A Fixed Point-Equilibrium Theorem with Applications

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Abstract

In this paper, using the Kakutani-Fan-Glicksberg fixed point theorem, we obtain an existence theorem of a point which is simultaneously fixed point for a given mapping and equilibrium point for a very general vector equilibrium problem. Finally some particular cases are discussed and three applications are given.

1 Introduction and preliminaries

If X and Y are topological spaces a multivalued mapping (or simply, a mapping) $T : X \multimap Y$ is said to be: (i) *upper semicontinuous* (in short, u.s.c) (respectively, *lower semicontinuous* (in short, l.s.c.)) if for every closed subset B of Y the set $\{x \in X : T(x) \cap B \neq \emptyset\}$ (respectively, $\{x \in X : T(x) \subseteq B\}$) is closed; (ii) *closed* if its graph (that is, the set $GrT = \{(x, y) \in X \times Y : \ y \in T(x), \ x \in X \}$) is a closed subset of $X \times Y$; (iii) *compact* if $T(X)$ is contained in a compact subset of Y.

The following lemma collects known facts about u.s.c. or l.s.c. mappings (see for instance [14] for assertions (i) and (ii), respectively [26] for assertion (iii)).

Lemma 1 *Let X and Y be topological spaces and $T : X \multimap Y$ be a mapping.*

(i) If Y is regular and T is u.s.c. with closed values, then T is closed.

(ii) If Y is compact and T is closed, then T is u.s.c..

(iii) T is l.s.c. if and only if for any $x \in X$, $y \in T(x)$ and any net $\{x_t\}$ converging to x, there exists a net $\{y_t\}$ converging to y, with $y_t \in T(x_t)$ for each t.

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Since the topological vector spaces are regular, by (i) and (ii) we infer that, if Y is a compact subset of a topological vector space, a closed-valued mapping $T : X \multimap Y$ is u.s.c. if an only if it is closed.

Let X be a nonempty subset of a topological vector space and $f : X \times X \to \mathbb{R}$ be a function with $f(x,x) \geq 0$ for all $x \in X$. Then the scalar equilibrium problem, in the sense of Blum and Oettli ([7]), is to find a $\widetilde{x} \in X$ such that $f(\widetilde{x},y) \geq 0$ for all $y \in X$. This problem includes fundamental mathematical problems like optimization problems, variational inequalities, fixed point problems, saddle point problems, problems of Nash equilibria, complementary problems (see [7]). In the last years the scalar equilibrium problem was extensively generalized in several ways to vector equilibrium for multivalued mappings (see [1], [2], [8-11], [13], [15-20] and the references therein).

In many of the papers mentioned above, for a suitable choice of the sets X and Z and of the mappings $F : X \times X \multimap Z$, $C : X \multimap Z$, the authors study, all or part of the following equilibrium problems:

(I) Find $\widetilde{x} \in X$ such that $F(\widetilde{x},y)\rho_i C(\widetilde{x})$, for all $y \in X$,

where, ρ_i $(i = \overline{1,4})$ are (binary) relations on 2^Z defined by:

(i) $A\rho_1 B \Leftrightarrow A \subseteq B$,

(ii) $A\rho_2 B \Leftrightarrow A \not\subseteq B$,

(iii) $A\rho_3 B \Leftrightarrow A \cap B \neq \emptyset$,

(iv) $A\rho_4 B \Leftrightarrow A \cap B = \emptyset$,

for $A, B \subseteq Z$.

In [4-6], [21], [24] and [25] the authors unify and extend all these problems considering an arbitrary relation ρ on 2^Z and looking for a point $\widetilde{x} \in X$ such that

(II) $F(\widetilde{x},y)\ \rho\ C(\widetilde{x})$, for all $y \in X$.

On the other hand, in [3], [10] and [21] is investigated the following problem, called vector quasi-equilibrium problem:

If F, C, ρ_i are as above and $T : X \multimap X$ is a suitable mapping, find $\widetilde{x} \in X$ such that

(III) $\widetilde{x} \in T(\widetilde{x})$ and $F(\widetilde{x},y)\rho_i\ C(\widetilde{x})$ for all $y \in T(\widetilde{x})$.

The following hybrid problem arises naturally:

Find a point $\widetilde{x} \in X$ such that

(IV) $\widetilde{x} \in T(\widetilde{x})$ and $F(\widetilde{x},y)\rho\ C(\widetilde{x})$ for all $y \in X$.

In the next section we give an existence theorem for this problem in the case when T is a Kakutani mapping and ρ is an arbitrary relation on 2^Z. In Section 3 problem (III) will be studied in the particular cases $\rho = \rho_1$ and $\rho = \rho_2$, respectively. Three applications will be given in the last section of the paper.

2 Main result

Lemma 2 ([19]) *Let X be a topological space, Y be a topological vector space and $S, T : X \multimap Y$ be two mappings. If S is u.s.c. with nonempty compact values and T is closed, then $S + T$ is a closed mapping.*

Lemma 3 *Let X be a topological space and Y be a Hausdorff topological vector space. If $f : X \to \mathbb{R}$ is a continuous function and $T : X \multimap Y$ a compact closed mapping, then the mapping $fT : X \multimap Y$ defined by $(fT)(x) = f(x)T(x)$ is closed.*

Proof. Let $(x, y) \in \overline{Gr(fT)}$. Then there exists a net $\{(x_t, y_t)\}_{t \in \Delta}$ in $Gr(fT)$ converging to (x, y). For each $t \in \Delta$ we have $y_t = f(x_t)z_t$, for some $z_t \in T(x_t)$. Since $\overline{T(X)}$ is compact there is a subnet $\{z_{t_\alpha}\}$ of $\{z_t\}$ converging to a point $z \in \overline{T(X)}$. Since the mapping T is closed, $z \in T(x)$. Hence, $y_{t_\alpha} \to f(x)z \in (fT)(x)$. The space Y being Hausdorff, $y = f(x)z$. It follows that $(x, y) \in Gr(fT)$ hence the mapping fT is closed. ■

Definition 1. ([12]) For a subset K of a vector space E and $x \in E$, the *outward set* of K at x is denoted and defined as follows:
$\mathbf{O}(K; x) = \bigcup_{\lambda \geq 1}(\lambda x + (1 - \lambda)K)$.

If A is a nonempty set and ρ is a relation on A we denote by ρ^c the complementary relation of ρ (that is, for any $a, b \in A$ exactly one of the following assertions $a\rho\, b, a\rho^c\, b$ holds).

Theorem 1. *Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space, Z be a nonempty set, ρ be a relation on 2^Z and $T : X \multimap X$, $F : X \times X \multimap Z$ and $C : X \multimap Z$ be three mappings satisfying the following conditions:*

(i) T is u.s.c. with nonempty compact convex values;

(ii) for each $x \in X$, the set $\{y \in X : F(x, y)\rho^c C(x)\}$ is convex;

(iii) for each $y \in X$, the set $\{x \in X : F(x, y)\rho C(x)\}$ is closed in X;

(iv) for each $x \in X$ and $y \in \mathbf{O}\big(T(x); x\big) \cap X$, $F(x, y)\rho C(x)$.

Then there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$ and $F(\widetilde{x}, y)\ \rho C(\widetilde{x})$ for all $y \in X$.

Proof. For $y \in X$, let $G_y = \{x \in X : F(x, y)\rho^c C(x)\}$. Let $G_0 = \{x \in X : x \notin T(x)\}$. Since the mapping T is closed, it follows readily that G_0 is open.

Suppose that the conclusion is false. Then for each $x \in X$, either $x \in G_0$ or $x \in G_y$, for some $y \in X$. Thus, $X = G_0 \bigcup \bigcup_{y \in X} G_y$. Since X is compact, there exists a finite set $\{y_1, \cdots, y_n\} \subset X$ such that $X = G_0 \cup \bigcup_{i=1}^n G_{y_i}$. For the sake of simplicity, we will put G_i instead of G_{y_i}. Let $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ be a partition of unity on X subordinated to the open cover $\{G_0, G_1, \cdots, G_n\}$. Recall that this means that

$$\begin{cases} \alpha_i : X \to [0, 1] \text{ is continuous, for each } i \in \{0, 1, \ldots, n\}; \\ \alpha_i(x) > 0 \Rightarrow x \in G_i; \\ \sum_{i=0}^n \alpha_i(x) = 1 \text{ for each } x \in X. \end{cases}$$

Define the mapping $S : X \multimap X$ by

$$S(x) = \alpha_0(x)T(x) + \alpha_1(x)y_1 + \cdot + \alpha_n(x)y_n.$$

It is clear that S has nonempty compact convex values. Since the mapping $x \mapsto \alpha_1(x)y_1 + \cdots + \alpha_n(x)y_n$ is closed, combining Lemmas 2 and 3 we infer that S is closed, hence u.s.c. By Kakutani-Fan-Glicksberg fixed point theorem, there exists $x_0 \in X$ such that $x_0 \in S(x_0)$. We shall prove that each of the cases $\alpha_0(x_0) = 0$, $\alpha_0(x_0) = 1$ and $\alpha_0(x_0) \in (0,1)$ leads to a contradiction.

Let $I = \{i \in \{1, \cdots, n\} : \alpha_i(x_0) > 0\}$. For each $i \in I$, $x_0 \in G_i$, hence $F(x_0, y_i)\rho^c C(x_0)$.

If $\alpha_0(x_0) = 0$, then

$$x_0 = \sum_{i\in I} \alpha_i(x_0)y_i \in co\{y_i : i \in I\}.$$

By (ii), it follows that $F(x_0, x_0)\rho^c\, C(x_0)$. On the other hand, since $x_0 \in \mathbf{O}\big(T(x_0); x_0\big) \cap X$, $F(x_0, x_0)\rho C(x_0)$. We have thus obtained a contradiction.

If $\alpha_0(x_0) = 1$, it follows that $x_0 \in S(x_0) = T(x_0)$. On the other hand, since $\alpha_0(x_0) > 0$, $x_0 \in G_0$, that is, $x_0 \notin T(x_0)$; a contradiction again.

If $\alpha_0(x_0) \in (0,1)$, then there exists $y_0 \in T(x_0)$ such that

$$x_0 = \alpha_0(x_0)y_0 + \sum_{i\in I} \alpha_i(x_0)y_i.$$

Dividing the previous relation by $1 - \alpha_0(x_0)$ and denoting by $\lambda = \frac{1}{1-\alpha_0(x_0)}$ we get

$$\lambda x_0 + (1-\lambda)y_0 = \sum_{i\in I} \frac{\alpha_i(x_0)}{1-\alpha_0(x_0)}\, y_i.$$

Since $\lambda x_0 + (1-\lambda)y_0 \in co\{y_i : i \in I\}$, by (ii) we have $F(x_0, \lambda x_0 + (1-\lambda)y_0)\rho^c\, C(x_0)$. On the other hand, since $\lambda > 1$, by (iv) we get $F(x_0, \lambda x_0 + (1-\lambda)y_0)\rho\, C(x_0)$. The contradiction obtained completes the proof. ■

Remark 1. Denote by S_ρ the set of all $\widetilde{x} \in X$ satisfying the conclusion of Theorem 1. Since $S_\rho = \{x \in X : x \in T(x)\} \cap \bigcap_{y\in X}\{x \in X : F(x,y)\rho\, C(x)\}$, by the requirements of the theorem it follows that S_ρ is a closed subset of X and, since X is compact, S_ρ is compact, too.

Recall that a mapping $T : X \multimap Y$ (X and Y topological spaces) is said to be selectionable if there exists a continuous function $g : X \to Y$ such that $g(x) \in T(x)$, for all $x \in X$. If X is a paracompact topological space and Y is a convex set in a Hausdorff topological vector space, by the selection theorem of Yannelis and Prabhakar [27], any mapping $T : X \multimap Y$ with nonempty convex values and open fibers is selectionable. Also, when X is paracompact and Y is Banach space, T is selectionable, if it is l.s.c. with closed convex values, according to the well-known Michael's selection theorem ([22]).

Corollary 2. *Let X be a nonempty compact convex subset of a Hausdorff locally convex topological vector space, Z be a nonempty set, ρ be a relation on 2^Z and $T : X \multimap X$, $F : X \times X \multimap Z$ and $C : X \multimap Z$ be three mappings satisfying conditions (ii), (iii) and (iv) in Theorem 1. If T is selectionable, then there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$ and $F(\widetilde{x}, y)\rho\, C(\widetilde{x})$ for all $y \in X$.*

Proof. Let $g : X \to Y$ be a continuous selection of T and the mapping $T_1 : X \multimap X$ defined by $T_1(x) = \{g(x)\}$. Apply Theorem 1 for the mappings T_1, F, C. ■

3 Particular cases of Theorem 1

In this section we show that if the relation ρ is one of the relations ρ_i considered in the first section of the paper, conditions (ii) and (iii) in Theorem 1 can be replaced by suitable conditions on the mappings F and C. Let us observe that each existence result concerning relation ρ_1 (respectively, ρ_2) yields an existence theorem for ρ_4 (respectively ρ_3), if we take into account the following equivalences: $F(x,y) \subseteq C(x) \Leftrightarrow F(x,y) \cap C^c(x) = \emptyset$ and $F(x,y) \nsubseteq C(x) \Leftrightarrow F(x,y) \cap C^c(x) \neq \emptyset$. For this reason we can fix our attention on relations ρ_1 and ρ_2, only.

Definition 2. ([23]) Let X and Z be convex sets in two vector spaces. A mapping $F : X \multimap Z$ is said to be:
(a) *quasiconvex* if $F(x_1) \cap C \neq \emptyset$ and $F(x_2) \cap C \neq \emptyset \Rightarrow F(x) \cap C \neq \emptyset$ for all convex sets $C \subseteq Z$, $x_1, x_2 \in X$ and $x \in \text{co}\{x_1, x_2\}$;
(b) *quasiconcave* if $F(x_1) \subseteq C$ and $F(x_2) \subseteq C \Rightarrow F(x) \subseteq C$ for all convex sets $C \subseteq Z$, $x_1, x_2 \in X$ and $x \in \text{co}\{x_1, x_2\}$.

It is clear that any convex (respectively, concave) mapping is quasiconvex (respectively, quasiconcave).

Definition 3. ([2]) Let X and Y be two nonempty convex subsets of two vector spaces and Z be a vector space. Let $F : X \times Y \multimap Z$ and $C : X \multimap Z$ be two mappings such that for each $x \in X$, $C(x)$ is a convex cone. We say that:

(i) F is $C(x)$-*quasiconvex* if for all $x \in X, y_1, y_2 \in Y$ and $\lambda \in [0,1]$, we have either $F(x,y_1) \subseteq F(x, \lambda y_1 + (1-\lambda)y_2) + C(x)$, or $F(x,y_2) \subseteq F(x, \lambda y_1 + (1-\lambda)y_2) + C(x)$.

(ii) F is $C(x)$-*quasiconvex-like* if for any $x \in X$, $y_1, y_2 \in Y$ and $\lambda \in [0,1]$, we have either $F(x, \lambda y_1 + (1-\lambda)y_2) \subseteq F(x,y_1) - C(x)$, or $F(x, \lambda y_1 + (1-\lambda)y_2) \subseteq F(x,y_2) - C(x)$.

In the next results we suppose that Z is a topological vector space.

Theorem 3. *Suppose that for $\rho = \rho_1$ conditions (i) and (iv) in Theorem 1 are fulfilled. Moreover suppose that:*

(ii) either

(ii_1) *for each $x \in X$, the mapping $F(x, \cdot)$ is quasiconvex and $Z \setminus C(x)$ is convex set; or*

(ii_2) *C has nonempty convex cone values and the mapping F is $C(x)$-quasiconvex;*

(iii) C is closed mapping and for each $y \in X$, $F(\cdot, y)$ is l.s.c.

Then there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$ and $F(\widetilde{x}, y) \subseteq C(\widetilde{x})$ for all $y \in X$.

Proof. We proof first that for $x \in X$, arbitrarily fixed, the set $M = \{y \in X : F(x,y) \nsubseteq C(x)\} = \{y \in X : F(x,y) \cap (Z \setminus C(x)) \neq \emptyset\}$ is convex. Clearly, this is the situation in case (ii_1). Suppose now that (ii_2) holds and let $y_1, y_2 \in M$ and $\lambda \in [0,1]$. Then $F(x,y_1) \nsubseteq C(x)$ and $F(x,y_2) \nsubseteq C(x)$. We want to show that $F(x, \lambda y_1 + (1-\lambda)y_2) \nsubseteq C(x)$. Suppose to the contrary that $F(x, \lambda y_1 + (1-\lambda)y_2) \subseteq C(x)$. Since F is $C(x)$-quasiconvex, for some $i \in \{1,2\}$ we obtain the following contradiction

$F(x,y_i) \subseteq F(x, \lambda y_1 + (1-\lambda)y_2) + C(x) \subseteq C(x) + C(x) = C(x)$.

We show now that for $y \in X$, arbitrarily fixed, the set $N = \{x \in X : F(x,y) \subseteq C(x)\}$ is closed in X. Indeed, if $x \in \overline{N}$ (the closure being considered relative to X) then there exists a net $\{x_t\}_{t\in\Delta}$ in N such that $x_t \to x$. Then, for each $t \in \Delta$, $F(x_t, y) \subseteq C(x_t)$. Let $z \in F(x,y)$. Since $F(\cdot, y)$ is l.s.c., by Lemma 1 (iii), there is a net $\{z_t\}_{t\in\Delta}$ in Z converging to z such that $z_t \in F(x_t, y)$ for all $t \in \Delta$. It follows that $z_t \in C(x_t)$ and since C is closed, $z \in C(x)$ hence $F(x,z) \subseteq C(x)$. This shows that the set N is closed.

The desired conclusion follows now from Theorem 1. ■

Example Let $X = [0,3]$, $Z = \mathbb{R}$ and the mappings $F : [0,3] \times [0,3] \multimap \mathbb{R}$, $C : [0,3] \multimap \mathbb{R}$ and $T : [0,3] \multimap [0,3]$ defined by

$$F(x,y) = [x+y, +\infty), \;\; C(x) = [2x-1, +\infty),$$

$$T(x) = \begin{cases} [-x+2, -2x+3] & \text{if } x \in [0,1), \\ [(x-2)^2, x] & \text{if } x \in [1,3]. \end{cases}$$

Observe that $F(x,y) \subseteq C(x)$ if and only if $x \le y+1$ and

$$\mathbf{O}\big(T(x); x\big) \cap X = \begin{cases} [0,x] & \text{if } x \in [0,1), \\ \{1\} & \text{if } x = 1, \\ [x,3] & \text{if } x \in (1,3]. \end{cases}$$

One readily verify that the mappings F, C, T satisfy all the requirements of Theorem 3. The unique $\widetilde{x}$ satisfying the conclusion of Theorem 3 is $\widetilde{x} = 1$.

Theorem 4. *Suppose that for $\rho = \rho_2$ conditions (i), and (iv) in Theorem 1 are fulfilled. Moreover suppose that:*

(ii) either

(ii_1) for each $x \in X$, the mapping $F(x,\cdot)$ is quasiconcave and $C(x)$ is convex set; or

(ii_2) C has nonempty convex cone values and the mapping F is $C(x)$-quasiconvex-like;

(iii) C has open graph and for each $y \in X$, $F(\cdot, y)$ is u.s.c.

Then there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$ and $F(\widetilde{x}, y) \nsubseteq C(\widetilde{x})$ for all $y \in X$.

The proof of the previous theorem is similar to that of Theorem 3. For the proof of the fact that, in the case $\rho = \rho_2$ condition (iii) in Theorem 4 implies the condition similarly denoted in Theorem 1 see, for example, the proof of Theorem 2.1 in [2].

4 Applications

We give in this section some applications for Theorems 3 and 4. The first one is a common fixed point theorem.

Theorem 5. *Let $(E, \langle \cdot, \cdot \rangle)$ be a real inner product space and X be a nonempty compact convex subset of E. Let $T : X \multimap X$ be a u.s.c. mapping with nonempty compact convex values and $f : X \to X$ be a continuous function. Suppose that*

$$\langle f(x) - x, y - x \rangle \geq 0, \text{ for each } x \in X \text{ and } y \in T(x). \tag{1}$$

Then, there exists $\widetilde{x} \in X$ such that $f(\widetilde{x}) = \widetilde{x} \in T(\widetilde{x})$.

Proof. Take in Theorem 3, $Z = \mathbb{R}$,

$$F(x, y) = [\|y - f(x)\| - \|x - f(x)\|, +\infty), \quad C(x) = [0, +\infty),$$

where $\| \cdot \|$ is the norm generated by the inner product $\langle \cdot, \cdot \rangle$.

Observe that $F(x, y) \subseteq C(x) \Leftrightarrow \|x - f(x)\| \leq \|y - f(x)\|$. Since the function $y \mapsto \|y - f(x)\|$ is quasiconvex, it is easy to see that F is $C(x)$-quasiconvex. Thus, condition (ii_2) in Theorem 3 holds.

Let $x \in X$ and $z \in \mathbf{O}\big(T(x); x\big) \cap X$. Then $z = \lambda x + (1 - \lambda)y$, for some $y \in T(x)$ and $\lambda \geq 1$. Taking into account (1), we have

$\|z - f(x)\|^2 = \| \, (x - f(x)) + (\lambda - 1)(x - y)\|^2 = \|x - f(x)\|^2 + 2(\lambda - 1)\langle x - f(x), x - y \rangle + (\lambda - 1)^2\|x - y\|^2 \geq \|x - f(x)\|^2$.

Thus, for $\rho = \rho_1$ condition (iv) in Theorem 1 is fulfilled. One readily check that all the other requirements of Theorem 3 are fulfilled. Consequently there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$ and $\|\widetilde{x} - f(\widetilde{x})\| \leq \|y - f(\widetilde{x})\|$, for each $y \in X$.

Taking $y = f(\widetilde{x})$ we get $\|\widetilde{x} - f(\widetilde{x})\| \leq 0$, that is $\widetilde{x} = f(\widetilde{x})$. So, $f(\widetilde{x}) = \widetilde{x} \in T(\widetilde{x})$. ∎

Theorem 6. *Let E be a locally convex Hausdorff topological vector space and X be a nonempty compact convex subset of E. Let $T : X \multimap X$ be a u.s.c. mapping with nonempty compact convex values and L be a continuous function from X to E^* endowed with weak*-topology. Suppose that:*

(i) $\langle L(x), x \rangle \geq 0$, for all $x \in X$;

(ii) $max_{y \in T(x)} \langle L(x), y \rangle \leq \langle L(x), x \rangle$, for all $x \in X$.

Then there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$ and $\langle L(\widetilde{x}), y \rangle \geq 0$, for all $y \in X$.

Proof. We take in Theorem 3, $Z = \mathbb{R}$ and for all $x, y \in X$,

$F(x, y) = \{\langle L(x), y \rangle\}$, $C(x) = [0, +\infty)$.

Then, $F(x, y) \subseteq C(x) \Leftrightarrow \langle L(x), y \rangle \geq 0$. We show that, for $\rho = \rho_1$ condition (iv) in Theorem 1 is fulfilled. Let $x \in X$ and $y \in T(x)$. By (ii), $\langle L(x), y \rangle \leq \langle L(x), x \rangle$. Then, for any $\lambda \geq 1$ for which $\lambda x + (1 - \lambda)y \in X$ we have $\langle L(x), \lambda x + (1 - \lambda)y \rangle = \langle L(x), y \rangle + \lambda[\langle L(x), x \rangle - \langle L(x), y \rangle] \geq 0$. It is easy to see that all the other requirements of Theorem 3 are satisfied. The desired conclusion follows by Theorem 3. ∎

As application of Theorem 4 we shall obtain an existence theorem for the solution of a quasivector optimization problem. But first we need recall some concepts. Let X be a nonempty compact convex of $\mathbb{R}^n$ and C be a proper, closed, pointed and convex cone of $\mathbb{R}^m$. A function $\varphi : X \to \mathbb{R}^m$ is said to be C-convex if, for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$, we have

$$\lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) - \varphi(\lambda x_1 + (1-\lambda)x_2) \in C.$$

Following [2], for a such function, we define the subdifferential of φ in $x \in X$, denoted by $\partial\varphi(x)$, as

$$\varphi(x) = \{u \in (\mathbb{R}^n, \mathbb{R}^m)^* : \varphi(y) - \varphi(x) - \langle u, y - x\rangle \in C, \forall y \in X\},$$

where $(\mathbb{R}^n, \mathbb{R}^m)^*$ and $\langle u, x\rangle$ denote the space of linear continuous function from $\mathbb{R}^n$ into $\mathbb{R}^m$ and the evaluation of $u \in (\mathbb{R}^n, \mathbb{R}^m)^*$ at $x \in \mathbb{R}^n$, respectively.

The following theorem and its proof are inspired from Theorem 4.1 in [3].

Theorem 7. *Let X be a nonempty compact convex subset of $\mathbb{R}^n$, C be a proper, closed, pointed and convex cone of $\mathbb{R}^m$, $T : X \multimap X$ be a u.s.c. mapping with nonempty compact convex values and $\varphi : X \to \mathbb{R}^m$ be a C-convex function. Suppose that:*

(i) $\partial\varphi$ is a u.s.c. mapping with nonempty compact convex values;

(ii) for each $x \in X$ and $y \in T(x)$, $\varphi(y) - \varphi(x) \notin C$.

Then there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$ and $\varphi(y) - \varphi(\widetilde{x}) \notin -int\ C$, for all $y \in X$.

Proof. We take in Theorem 4, $Z = \mathbb{R}$ and for any $x, y \in X$,

$$F(x,y) = \langle\partial\varphi(x), y - x\rangle = \{\langle u, y - x\rangle : u \in \partial\varphi(x)\},\ C(x) = -int\ C.$$

Then, $F(x,y) \nsubseteq C(x) \Leftrightarrow \exists u \in \varphi(x) : \langle u, y - x\rangle \notin -int\ C$.

Since the mapping $\partial\varphi$ is u.s.c. with compact values, by Theorem 1 in [16], it follows that for each $y \in X$, $F(\cdot, y)$ is u.s.c.. We claim that $\langle u, y - x\rangle \notin C$, whenever $x \in X$, $y \in T(x)$ and $u \in \partial\varphi(x)$. Supposing the contrary, we infer that

$$\varphi(y) - \varphi(x) \in \langle u, y - x\rangle + C \subseteq C,$$

and this contradicts (ii). For x, y, u as above and $\lambda > 1$ such that $\lambda x + (1-\lambda)y \in X$ we have

$$\langle u, (\lambda x + (1-\lambda)y) - x\rangle = (1-\lambda)\langle u, y - x\rangle \notin -C,$$

and consequently,

$$\langle u, (\lambda x + (1-\lambda)y) - x\rangle \notin -int\ C. \tag{2}$$

Obviously (2) holds too, when $\lambda = 1$. This proves that, for $\rho = \rho_2$ condition (iv) in Theorem 1 is fulfilled. One can be easily check that the other requirements of Theorem 4 are also satisfied. Then, according to this theorem, there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$ and

$$\forall y \in X\ \exists u \in \partial\varphi(\widetilde{x}) : \langle u, y - \widetilde{x}\rangle \notin -int\ C. \tag{3}$$

Since $u \in \partial\varphi(\widetilde{x})$, we have

$$\varphi(y) - \varphi(\widetilde{x}) - \langle u, y - \widetilde{x}\rangle \in C. \tag{4}$$

Combining (3) and (4), we get $\varphi(y) - \varphi(\widetilde{x}) \notin -int\ C$, for all $y \in X$. ∎

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