(r,f), m(r,f)$ and $T(r,f)$, respectively.*

Lemma 2.3. (see [1, Lemma 1].) *Suppose that f is a non-constant meromorphic function and that $X = \{z : \alpha < \arg z < \beta\}$ is an angular domain, where $0 < \beta - \alpha \le 2\pi$. Let $P(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_p (a_0 \ne 0)$ be a polynomial of f with degree p, where the coefficients $a_j (j = 0, 1, \ldots, p)$ are constants, and let $b_j (j = 1, 2, \ldots, q)$ be $q (q \ge p+1)$ distinct finite complex numbers. Then*

$$D_{\alpha,\beta}\left(r, \frac{P(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) = R(r,f).$$

Lemma 2.4. (see [1, Lemma 2].) *Let f and g be two distinct transcendental meromorphic functions that share four distinct values $a_j (j = 1,2,3,4)$ IM in one angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \le 2\pi$. Then*

(i) *$S_{\alpha,\beta}(r,f) = S_{\alpha,\beta}(r,g) + R(r,f), S_{\alpha,\beta}(r,g) = S_{\alpha,\beta}(r,f) + R(r,g)$;*

(ii) *$\sum_{j=1}^{4} \overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j}) = 2S_{\alpha,\beta}(r,f) + R(r,f)$;*

(iii) *$\overline{C}_{\alpha,\beta}(r, \frac{1}{f-b}) = S_{\alpha,\beta}(r,f) + R(r,f)$, $\overline{C}_{\alpha,\beta}(r, \frac{1}{g-b}) = S_{\alpha,\beta}(r,g) + R(r,g)$, where $b \neq a_j (j = 1,2,3,4)$ and $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-b}) = \overline{C}_{\alpha,\beta}(r,f)$ when $b = \infty$;*

(iv) *$C^*_{\alpha,\beta}(r, \frac{1}{f'}) = R(r,f), C^*_{\alpha,\beta}(r, \frac{1}{g'}) = R(r,g)$, where $C^*_{\alpha,\beta}(r, \frac{1}{f'})$ and $C^*_{\alpha,\beta}(r, \frac{1}{g'})$ are respectively the counting functions of the zeros of f' that are not zeros of $f - a_j (j = 1,2,3,4)$, and the zeros of g' that are not zeros of $g - a_j (j = 1,2,3,4)$;*

(v) *$\sum_{j=1}^{4} C^{**}_{\alpha,\beta}(r, f(z) = a_j = g(z)) = R(r.f)$, where $C^{**}_{\alpha,\beta}(r, f(z) = a_j = g(z))$ is the counting function for common multiple zeros of $f - a_j$ and $g - a_j (j = 1,2,3,4)$, counting the smaller one of the two multiplicities at each of the points.*

Lemma 2.5. *Under the assumption of Lemma 2.4. Let*

$$\varphi = \frac{f'g'(f-g)^2}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)(g-a_1)(g-a_2)(g-a_3)(g-a_4)}. \quad (3)$$

Then $S_{\alpha,\beta}(r,\varphi) = R(r,f) + R(r,g)$.

Proof. Suppose $z_0 \in X$ and $f(z_0) = a_1$ (or a_2, a_3, a_4) with multiplicity p and $g(z_0) = a_1$ (or a_2, a_3, a_4) with multiplicity q. From (3), we can get

$$\varphi(z) = O\left((z - z_0)^{2\min(p,q)-2}\right). \quad (4)$$

Hence φ is an analytic function in X. By Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
S_{\alpha,\beta}(r,\varphi) &= D_{\alpha,\beta}(r,\varphi) \\
&\leq D_{\alpha,\beta}\left(r, \frac{f'}{(f-a_2)(f-a_3)(f-a_4)}\right) \\
&\quad + D_{\alpha,\beta}\left(r, \frac{f'}{(f-a_1)(f-a_2)(f-a_3)}\right) \\
&\quad + D_{\alpha,\beta}\left(r, \frac{f'}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)}\right) \\
&\quad + D_{\alpha,\beta}\left(r, \frac{f'P_1(f)}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)}\right) \\
&\quad + D_{\alpha,\beta}\left(r, \frac{g'}{(g-a_2)(g-a_3)(g-a_4)}\right) \\
&\quad + D_{\alpha,\beta}\left(r, \frac{g'}{(g-a_1)(g-a_2)(g-a_3)}\right) \\
&\quad + D_{\alpha,\beta}\left(r, \frac{g'}{(g-a_1)(g-a_2)(g-a_3)(g-a_4)}\right) \\
&\quad + D_{\alpha,\beta}\left(r, \frac{g'P_2(g)}{(g-a_1)(g-a_2)(g-a_3)(g-a_4)}\right) + O(1) \\
&= R(r,f) + R(r,g),
\end{aligned}$$

where $P_1(f)$ is a polynomial of degree no more than 2 in f and $P_2(g)$ is a polynomial of degree no more than 2 in g.

Thus, we complete the proof of this lemma. ■

3 Proof of Theorem 1.8

Proof. Suppose that $f \not\equiv g$ and none of the $a_j (j = 1, 2, 3, 4)$ is ∞. Let φ be the function expressed in Lemma 2.5. Then $\varphi \not\equiv 0$. From (1), we have $R(r, f) = O(\log(rT(r, f))) = o(S_{\alpha,\beta}(r, f))$ and $R(r, g) = O(\log(rT(r, g))) = o(S_{\alpha,\beta}(r, f))$. Hence, we have $R(r) = o(S_{\alpha,\beta}(r, f))$, where $R(r) = \max\{R(r, f), R(r, g)\}$. By Lemma 2.4 (iii), we have

$$D_{\alpha,\beta}\left(r, \frac{1}{f - b_j}\right) = R(r, f), \quad D_{\alpha,\beta}\left(r, \frac{1}{g - b_j}\right) = R(r, g), \tag{5}$$

for any $b_j \in S (j = 1, 2, \ldots, m)$.

Set

$$\varphi_1 := \frac{(g - b_1) \cdots (g - b_m)}{(f - b_1) \cdots (f - b_m)} \cdot \left(\frac{g'(f - g)}{(g - a_1) \cdots (g - a_4)}\right)^m$$

and

$$\varphi_2 := \frac{(f - b_1) \cdots (f - b_m)}{(g - b_1) \cdots (g - b_m)} \cdot \left(\frac{f'(f - g)}{(f - a_1) \cdots (f - a_4)}\right)^m.$$

By Lemma 2.3 and (5), we can get that

$$D_{\alpha,\beta}\left(r, \frac{1}{f - b_j} \cdot \frac{g'(f - g)(g - b_j)}{(g - a_1) \cdots (g - a_4)}\right) = R(r)$$

and

$$D_{\alpha,\beta}\left(r, \frac{1}{g - b_j} \cdot \frac{f'(f - g)(f - b_j)}{(f - a_1) \cdots (f - a_4)}\right) = R(r).$$

From the definitions of φ_1 and φ_2, we have $D_{\alpha,\beta}(r, \varphi_j) = R(r), j = 1, 2$. By Lemma 2.4(iii), we see that "almost all" of poles and b_j-points of f and g in the angular domain X are simple. Since f, g share the four distinct values $a_j, j = 1, 2, 3, 4$ in the angular domain X and $\overline{E}_X(S, f) \subset \overline{E}_X(S, g)$, we can easily get that $C_{\alpha,\beta}(r, \varphi_1) = R(r)$. Therefore, we have

$$S_{\alpha,\beta}(r, \varphi_1) = R(r). \tag{6}$$

Since $\varphi_1\varphi_2 \equiv \varphi^m$, we can have

$$S_{\alpha,\beta}(r, \varphi_2) = R(r). \tag{7}$$

Let $S_X^{pq}(a_j)$ be the set of those a_j-points of f and g in the angular domain X such that the multiplicities of f and g at these points are p and q, respectively. For any $z_0 \in S_X^{pq}(a_1)$, by simple computation, we have

$$\varphi_1(z_0) = \left(q \cdot \frac{f'(z_0) - g'(z_0)}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)}\right)^m$$

and

$$\varphi_2(z_0) = \left(p \cdot \frac{f'(z_0) - g'(z_0)}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)}\right)^m.$$

Hence

$$\frac{1}{q^m}\varphi_1(z_0) - \frac{1}{p^m}\varphi_2(z_0) = 0. \tag{8}$$

Similarly, we can see that (8) holds for any $z_0 \in S_X^{pq}(a_j), j = 2, 3, 4$.

Now we discuss two cases as follows.

Case 1. Suppose that $\varphi^{pq} := \frac{1}{q^m}\varphi_1 - \frac{1}{p^m}\varphi_2 \not\equiv 0$, for all positive integers p, q.

For the sake of convenience, we denote by $C_{\alpha,\beta}^{pq}(r, \frac{1}{f-a_j})$ the counting function of f in X with respect to the set $S_X^{pq}(a_j)$, denote by $\overline{C}_{\alpha,\beta}^{pq}(r, \frac{1}{f-a_j})$ the corresponding reduced counting function. Thus, we have

$$C_{\alpha,\beta}\left(r, \frac{1}{f - a_j}\right) = \sum_{p,q=1}^{\infty} C_{\alpha,\beta}^{pq}\left(r, \frac{1}{f - a_j}\right)$$

and

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f - a_j}\right) = \sum_{p,q=1}^{\infty} \overline{C}_{\alpha,\beta}^{pq}\left(r, \frac{1}{f - a_j}\right).$$

From the above two equations, (5),(6) and (7), we can see that $S_{\alpha,\beta}(r, \varphi^{pq}) = R(r, f) + R(r, g)$. And by (8) each zero of $f - a_j$ is a zero of φ^{pq}, so with the help of Lemma 2.1 and $\varphi^{pq} \not\equiv 0$, we can get

$$\begin{aligned}
\overline{C}_{\alpha,\beta}^{pq}\left(r, \frac{1}{f - a_j}\right) &\le \overline{C}_{\alpha,\beta}^{pq}\left(r, \frac{1}{\varphi^{pq}}\right) \le S_{\alpha,\beta}\left(r, \frac{1}{\varphi^{pq}}\right) \\
&\le S_{\alpha,\beta}(r, \varphi^{pq}) + O(1) = R(r, f) + R(r, g),
\end{aligned}$$

for some p, q. By Lemma 2.4 (ii), we have $S_{\alpha,\beta}(r, f) + R(r, f) = S_{\alpha,\beta}(r, g) + R(r, g)$,we can get $S_{\alpha,\beta}(r, f) = S_{\alpha,\beta}(r, g) + R(r)$. Therefore, we have

$$\begin{aligned}
\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f - a_j}\right) &= \sum_{\max(p,q)\ge 5} \overline{C}_{\alpha,\beta}^{pq}\left(r, \frac{1}{f - a_j}\right) + R(r, f) \\
&\le \frac{1}{5}\left(\sum_{\max(p,q)\ge 5} C_{\alpha,\beta}^{pq}\left(r, \frac{1}{f - a_j}\right) + \sum_{\max(p,q)\ge 5} C_{\alpha,\beta}^{pq}\left(r, \frac{1}{g - a_j}\right)\right) \\
&\quad + R(r, f) \\
&\le \frac{1}{5}\left(C_{\alpha,\beta}\left(r, \frac{1}{f - a_j}\right) + C_{\alpha,\beta}\left(r, \frac{1}{g - a_j}\right)\right) + R(r, f) \\
&\le \frac{2}{5}S_{\alpha,\beta}(r, f) + R(r), \quad j = 1, 2, 3, 4.
\end{aligned}$$

By the above inequality and Lemma 2.4(ii), we can get

$$2S_{\alpha,\beta}(r, f) \le \frac{8}{5}S_{\alpha,\beta}(r, f) + R(r). \tag{9}$$

Thus, we can get a contradiction.

Case 2. Suppose that $\varphi^{pq} := \frac{1}{q^m}\varphi_1 - \frac{1}{p^m}\varphi_2 \equiv 0$, for some positive integers p, q. From the definitions of φ_1 and φ_2, we have

$$\left(\frac{p}{q}\right)^m \cdot \frac{(g-b_1)^2\cdots(g-b_m)^2}{(f-b_1)^2\cdots(f-b_m)^2} \equiv \left(\frac{f'(g-a_1)\cdots(g-a_4)}{g'(f-a_1)\cdots(f-a_4)}\right)^m. \tag{10}$$

Next we take the two subcases in the following into consideration:

Subcase 2.1. Suppose that $p \neq q$. Without loss of generality, we may assume that $p < q$. For some two positive integers p_1 and q_1, if $z_1 \in S_X^{p_1 q_1}(a_j)$ for some $j \in \{1,2,3,4\}$, then (10) implies that $\frac{p}{q} = \frac{p_1}{q_1}$. Hence $q_1 > p_1 \geq 1$, and $q_1 \geq 2$ which means that any a_j-points $(j = 1,2,3,4)$ of g in X are multiple. By Lemma 2.4, we can get

$$\begin{aligned} 2S_{\alpha,\beta}(r,g) &= \sum_{j=1}^{4} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{g-a_j}\right) + R(r,g) \\ &\leq \frac{1}{2}\sum_{j=1}^{4} C_{\alpha,\beta}\left(r, \frac{1}{g-a_j}\right) + R(r,g) \\ &\leq 2S_{\alpha,\beta}(r,g) + R(r,g), \end{aligned}$$

which leads to the following equations

$$S_{\alpha,\beta}(r,g) = C_{\alpha,\beta}\left(r, \frac{1}{g-a_j}\right) + R(r), \tag{11}$$

and

$$C_{\alpha,\beta}\left(r, \frac{1}{g-a_j}\right) = 2\overline{C}_{\alpha,\beta}\left(r, \frac{1}{g-a_j}\right) + R(r). \tag{12}$$

From (11) and (12), we can see that "almost all" of a_j-points of g have multiplicity 2, and "almost all" of a_j-points of f are simple in X. Without loss of generality, we may assume that f and g attain the values a_3 and a_4 in X. Set

$$\phi_1 := \frac{2f'(f-a_4)}{(f-a_1)(f-a_2)(f-a_3)} - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)(g-a_3)}$$

and

$$\phi_2 := \frac{2f'(f-a_3)}{(f-a_1)(f-a_2)(f-a_4)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)(g-a_4)}.$$

Since $\phi_i (i = 1,2)$ is analytic at the poles of f and of g and also at those common a_j-points of f and g which have multiplicity 1 with respect to f and multiplicity 2 with respect to g, by Lemma 2.3, we have $S_{\alpha,\beta}(r, \phi_i) = R(r,f), i = 1,2$. If $\phi_i \not\equiv 0$, then $C_{\alpha,\beta}\left(r, \frac{1}{f-a_4}\right) \leq C_{\alpha,\beta}\left(r, \frac{1}{\phi_1}\right) = R(r,f)$, which contradicts to equation (11). Then $\phi_1 \equiv 0$. Similarly, we have $\phi_2 \equiv 0$. Therefore, from the definitions of ϕ_1 and ϕ_2, we have

$$\left(\frac{f-a_4}{f-a_3}\right)^2 \equiv \left(\frac{g-a_4}{g-a_3}\right)^2. \tag{13}$$

Since $f \not\equiv g$, and from (13), we have

$$\frac{f-a_4}{f-a_3} \equiv -\frac{g-a_4}{g-a_3},$$

which implies that f and g share a_3, a_4 CM in X. Since f and g assume the value a_3 there exist positive integers p_1, q_1 such that $S^{p_1 q_1}(a_3) \neq \emptyset$. From the considerations above we get $q_1 > p_1$, contradicting the fact that f and g share a_3 CM.

Subcase 2.2. Suppose that $p = q$.

In this subcase, (10) becomes

$$\frac{(g-b_1)^2 \cdots (g-b_m)^2}{(f-b_1)^2 \cdots (f-b_m)^2} \equiv \left(\frac{f'(g-a_1)\cdots(g-a_4)}{g'(f-a_1)\cdots(f-a_4)}\right)^m.$$

which implies that f and g share the four values $a_j (j = 1, 2, 3, 4)$ CM in X. From the conditions of Theorem 1.8 and applying Theorem 1.7, g is a Möbius transformation of f. Furthermore, two of the four values, say a_1, a_2 are Picard exceptional values of f and g in X. Set

$$\chi_1 := \frac{f'(f-a_4)}{(f-a_1)(f-a_2)(f-a_3)} - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)(g-a_3)}$$

and

$$\chi_2 := \frac{f'(f-a_3)}{(f-a_1)(f-a_2)(f-a_4)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)(g-a_4)}.$$

Using the analogous argue of subcase 2.1 for χ_1, χ_2, we can get

$$\frac{f-a_3}{f-a_4} \equiv -\frac{g-a_3}{g-a_4}.$$

We define the Möbius transformations T, M and L by

$$T(\omega) := \frac{w-a_3}{w-a_4}, \qquad M(w) := -w \qquad \textit{and} \qquad L := T^{-1} \circ M \circ T.$$

Then we have

$$T \circ f = -T \circ g, \qquad \textit{hence} \qquad g = L \circ f.$$

Obviously a_3 and a_4 are the fixed points of L. Therefore, there exist no fixed points of L in the set S. Let some $b \in S$ be given. Then in vies of $b \neq a_1, a_2$ there exists a $z_0 \in \mathbb{C}$ such that $b = f(z_0)$, and from $\overline{E}_X(S, f) \subseteq \overline{E}_X(S, g)$ we obtain

$$L(b) = L(f(z_0)) = g(z_0) \in S.$$

So S is invariant under L. Furthermore, we have $L \circ L = I$ where I denotes the identical transformation. Hence we can conclude that S must contain an even number of values. Thus, we complete the proof of Theorem 1.8. ∎

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