Multiplicity of solutions for anisotropic quasilinear elliptic equations with variable exponents

Denisa Stancu-Dumitru

Abstract

We study an anisotropic partial differential equation on a bounded domain $\Omega \subset \mathbb{R}^N$. We prove the existence of at least two nontrivial weak solutions using as main tools the mountain pass lemma and Ekeland's variational principle.

1 Introduction

Equations involving variable exponent growth conditions have been extensively studied in the last decade. We just remember the recent advances in [10, 12, 13, 1, 2, 25, 16, 26, 27, 20, 18, 30, 28, 29]. The large number of papers studying problems involving variable exponent growth conditions is motivated by the fact that this type of equations can serve as models in the theory of electrorhological fluids [17, 35, 36, 5, 1], image processing [4] or the theory of elasticity [40].

Typical models of elliptic equations with variable exponent growth conditions appeal to the so called $p(x)$-Laplace operator, i.e.

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)\,,$$

where $p(x)$ is a function satisfying $p(x) > 1$ for each x. Recently, Mihăilescu-Pucci-Rădulescu extended in [24] the study involving the $p(x)$-Laplace operator

Received by the editors August 2009.
Communicated by P. Godin.
2000 *Mathematics Subject Classification :* 35D05, 35J60, 35J70, 58E05, 68T40, 76A02.
Key words and phrases : anisotropic equation, variable exponent, weak solution, mountain-pass theorem, Ekeland's variational principle.

to the case of anisotropic equations with variable exponent growth conditions, where the differential operator considered has the form

$$\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u)\,, \tag{1}$$

with $p_i(x)$ functions satisfying $\inf_x p_i(x) > 1$ for each $i \in \{1, ..., N\}$. Undoubtedly, in the particular case when $p_i(x) = p(x)$ for each $i \in \{1, ..., N\}$ the above differential operator becomes $\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p(x)-2}\partial_{x_i} u)$ and has similar properties with the $p(x)$-Laplace operator. On the other hand, the anisotropic equations with variable exponent growth conditions enable the study of equations with more complicated nonlinearities since the differential operator (1) allows a distinct behavior for partial derivatives in various directions.

Motivated by the above discussion, the goal of this paper is to investigate a problem of the type

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) = f(x, u), & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, p_i are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x)$ for any $x \in \overline{\Omega}$ and $i \in \{1, ..., N\}$. Our main result on problem (2) will supplement the results in [23, 24, 22, 21] obtained for similar anisotropic equations.

2 Preliminary results on variable exponent spaces

Assume $\Omega \subset \mathbb{R}^N$ is an open domain.

Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ we define

$$p^+ = \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\Omega} p(x)\,.$$

For each $p \in C_+(\overline{\Omega})$, we recall the definition of the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}\,.$$

This space becomes a Banach space [19, Theorem 2.5] with respect to the *Luxemburg norm*, that is

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}\,.$$

Moreover, $L^{p(\cdot)}(\Omega)$ is a reflexive space [19, Corollary 2.7] provided that $1 < p^- \leq p^+ < \infty$. Furthermore, on such kind of spaces a Hölder type inequality is valid

[19, Theorem 2.1]. More exactly, denoting by $L^{q(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{q(x)}=1$ for any $x\in\overline{\Omega}$, for each $u\in L^{p(\cdot)}(\Omega)$ and each $v\in L^{q(\cdot)}(\Omega)$ the Hölder type inequality reads as follows

$$\left|\int_\Omega uv\,dx\right|\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(\cdot)}|v|_{q(\cdot)}\,. \tag{3}$$

An immediate consequence of Hölder's inequality is connected with some inclusions between various Lebesgue spaces involving variable exponent growth [19, Theorem 2.8]: if $0<|\Omega|<\infty$ and p_1, p_2 are variable exponents, so that $p_1(x)\le p_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{p_2(\cdot)}(\Omega)\hookrightarrow L^{p_1(\cdot)}(\Omega)$, whose norm does not exceed $|\Omega|+1$.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)}$: $L^{p(\cdot)}(\Omega)\to\mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u)=\int_\Omega |u|^{p(x)}\,dx\,,$$

provided that $p^+<\infty$. Spaces with $p^+=\infty$ have been studied by Edmunds, Lang and Nekvinda [6].

We point out some relations which can be established between the Luxemburg norm and the modular. If (u_n), $u\in L^{p(\cdot)}(\Omega)$ and $p^+<\infty$ then the following relations hold true

$$|u|_{p(\cdot)}>1\quad\Rightarrow\quad |u|_{p(\cdot)}^{p^-}\le\rho_{p(\cdot)}(u)\le|u|_{p(\cdot)}^{p^+} \tag{4}$$

$$|u|_{p(\cdot)}<1\quad\Rightarrow\quad |u|_{p(\cdot)}^{p^+}\le\rho_{p(\cdot)}(u)\le|u|_{p(\cdot)}^{p^-} \tag{5}$$

$$|u_n-u|_{p(\cdot)}\to 0\quad\Leftrightarrow\quad\rho_{p(\cdot)}(u_n-u)\to 0\,. \tag{6}$$

Next, we define the *variable exponent Sobolev space* $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|=|\nabla u|_{p(\cdot)}\,.$$

The space $(W_0^{1,p(\cdot)}(\Omega),\|\cdot\|)$ is a separable and reflexive Banach space, provided that $1<p^-\le p^+<\infty$. We recall that if Ω is a bounded, open domain in $\mathbb{R}^N$, $q\in C_+(\overline{\Omega})$ and $q(x)<p^\star(x)$ for all $x\in\overline{\Omega}$ then the embedding

$$W_0^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$$

is compact and continuous, where $p^\star(x)=\frac{Np(x)}{N-p(x)}$ if $p(x)<N$ or $p^\star(x)=+\infty$ if $p(x)\ge N$. We refer to [31, 6, 7, 8, 11, 14, 19] for further properties of variable exponent Lebesgue-Sobolev spaces.

Finally, we recall the definition and properties of the anisotropic variable exponent Sobolev spaces as they were introduced in [24]. With that end in view, we assume in the sequel that Ω is a bounded open domain in $\mathbb{R}^N$ and we denote by $\overrightarrow{p}(\cdot) : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot))$. We define $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, the *anisotropic variable exponent Sobolev space,* as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}\,.$$

In the case when $p_i(\cdot) \in C_+(\overline{\Omega})$ are constant functions for any $i \in \{1, ..., N\}$ the resulting anisotropic Sobolev space is denoted by $W_0^{1,\overrightarrow{p}}(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1, ..., p_N)$. The theory of this type of spaces was developed in [15, 32, 33, 34, 37, 38]. It was argued in [24] that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a reflexive Banach space.

On the other hand, in order to facilitate the manipulation of the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we introduced $\overrightarrow{P}_+$, $\overrightarrow{P}_-$ in $\mathbb{R}^N$ as

$$\overrightarrow{P}_+ = (p_1^+, ..., p_N^+), \quad \overrightarrow{P}_- = (p_1^-, ..., p_N^-),$$

and $P_+^+, P_-^+, P_-^- \in \mathbb{R}^+$ as

$$P_+^+ = \max\{p_1^+, ..., p_N^+\}, \quad P_-^+ = \max\{p_1^-, ..., p_N^-\}, \quad P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1 \tag{7}$$

and define $P_-^\star \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^\star = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \quad P_{-,\infty} = \max\{P_-^+, P_-^\star\}.$$

Finally, we recall a result regarding the compact embedding between $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and variable exponent Lebesgue spaces (see, [24, Theorem 1]):

Theorem 1. *Assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary. Assume relation (7) is fulfilled. For any $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{-,\infty} \quad for\ all \quad x \in \overline{\Omega}, \tag{8}$$

the embedding

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is continuous and compact.

3 The main result

In this paper we study problem (2) in the particular case

$$f(x,t) = A|t|^{a(x)-2}t + B|t|^{b(x)-2}t,$$

where $a : \overline{\Omega} \to \mathbb{R}$, $b : \overline{\Omega} \to \mathbb{R}$ are continuous functions such that

$$1 < a^- < a^+ < P_-^- \leq P_+^+ < b^- < b^+ < \min\{N, P_{-,\infty}\} \tag{9}$$

and $A, B > 0$. More precisely, we consider the following problem

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i}(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u) = A|u|^{a(x)-2}u + B|u|^{b(x)-2}u, & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega. \end{cases} \tag{10}$$

We seek solutions for problem (10) belonging to the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the sense given below.

Definition 1. We say that $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a *weak solution for problem* (10) if

$$\int_\Omega \left\{ \sum_{i=1}^N \left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u \; \partial_{x_i}v \right) - A|u|^{a(x)-2}uv - B|u|^{b(x)-2}uv \right\} dx = 0,$$

for all $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

The main result of this paper is given by the following theorem.

Theorem 2. *There exists $\mu > 0$ such that, for any $A \in (0,\mu)$ and any $B \in (0,\mu)$, problem (10) has at least two distinct nontrivial weak solutions.*

We point out that the result of Theorem 2 can be regarded as a generalization of Theorem 1 in [20], where a similar problem involving the $p(x)$-Laplace operator was studied.

4 Proof of Theorem 2

We start by introducing the energy functional corresponding to problem (10), that is $J : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$,

$$J(u) = \int_\Omega \left\{ \sum_{i=1}^N \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)} - A\frac{|u|^{a(x)}}{a(x)} - B\frac{|u|^{b(x)}}{b(x)} \right\} dx. \tag{11}$$

Standard arguments assure that $J \in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ and its Fréchet derivative is given by

$$\langle J'(u), v \rangle = \int_\Omega \left\{ \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u \; \partial_{x_i}v - A|u|^{a(x)-2}uv - B|u|^{b(x)-2}uv \right\} dx, \tag{12}$$

for all $u,\ v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Thus, the weak solutions of problem (10) are exactly the critical points of J. We shall prove that the functional J possesses two distinct critical points using as main tools the Mountain Pass Theorem (see, e.g. [3] or [39]) and Ekeland's Variational Principle (see, e.g. [9]).

The following lemma will be essential in proving our main result.

Lemma 1. *The following assertions hold.*

(i) There exists $\mu > 0$ such that for any $A,\ B \in (0,\mu)$ we can find $\rho_0 > 0$ and $a > 0$ such that

$$J(u) \geq a > 0, \quad \forall\, u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \ \ \textit{with}\ \|u\|_{\overrightarrow{p}(\cdot)} = \rho_0.$$

(ii) There exists $\varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$\lim_{t\to\infty} J(t\varphi) = -\infty.$$

(iii) There exists $\Phi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $\Phi \geq 0,\ \Phi \neq 0$ and

$$J(t\Phi) < 0,$$

for $t > 0$ small enough.

Proof. (i) By condition (9) we have $1 < a(x) < b(x) < P_{-,\infty}$, for all $x \in \overline{\Omega}$ and, consequently, Theorem 1 assures that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is continuously and compactly embedded in $L^{a(x)}(\Omega)$ and $L^{b(x)}(\Omega)$.

The fact that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is continuously and compactly embedded in $L^{a(x)}(\Omega)$ assures that there exists a positive constant C_1 such that

$$|u|_{a(\cdot)} \leq C_1 \cdot \|u\|_{\overrightarrow{p}(\cdot)}, \quad \forall\, u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{13}$$

Similarly, $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is continuously and compactly embedded in $L^{b(x)}(\Omega)$ and this guarantees that there exists a positive constant C_2 such that

$$|u|_{b(\cdot)} \leq C_2 \cdot \|u\|_{\overrightarrow{p}(\cdot)}, \quad \forall\, u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{14}$$

We fix $\rho_0 \in (0,1)$ such that $\rho_0 < \min\left\{\frac{1}{C_1}, \frac{1}{C_2}\right\}$. Then relations (13) and (14) imply

$$|u|_{a(\cdot)} < 1, \ \text{ for all } \ u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \ \text{ with } \ \|u\|_{\overrightarrow{p}(\cdot)} = \rho_0$$

and

$$|u|_{b(\cdot)} < 1, \ \text{ for all } \ u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \ \text{ with } \ \|u\|_{\overrightarrow{p}(\cdot)} = \rho_0.$$

Furthermore, relation (5) yields

$$\int_\Omega |u|^{a(x)}\, dx \leq |u|^{a^-}_{a(\cdot)}, \ \text{ for all } \ u \in W_0^{1,\overrightarrow{p}(\cdot)} \ \text{ with } \|u\|_{\overrightarrow{p}(\cdot)} = \rho_0 \tag{15}$$

and

$$\int_\Omega |u|^{b(x)}\, dx \leq |u|^{b^-}_{b(\cdot)}, \ \text{ for all } \ u \in W_0^{1,\overrightarrow{p}(\cdot)} \ \text{ with } \|u\|_{\overrightarrow{p}(\cdot)} = \rho_0. \tag{16}$$

Relations (13) and (15) imply

$$\int_\Omega |u|^{a(x)}\,dx \le C_1^{a^-}\,\|u\|^{a^-}_{\overrightarrow{p}(\cdot)}, \quad \text{for all } \ u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \ \text{ with } \ \|u\|_{\overrightarrow{p}(\cdot)} = \rho_0. \quad (17)$$

By relations (14) and (16) we get

$$\int_\Omega |u|^{b(x)}\,dx \le C_2^{b^-}\,\|u\|^{b^-}_{\overrightarrow{p}(\cdot)}, \quad \text{for all } \ u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \ \text{ with } \ \|u\|_{\overrightarrow{p}(\cdot)} = \rho_0. \quad (18)$$

Using relation (5), for all $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(\cdot)} < 1$, we obtain

$$\frac{\|u\|^{P_+^+}_{\overrightarrow{p}(\cdot)}}{N^{P_+^+-1}} = N\left(\frac{\sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}}{N}\right)^{P_+^+} \le \sum_{i=1}^N |\partial_{x_i}u|^{P_+^+}_{p_i(\cdot)} \le$$
$$\sum_{i=1}^N |\partial_{x_i}u|^{p_i^+}_{p_i(\cdot)} \le \sum_{i=1}^N \int_\Omega |\partial_{x_i}u|^{p_i(x}\,dx. \quad (19)$$

Relations (19), (18) and (17) show that for any $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(\cdot)} = \rho_0$ we have

$$\begin{aligned} J(u) &\ge \frac{1}{P_+^+ N^{P_+^+-1}}\|u\|^{P_+^+}_{\overrightarrow{p}(\cdot)} - \frac{A}{a^-}\,C_1^{a^-}\,\|u\|^{a^-}_{\overrightarrow{p}(\cdot)} - \frac{B}{b^-}\,C_2^{b^-}\,\|u\|^{b^-}_{\overrightarrow{p}(\cdot)} \\ &= \frac{1}{P_+^+ N^{P_+^+-1}}\rho_0^{P_+^+} - \frac{A}{a^-}\,C_1^{a^-}\,\rho_0^{a^-} - \frac{B}{b^-}\,C_2^{b^-}\,\rho_0^{b^-} \\ &= \rho_0^{a^-}\left(\frac{1}{2P_+^+ N^{P_+^+-1}}\,\rho_0^{P_+^+-a^-} - \frac{A}{a^-}\,C_1^{a^-}\right) + \\ &\qquad \rho_0^{P_+^+}\left(\frac{1}{2P_+^+ N^{P_+^+-1}} - \frac{B}{b^-}\,C_2^{b^-}\,\rho_0^{b^- - P_+^+}\right). \end{aligned}$$

Defining

$$\mu_1 = \frac{1}{4P_+^+ N^{P_+^+-1}}\rho_0^{P_+^+-a^-}\,\frac{a^-}{C_1^{a^-}} \quad \text{and} \quad \mu_2 = \frac{1}{4P_+^+ N^{P_+^+-1}}\rho_0^{P_+^+-b^-}\,\frac{b^-}{C_2^{b^-}},$$

simple computations show that

$$\rho_0^{a^-}\left(\frac{1}{2P_+^+ N^{P_+^+-1}}\,\rho_0^{P_+^+-a^-} - \frac{A}{a^-}\,C_1^{a^-}\right) \ge \frac{1}{4P_+^+ N^{P_+^+-1}}\rho_0^{P_+^+}, \quad \forall\, A \in (0,\mu_1),$$

and

$$\rho_0^{P_+^+}\left(\frac{1}{2P_+^+ N^{P_+^+-1}} - \frac{B}{b^-}\,C_2^{b^-}\,\rho_0^{b^- - P_+^+}\right) \ge \frac{1}{4P_+^+ N^{P_+^+-1}}\rho_0^{P_+^+}, \quad \forall\, B \in (0,\mu_2).$$

Consequently, defining

$$\mu := \min\{\mu_1, \mu_2\}, \quad (20)$$

and

$$a := \frac{1}{4P_+^+ N^{P_+^+ - 1}} \rho_0^{P_+^+}$$

we conclude that for any $A \in (0, \mu)$ and any $B \in (0, \mu)$ we have

$$J(u) \geq a > 0,$$

for all $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(\cdot)} = \rho_0$, where ρ_0 was fixed such that $\rho_0 \in (0, \min\{1, 1/C_1, 1/C_2\})$ at the beginning of the proof of (i).

(ii) Let $\varphi \in C_0^\infty(\Omega),\ \varphi \geq 0,\ \varphi \neq 0$ and $t > 1$. We have

$$\begin{aligned} J(t\varphi) &= \int_\Omega \left\{ \sum_{i=1}^N \frac{t^{p_i(x)}}{p_i(x)} |\partial_{x_i} \varphi|^{p_i(x)} - A \frac{t^{a(x)}}{a(x)} |\varphi|^{a(x)} - B \frac{t^{b(x)}}{b(x)} |\varphi|^{b(x)} \right\} dx \\ &\leq \frac{t^{P_+^+}}{P_-^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i} \varphi|^{p_i(x)}\, dx - A \frac{t^{a^-}}{a^+} \int_\Omega |\varphi|^{a(x)}\, dx - B \frac{t^{b^-}}{b^+} \int_\Omega |\varphi|^{b(x)}\, dx \\ &\leq \frac{t^{P_+^+}}{P_-^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i} \varphi|^{p_i(x)}\, dx - B \frac{t^{b^-}}{b^+} \int_\Omega |\varphi|^{b(x)}\, dx. \end{aligned}$$

Since $b^- > P_+^+$, by (9) we deduce that $\lim\limits_{t \to \infty} J(t\varphi) = -\infty$ and, thus, (ii) is proved.

(iii) Let $\Phi \in C_0^\infty(\Omega),\ \Phi \geq 0,\ \Phi \neq 0$ and $t \in (0, 1)$. We conclude that

$$\begin{aligned} J(t\Phi) &= \int_\Omega \left\{ \sum_{i=1}^N \frac{t^{p_i(x)}}{p_i(x)} |\partial_{x_i} \Phi|^{p_i(x)} - A \frac{t^{a(x)}}{a(x)} |\Phi|^{a(x)} - B \frac{t^{b(x)}}{b(x)} |\Phi|^{b(x)} \right\} dx \\ &\leq \frac{t^{P_-^-}}{P_-^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i} \Phi|^{p_i(x)}\, dx - A \frac{t^{a^+}}{a^+} \int_\Omega |\Phi|^{a(x)}\, dx - B \frac{t^{b^+}}{b^+} \int_\Omega |\Phi|^{b(x)}\, dx \\ &\leq \frac{t^{P_-^-}}{P_-^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i} \Phi|^{p_i(x)}\, dx - A \frac{t^{a^+}}{a^+} \int_\Omega |\Phi|^{a(x)}\, dx < 0 \end{aligned}$$

for $t < \delta^{1/(P_-^- - a^+)}$ with

$$0 < \delta < \min \left\{ 1, \frac{A P_-^- \int_\Omega |\Phi|^{a(x)}\, dx}{a^+ \sum_{i=1}^N \int_\Omega |\partial_{x_i} \Phi|^{p_i(x)}\, dx} \right\}.$$

It follows that (iii) is proved.

Thus, the proof of Lemma 1 is complete. ∎

Proof of Theorem 2. Let $\mu > 0$ be defined as in (20) and $A \in (0, \mu),\ B \in (0, \mu)$.

Using Lemma 1 (i) and (ii) and the Mountain Pass Theorem (see, e.g. [3]) we deduce that there exists a sequence $\{u_n\}$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$J(u_n) \to \overline{c} \ \text{ and } \ J'(u_n) \to 0 \ \text{ in } \ \left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \right)^\star, \tag{21}$$

where $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \right)^\star$ is the dual space of $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

First, we show that $\{u_n\}$ is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Assume by contradiction the contrary. Then, passing if necessary to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\|_{\overrightarrow{p}(\cdot)} \to \infty$ as $n \to \infty$. Thus, we may consider that $\|u_n\|_{\overrightarrow{p}(\cdot)} > 1$ for any integer n. Relations (21) and the above considerations imply that for n large enough it holds that

$$\begin{aligned} 1+\overline{c}+\|u_n\|_{\overrightarrow{p}(\cdot)} &\geq J(u_n) - \frac{1}{b^-}\langle J'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{b^-}\right)\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\,dx + \\ &\qquad A\left(\frac{1}{b^-} - \frac{1}{a^-}\right)\int_\Omega |u_n|^{a(x)}\,dx. \end{aligned}$$

For each n and $i \in \{1, ..., N\}$ we define

$$\xi_{n,i} = \begin{cases} P_+^+, & \text{if } |\partial_{x_i} u_n|_{p_i(\cdot)} < 1, \\ P_-^-, & \text{if } |\partial_{x_i} u_n|_{p_i(\cdot)} > 1. \end{cases}$$

Some elementary computations show that for all $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we have

$$\frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-}}{N^{P_-^- -1}} = N\left(\frac{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}}{N}\right)^{P_-^-} \leq \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}^{P_-^-} \tag{22}$$

On the other hand, we point out that

$$\int_\Omega |u|^{a(x)}\,dx \leq |u|_{a(\cdot)}^{a^+} + |u|_{a(\cdot)}^{a^-}, \text{ for all } u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{23}$$

Since, by Theorem 1, $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is compactly embedded in $L^{a(x)}(\Omega)$ and relation (23) holds true it follows that there exists a positive constant C_3 such that

$$\int_\Omega |u|^{a(x)}\,dx \leq C_3(\|u\|_{\overrightarrow{p}(\cdot)}^{a^+} + \|u\|_{\overrightarrow{p}(\cdot)}^{a^-}), \text{ for all } u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{24}$$

Since $a^- < b^-$ (see relation (9)), using relation (24) we find that there exists a positive constant C_4 such that

$$A\left(\frac{1}{b^-} - \frac{1}{a^-}\right)\int_\Omega |u|^{a(x)}\,dx \geq -C_4\left(\|u\|_{\overrightarrow{p}(\cdot)}^{a^+} + \|u\|_{\overrightarrow{p}(\cdot)}^{a^-}\right), \text{ for all } u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{25}$$

Using relations (4), (5), (22) and (25) we infer that for n large enough we have

$$
\begin{aligned}
&1+\overline{c}+\|u\|_{\overrightarrow{p}(\cdot)} \\
&\quad\geq \left(\frac{1}{P_+^+}-\frac{1}{b^-}\right)\int_\Omega \sum_{i=1}^N |\partial_{x_i}u_n|^{p_i(x)}\,dx + A\left(\frac{1}{b^-}-\frac{1}{a^-}\right)\int_\Omega |u_n|^{a(x)}\,dx \\
&\quad\geq \left(\frac{1}{P_+^+}-\frac{1}{b^-}\right)\sum_{i=1}^N |\partial_{x_i}u_n|_{p_i(\cdot)}^{\xi_{n,i}} - C_4\left(\|u\|_{\overrightarrow{p}(\cdot)}^{a^+}+\|u\|_{\overrightarrow{p}(\cdot)}^{a^-}\right) \\
&\quad\geq \left(\frac{1}{P_+^+}-\frac{1}{b^-}\right)\sum_{i=1}^N |\partial_{x_i}u_n|_{p_i(\cdot)}^{P_-^-} - \left(\frac{1}{P_+^+}-\frac{1}{b^-}\right) \\
&\qquad\qquad \sum_{\{i;\,\xi_{n,i}=P_+^+\}}\left(|\partial_{x_i}u_n|_{p_i(\cdot)}^{P_-^-}-|\partial_{x_i}u_n|_{p_i(\cdot)}^{P_+^+}\right) - C_4\left(\|u\|_{\overrightarrow{p}(\cdot)}^{a^+}+\|u\|_{\overrightarrow{p}(\cdot)}^{a^-}\right) \\
&\quad\geq \left(\frac{1}{P_+^+}-\frac{1}{b^-}\right)\frac{1}{N^{P_-^- -1}}\|u_n\|_{\overrightarrow{p}(\cdot)}^{P_-^-} - N\left(\frac{1}{P_+^+}-\frac{1}{b^-}\right) \\
&\qquad\qquad -C_4\left(\|u\|_{\overrightarrow{p}(\cdot)}^{a^+}+\|u\|_{\overrightarrow{p}(\cdot)}^{a^-}\right),
\end{aligned}
$$

where C_4 is a positive constant.

Taking into account that condition (9) holds true, dividing the above inequality by $\|u_n\|_{\overrightarrow{p}(\cdot)}^{P_-^-}$ and passing to the limit as $n\to\infty$ we obtain a contradiction.

It follows that $\{u_n\}$ is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. This information and the fact that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a reflexive space implies that there exist a subsequence, still denoted by $\{u_n\}$, and $u_1\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $\{u_n\}$ converges weakly to u_1 in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Since, by Theorem 1, the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is compactly embedded in $L^{a(x)}(\Omega)$ and $L^{b(x)}(\Omega)$, we conclude that $\{u_n\}$ converges strongly to u_1 in $L^{a(x)}(\Omega)$ and $L^{b(x)}(\Omega)$. Then, by inequality (3), we deduce

$$\lim_{n\to\infty}\int_\Omega |u_n|^{a(x)-2}\,u_n(u_n-u_1)\,dx=0\,,$$

and

$$\lim_{n\to\infty}\int_\Omega |u_n|^{b(x)-2}\,u_n(u_n-u_1)\,dx=0\,.$$

On the other hand, by relation (21) we have

$$\lim_{n\to\infty}\langle J'(u_n),u_n-u_1\rangle=0.$$

Thus, by using the above equations, we get

$$\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega |\partial_{x_i}u_n|^{p_i(x)-2}\,\partial_{x_i}u_n(\partial_{x_i}u_n-\partial_{x_i}u_1)\,dx=0\,. \tag{26}$$

Relation (26) and the fact that $\{u_n\}$ converges weakly to u_1 in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ imply

$$\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega \left(|\partial_{x_i}u_n|^{p_i(x)-2}\,\partial_{x_i}u_n-|\partial_{x_i}u_1|^{p_i(x)-2}\,\partial_{x_i}u_1\right)(\partial_{x_i}u_n-\partial_{x_i}u_1)\,dx=0\,. \tag{27}$$

Next, we recall that the following elementary inequality

$$\left(|\eta|^{t-2}\eta - |\varrho|^{t-2}\varrho\right)(\eta - \varrho) \geq 2^{-t}|\eta - \varrho|^t, \text{ for all } \eta, \varrho \in \mathbb{R}, \tag{28}$$

is valid for all $t \geq 2$.

Applying the above inequality in relation (27) we get

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n - \partial_{x_i} u_1|^{p_i(x)}\, dx = 0\,,$$

and, consequently, $\{u_n\}$ converges strongly to u_1 in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Then, by relation (21), we get

$$J(u_1) = \overline{c} > 0 \quad \text{and} \quad J'(u_1) = 0,$$

that is, u_1 is a nontrivial weak solution of problem (10).

Next, we prove that there exists a second weak solution $u_2 \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $u_2 \neq u_1$.

By Lemma 1 (i), on the boundary of the ball centered at the origin and of radius ρ_0 in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, denoted by $B_{\rho_0}(0)$, we have

$$\inf_{\partial B_{\rho_0}(0)} J > 0.$$

On the other hand, by Lemma 1 (iii), there exists $\Phi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $\Phi \geq 0$, $\Phi \neq 0$ and $J(t\Phi) < 0$, for $t > 0$ small enough. Moreover, by relations (17), (18), (19) and (5) we obtain that for any $u \in B_{\rho_0}(0)$, the inequality

$$J(u) \geq \frac{1}{P_+^+ N^{P_+^+ - 1}} \|u\|_{\overrightarrow{p}(\cdot)}^{P_+^+} - \frac{A}{a^-} C_1^{a^-} \|u\|_{\overrightarrow{p}(\cdot)}^{a^-} - \frac{B}{b^-} C_2^{b^-} \|u\|_{\overrightarrow{p}(\cdot)}^{b^-} \tag{29}$$

holds true and it follows that

$$-\infty < \underline{c} := \inf_{B_{\rho_0}(0)} J < 0. \tag{30}$$

Particularly, we have found that J is bounded from below.

Let ε such that

$$0 < \varepsilon < \inf_{\partial B_{\rho_0}(0)} J - \inf_{B_{\rho_0}(0)} J. \tag{31}$$

The same arguments as in the proof of Lemma 3.4 in [25] can be used in order to show that J is weakly lower semi-continuous on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Now, we concentrate our attention on functional $J : \overline{B_{\rho_0}(0)} \to \mathbb{R}$. Since $J \in C^1\left(\overline{B_{\rho_0}(0)}, \mathbb{R}\right)$, J is bounded from below in $\overline{B_{\rho_0}(0)}$ and J is weakly lower semi-continuous on $\overline{B_{\rho_0}(0)}$, we can apply Ekeland's Variational Principle for J (see [9]) in order to obtain that there exists $u_\varepsilon \in \overline{B_{\rho_0}(0)}$ such that:

1) $J(u_\varepsilon) \leq \inf\limits_{\overline{B_{\rho_0}(0)}} J + \varepsilon$

and

2) $J(u_\varepsilon) < J(u) + \varepsilon \cdot \|u - u_\varepsilon\|_{\overrightarrow{p}(\cdot)}, \ \forall \ u \in \overline{B_{\rho_0}(0)}$ with $u \neq u_\varepsilon$.

Actually, we have $J(u_\varepsilon) \leq \inf\limits_{\overline{B_{\rho_0}(0)}} J + \varepsilon \leq \inf\limits_{B_{\rho_0}(0)} J + \varepsilon < \inf\limits_{\partial B_{\rho_0}(0)} J$, since the last inequality holds true, then we get $u_\varepsilon \in B_{\rho_0}(0)$.

Now, we let $I : \overline{B_{\rho_0}(0)} \to \mathbb{R}$ defined by

$$I(u) = J(u) + \varepsilon \cdot \|u - u_\varepsilon\|_{\overrightarrow{p}(\cdot)}, \quad \forall \ u \in \overline{B_{\rho_0}(0)}.$$

It is clear that u_ε is a minimum point of functional I and thus

$$\frac{I(u_\varepsilon + t \cdot v) - I(u_\varepsilon)}{t} \geq 0, \tag{32}$$

for a small $t > 0$ and $v \in B_1(0)$. By relation (32), we deduce that

$$\frac{J(u_\varepsilon + t \cdot v) - J(u_\varepsilon)}{t} + \varepsilon \|v\|_{\overrightarrow{p}(\cdot)} \geq 0,$$

for a small $t > 0$ and $v \in B_1(0)$.

Passing, in the above inequality, to the limit as $t \to 0$, it follows that $\langle J'(u_\varepsilon), v\rangle + \varepsilon\|v\|_{\overrightarrow{p}(\cdot)} > 0$ and we infer that $\|J'(u_\varepsilon)\| \leq \varepsilon$. This implies the existence of a sequence $\{z_n\}$ in $B_{\rho_0}(0)$ such that

$$J(z_n) \to \underline{c} \quad \text{and} \quad J'(z_n) \to 0. \tag{33}$$

It is obvious that $\{z_n\}$ is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Since $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a reflexive space and $\{z_n\}$ is a bounded sequence, then there exists $u_2 \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that, up to a subsequence, $\{z_n\}$ converges weakly to u_2 in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Using similar arguments as in the case of the weak solution u_1, we can show that $\{z_n\}$ converges strongly to u_2 in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Then, since $J \in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ and relation (33) holds true, we obtain

$$J(u_2) = \underline{c} < 0 \quad \text{and} \quad J'(u_2) = 0, \tag{34}$$

that is, u_2 is a nontrivial weak solution for problem (10).

Finally, we conclude that $u_1 \neq u_2$ since

$$J(u_1) = \overline{c} > 0 > \underline{c} = J(u_2).$$

Thus, Theorem 2 is completely proved. ■

Acknowledgements. This work was partially supported by the strategic grant POSDRU/88/1.5/S/49516, Project ID 49516 (2009), co-financed by the european Social Fund- Investing in People, within the Sectorial Operational Programme Human Resources Development 2007-2013.

References

[1] E. Acerbi and G. Mingione, Regularity results for a class of functionals with nonstandard growth, *Arch. Rational Mech. Anal.* **156** (2001), 121-140.

[2] E. Acerbi and G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, *J. Reine Angew. Math.* **584** (2005), 117-148.

[3] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory, *J. Funct. Anal.* **14** (1973), 349–381.

[4] Y. Chen, S. Levine and R. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. Appl. Math.* **66** (2006), 1383-1406.

[5] L. Diening, *Theorical and numerical results for electrorheological fluids*, Ph.D. thesis, University of Frieburg, Germany, 2002.

[6] D. E. Edmunds, J. Lang, and A. Nekvinda, On $L^{p(x)}$ norms, *Proc. Roy. Soc. London Ser. A* **455** (1999), 219-225.

[7] D. E. Edmunds and J. Rákosník, Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. Roy. Soc. London Ser. A* **437** (1992), 229-236.

[8] D. E. Edmunds and J. Rákosník, Sobolev embedding with variable exponent, *Studia Math.* **143** (2000), 267-293.

[9] I. Ekeland, On the variational principle, *J. Math. Anal. App.* **47** (1974), 324-353.

[10] X. Fan, Remarks on eigenvalue problems involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.* **352** (2009), 85-98.

[11] X. Fan, J. Shen and D. Zhao, Sobolev Embedding Theorems for Spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **262** (2001), 749-760.

[12] X. L. Fan and Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.* **52** (2003), 1843-1852.

[13] X. Fan, Q. Zhang and D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, *J. Math. Anal. Appl.* **302** (2005), 306-317.

[14] X. L. Fan and D. Zhao, On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424-446.

[15] I. Fragalà, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasilinear equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** (2004) 751-734.

[16] M. Ghergu and V. Rădulescu, *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Its Applications, vol. 37, Oxford University Press, 2008.

[17] T. C. Halsey, Electrorheological fluids, *Science* **258** (1992), 761-766.

[18] P. Harjulehto, P. Hästö, Ú. V. Lê and M. Nuortio, *Overview of differential equations with non-standard growth*, preprint.

[19] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41** (1991), 592-618.

[20] M. Mihăilescu, Existence and multiplicity of solutions for an elliptic equation with $p(x)$-growth conditions, *Glasgow Math. J* **48** (2006), 411-418.

[21] M. Mihăilescu and G. Moroşanu, Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions, *Applicable Analysis* **89** (2) (2010), 257–271.

[22] M. Mihăilescu and G. Moroşanu, On an eigenvalue problem for an anisotropic elliptic equation involving variable exponents, *Glasgow Mathematical Journal* **52** (2010), 517–527.

[23] M. Mihăilescu, P. Pucci and V. Rădulescu, Nonhomogeneous boundary value problems in anisotropic Sobolev spaces, *C. R. Acad. Sci. Paris* **345** (2007), 561-566.

[24] M. Mihăilescu, P. Pucci, and V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.* **340** (2008), 687-698.

[25] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **462** (2006), 2625-2641.

[26] M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proceedings Amer. Math. Soc.* **135** (2007), No. 9, 2929-2937.

[27] M. Mihăilescu and V. Rădulescu, Continuous spectrum for a class of nonhomogeneous differential operators, *Manuscripta Mathematica* **125** (2008), 157-167.

[28] M. Mihăilescu and V. Rădulescu, Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces, *Ann. Inst. Fourier* **58** (2008), 2087-2111.

[29] M. Mihăilescu and V. Rădulescu, Spectrum in an unbounded interval for a class of nonhomogeneous differential operators, *Bull. London Math. Soc.* **40** (2008), 972-984.

[30] M. Mihăilescu and D. Stancu-Dumitru, On an eigenvalue problem involving the $p(x)$-Laplace operator plus a non-local term, *Differential Equations & Applications* **1** (2009), 367-378.

[31] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983.

[32] S. M. Nikol'skii, On imbedding, continuation and approximation theorems for differentiable functions of several variables, *Russian Math. Surveys* **16** (1961) 55-104.

[33] J. Rákosník, Some remarks to anisotropic Sobolev spaces I, *Beiträge Anal.* **13** (1979) 55-68.

[34] J. Rákosník, Some remarks to anisotropic Sobolev spaces II, *Beiträge Anal.* **15** (1981) 127-140.

[35] M. Ruzicka, Flow of shear dependent electrorheological fluids, *C. R. Acad. Sci. Paris Ser. I Math.* **329** (1999), 393-398.

[36] M. Ruzicka, *Electrorheological Fluids Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2002.

[37] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, *Ricerche Mat.* **18** (1969) 3-24.

[38] L. Ven'-tuan, On embedding theorems for spaces of functions with partial derivatives of various degree of summability, *Vestnik Leningrad. Univ.* **16** (1961) 23-37.

[39] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[40] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR Izv.* **29** (1987), 33-66.

Department of Mathematics, University of Craiova,
200585 Craiova, Romania
E-mail address: denisa.stancu@yahoo.com