Global well-posedness and scattering of a 2D Schrödinger equation with exponential growth

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Abstract

We investigate the initial value problem for a semi-linear Schrödinger equation with exponential-growth nonlinearity. We establish global well-posedness and scattering in the energy space.

1 Introduction

In this work, we study the initial value Schrödinger equation with exponential growth nonlinearity

$$i\partial_t u + \Delta_x u = \sigma f(u) \quad \text{on} \quad \mathbb{R}_t \times \mathbb{R}_x^2 \tag{1.1}$$

with data

$$u_0 := u(0,.) \in H^1(\mathbb{R}^2). \tag{1.2}$$

Where $\sigma \in \{-1, 1\}$, $u := u(t,x)$ is a complex-valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, and

$$f(u) := \lambda u (1 + 4\pi |u|^2)^{\frac{\alpha}{2}-1} \Big(\mathrm{e}^{(1+4\pi|u|^2)^{\frac{\alpha}{2}}} - e \Big), \lambda > 0, \alpha > 0. \tag{1.3}$$

*The author is supported by the Laboratory of PDE and Applications of the Faculty of Sciences of Tunis.

Received by the editors November 2008 - In revised form in March 2009.

Communicated by P. Godin.

2000 *Mathematics Subject Classification :* 35-xx, 35Q55, 35B60, 35B33, 37K07.

Key words and phrases : Nonlinear Schrödinger equation, well-posedness, Moser-Trudinger inequality, scattering.

A solution u of (1.1), satisfies formally conservation of the mass

$$M(u,t) := \|u(t)\|^2_{L^2(\mathbb{R}^2)}, \tag{1.4}$$

and the Hamiltonian

$$H(u,t) := \|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)} + \sigma\frac{\lambda}{2\pi\alpha}\|\mathrm{e}^{(1+4\pi|u(t)|^2)^{\frac{\alpha}{2}}} - e(1+4\pi|u(t)|^2)^{\frac{\alpha}{2}}\|_{L^1(\mathbb{R}^2)}. \tag{1.5}$$

We call energy of u,

$$E(u,t) := M(u,t) + H(u,t). \tag{1.6}$$

Before going further, we recall a few historic facts about well-posedness of the monomial defocusing semilinear Schrödinger equation

$$i\partial_t u + \Delta_x u = |u|^{p-1}u, \quad p > 1, \quad u : (-T^*, T^*) \times \mathbb{R}^d \to \mathbb{C} \tag{1.7}$$

denoted $NLS_p(\mathbb{R}^d)$ which was widely investigated. A solution u to (1.7) satisfies conservation of mass and Hamiltonian

$$H_p(u(t)) := \|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)} + \frac{2}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}(t,x)dx.$$

Moreover, for $\lambda > 0$,

$$u_\lambda : (-T^*\lambda^2, T^*\lambda^2) \times \mathbb{R}^d \to \mathbb{C}$$
$$(t,x) \longmapsto \lambda^{\frac{2}{1-p}} u(\lambda^{-2}t, \lambda^{-1}x)$$

solves (1.7). Note also that for $s_c := \frac{d}{2} - \frac{2}{p-1}$, the norm of $\dot{H}^{s_c}(\mathbb{R}^d)$ is relevant in the well-posedness theory of (1.7) since it's invariant under the mapping $f \longmapsto \lambda^{\frac{2}{1-p}} f(\lambda^{-1}.), \lambda > 0$.

We limit our discussion to $0 \leq s_c \leq 1$ [1]

1. $NLS_p(\mathbb{R}^d)$ **local well-posedness in** $H^s(\mathbb{R}^d)$. It is now known ([11],[19],[12]) that

   (a) If $s > s_c$, (1.7) is locally well-posed in H^s, with an existence interval depending only upon $\|u_0\|_{H^s}$.

   (b) For $s = s_c$, (1.7) is locally well-posed in H^s, with an existence interval depending upon $\mathrm{e}^{it\Delta}u_0$.

   (c) If $s < s_c$, (1.7) is ill-posed in H^s (see [7, 13, 8, 2, 35]).

   So, It's naturel to refer to H^{s_c} as the critical regularity for (1.7).

2. $NLS_p(\mathbb{R}^d)$ **global well-posedness** .

   (a) *The energy subcritical case,* $s_c < 1$. Using local well-posedness and the conservation laws of Hamiltonian and mass, we obtain global well-posedness of (1.7) in H^1. It is expected that the local H^{s_c} solutions of (1.7) extend to global solutions. For certain choice of p, d, there are

[1] If $s_c > 1$, (1.7) is locally well-posed in H^s, for $s > s_c$.

results (see for instance [3, 4, 16, 17, 38]) which show that H^s initial data u_0 evolve into global solutions of (1.7) for $s \in (\tilde{s}_{p,d}, 1)$ with $s < \tilde{s}_{p,d} < 1$ and $\tilde{s}_{p,d}$ close to 1 and away from s_c. For all problems with $0 \leq s_c < 1$, global well-posedness in the scale invariant space H^{s_c} is unknown but conjured to hold. Moreover the solutions scatter when $p > p_* := 1 + \frac{4}{d}$ [19, 28].

(b) *The energy critical case,* $s_c = 1$. Since the local existence interval does not depend only on $\|u_0\|_{H^1}$, an iteration of the local well-posedness theory fails to prove global well-posedness. But using new ideas of Bourgain in [4] (see also [5]) (which treated the radial case in dimension 3) and a new interaction Morawetz inequality [17] the energy critical case of (1.7) is now completely resolved [34, 39, 32]. Finite energy initial data u_0 evolve into global solution u with finite spacetime size $\|u\|_{L_{t,x}^{\frac{2(2+d)}{d-2}}} < \infty$ and scatter.

(c) *The energy supercritical case,* $s_c > 1$. Global well-posedness for the defocusing energy supercritical $NLS_p(\mathbb{R}^d)$ is an outstanding open problem (see [8, 2, 35] for some partial results).

3. **The two space dimensions case**. The initial value problem $NLS_p(\mathbb{R}^2)$ is energy subcritical for all $p > 1$. So it's natural to consider problems with exponential nonlinearities, which have several applications, as for example the self trapped beams in plasma (see [25]). Cazenave considered in [9] the Schrödinger equation with decreasing exponential nonlinearity and showed global well-posedness and scattering. With increasing exponentials the situation is more complicated because there's no a priori L^∞ control of the nonlinear term. Moreover the two dimensional case is interesting because of its relation to the critical Moser-Trudinger inequalities (see [1, 33]). The two dimensional Schrödinger problems with exponential growth nonlinearities was studied, for small cauchy data, by Nakamoura and Ozawa in [30]. They proved global well-posedness and scattering.
Later on, Colliander-Ibrahim-Majdoub-Masmoudi considered the Schrödinger equation (see [15]),

$$i\partial_t u + \Delta_x u = \sigma u(\mathrm{e}^{4\pi|u|^2} - 1) \quad \text{on} \quad \mathbb{R}_t \times \mathbb{R}_x^2,\ \sigma \in \{-1, 1\}. \tag{1.8}$$

They obtained local well-posedness (resp global well-posedness in the defocusing case) for $H(u_0) \leq 1$ and an instability result for $H(u_0) > 1$ (similar results was proved in the case of wave equation [23, 24]). Subtracting the cubic term of (1.8) nonlinearity, Ibrahim-Majdoub-Masmoudi-Nakanishi proved recently in [21] scattering of

$$i\partial_t u + \Delta_x u = u(\mathrm{e}^{4\pi|u|^2} - 1 - 4\pi|u|^2) \quad \text{on} \quad \mathbb{R}_t \times \mathbb{R}_x^2 \tag{1.9}$$

in the case $H(u_0) < 1$. They used a new interaction Morawetz estimate proved independently by Colliander et al. and Planchon-Vega [14, 31]. The case $H(u_0) = 1$ is an open problem. (Similar results was proved in the case of wave equation [21, 29]). In the light of [15, 21], we consider the Schrödinger equation (1.1) with exponential nonlinearity for $\alpha < 2$. Note that if we fix $\lambda = \mathrm{e}^{-1}$ and $\alpha = 2$, we find exactly (1.8). The case $\alpha > 2$ is an open problem. We show local well-posedness (resp global well-posedness in the defocusing case) in the space $\mathcal{C}([0,T], H^1(\mathbb{R}^2)) \cap L^4([0,T], W^{1,4}(\mathbb{R}^2))$, $T > 0$ (resp in $\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)) \cap L^4_{loc}(W^{1,4}(\mathbb{R}^2)))$. We prove uniqueness of solution in the energy space $C([0,T], H^1(\mathbb{R}^2))$. Similar well-posedness results were proved for the nonlinear Klein-Gordon equation with exponential nonlinearity in [26]. Then we subtract the cubic term of (1.1) nonlinearity and prove scattering in the energy space.

Remark 1.1. *Comparing this work with [15, 21] we note that*

1. *For $\alpha = 2$, well-posedness in the energy space was proved in [15] under the condition $H(u_0) \leq 1$ and using a logarithmic inequality (see [22]). But for $0 \leq \alpha < 2$, we establish well-posedness in the energy space without any restrictive condition, moreover we don't use any logarithmic inequality.*

2. *For $\alpha = 2$, scattering in the energy space was proved in [21] under the condition $H(u_0) < 1$, but for $0 \leq \alpha < 2$, we establish scattering in the energy space without any restrictive condition.*

3. *In order to prove scattering we have subtracted the cubic part from our nonlinearity f to avoid the critical exponent p_*.*

In the following subsection we give our main results.

1.1 Main results

Our first result is the following local well-posedness Theorem obtained by a classical fixed point argument.

Theorem 1.2. *There exists $T > 0$ and a unique solution u to* (1.1)-(1.2) *in the class*

$$\mathcal{C}([0,T], H^1(\mathbb{R}^2)).$$

Moreover $u \in L^4([0,T], W^{1,4}(\mathbb{R}^2))$, and satisfies for all $0 \leq t < T$, $M(u,t) = M(u,0), H(u,t) = H(u,0)$. We recall the Hamiltonian

$$H(u,t) := \|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)} + \sigma \frac{\lambda}{2\pi\alpha} \|\mathrm{e}^{(1+4\pi|u(t)|^2)^{\frac{\alpha}{2}}} - e(1+4\pi|u(t)|^2)^{\frac{\alpha}{2}}\|_{L^1(\mathbb{R}^2)}.$$

Where $0 < \alpha < 2, \lambda > 0$, and $\sigma \in \{-1, 1\}$.

In the defocusing case, using the local theory we derive global well-posedness in the energy space. The crucial point here is that the local existence time depends only on the size of the data and not on its profile.

Theorem 1.3. *In the defocusing case* $(\sigma = 1)$*, the cauchy problem* (1.1)-(1.2) *has a unique global solution u in the class*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)).$$

Moreover, $u \in L^4_{loc}(\mathbb{R}, W^{1,4}(\mathbb{R}^2))$*, and for all* $0 \leq t < T$, $M(u,t) = M(u,0)$, $H(u,t) = H(u,0)$.

In order to avoid a scattering critical exponent $p_* := 1 + \frac{4}{d}$ (see [28]), we subtract the cubic part from the nonlinearity f and we prove scattering in the energy space. In fact we show that every global solution of (1.1) is asymptotic, as $t \to \pm\infty$, to a solution of the linear Schrödinger equation

$$i\partial_t v + \Delta v = 0. \tag{1.10}$$

In other words, the effect of the nonlinearity is negligible for large times. Precisely we have the following scattering result

Theorem 1.4. *Assume that*

$$f(u) := \lambda u(1+4\pi|u|^2)^{\frac{\alpha}{2}-1}\Big(\mathrm{e}^{(1+4\pi|u|^2)^{\frac{\alpha}{2}}} - e(1+4\pi|u|^2)^{\frac{\alpha}{2}}\Big), \alpha \in [0,2[, \lambda \geq 0. \tag{1.11}$$

For any global solution $u \in C(\mathbb{R}, H^1)$ *of* (1.1)*, there exist unique free solutions* $u_\pm$ *of* (1.10) *such that*

$$\|(u-u_\pm)(t)\|_{H^1(\mathbb{R}^2)} \longrightarrow 0 \quad as \quad t \to \pm\infty.$$

Moreover, the map

$$H^1(\mathbb{R}^2) \longrightarrow H^1(\mathbb{R}^2) \qquad u(0) \longmapsto u_\pm(0) \tag{1.12}$$

is an homeomorphism.

Remark 1.5. *We note that*

1. *A similar result was proved for,* $\alpha = 2, \lambda = \mathrm{e}^{-1}$*, under the condition* $H(u_0) < 1$ *and where the homeomorphism* (1.12) *is not global but from* $\{\phi \in H^1(\mathbb{R}^2); H(\phi) < 1\}$ *onto* $\{\phi \in H^1(\mathbb{R}^2); \|\nabla\phi\|_{L^2(R^2)} < 1\}$*. See [21] for more details.*

2. *A complete scattering theory is available in the case* $p_* = 1 + \frac{4}{d}$*, in the conformal space of functions* $f \in H^1(\mathbb{R}^d)$ *such that* $\int |x|^2|f(x)|^2 dx < \infty$ *(see [18, 37, 20]).*

In what follows, we collect some estimates needed in the sequel.

1.2 Tools

In order to control the solution of (1.1), we will use the following Strichartz estimate (see [10]).

Proposition 1.6. (Strichartz estimate) *Let $I \subset \mathbb{R}$ be a time slab, $t_0 \in I$ and $G \in L^{\alpha'}(I, W^{1,\beta'}(\mathbb{R}^2))$. There exists a positive real number C such that if $u := u(t,x)$ a solution in $\mathcal{C}(I, H^1(\mathbb{R}^2))$ of the linear problem*

$$i\partial_t u + \Delta_x u = G, \qquad u(t_0, .) \in H^1(\mathbb{R}^2),$$

then

$$\|u\|_{L^q(I,W^{1,r}(\mathbb{R}^2))} \leq C\Big(\|u(t_0,.)\|_{H^1(\mathbb{R}^2)} + \|G\|_{L^{\alpha'}(I,W^{1,\beta'}(\mathbb{R}^2))}\Big). \tag{1.13}$$

Where $1 \leq r, \beta < \infty$ and

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}, \frac{1}{\alpha} + \frac{1}{\alpha'} = \frac{1}{\beta} + \frac{1}{\beta'} = 1.$$

In particular we have the following energy estimate.

Proposition 1.7. (Energy estimate) *With the same hypothesis we have*

$$\sup_{t\in I} \|u(t,.)\|_{H^1(\mathbb{R}^2)} \leq C\Big(\|u(t_0,.)\|_{H^1(\mathbb{R}^2)} + \|G\|_{L^1(I,H^1(\mathbb{R}^2))}\Big). \tag{1.14}$$

In order to control the nonlinear part of the energy in the $L^1_t(H^1_x)$, we will use the following Moser-Trudinder inequality [1, 27, 36].

Proposition 1.8. (Moser-Trudinger inequality) *Let $\alpha \in (0, 4\pi)$, a constant $\mathcal{C}_\alpha$ exists such that for all $u \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$, we have*

$$\int_{\mathbb{R}^2} \Big(e^{\alpha|u(x)|^2} - 1\Big) dx \leq \mathcal{C}_\alpha \|u\|^2_{L^2(\mathbb{R}^2)}. \tag{1.15}$$

Moreover, (1.15) *is false if $\alpha \geq 4\pi$.*

Remark 1.9. *$\alpha = 4\pi$ becomes admissible if we take $\|u\|_{H^1(\mathbb{R}^2)} \leq 1$ rather than $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. In this case*

$$\mathcal{K} := \sup_{\|u\|_{H^1(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \Big(e^{4\pi|u(x)|^2} - 1\Big) dx < \infty \tag{1.16}$$

and this is false for $\alpha > 4\pi$. See [33] for more details.

These estimates will be coupled with the following absorption result

Lemma 1.10. (Bootstrap Lemma) *Let $T > 0$ and $X \in C([0,T], \mathbb{R}_+)$ such that*

$$X \leq a + bX^\theta, \text{ on } [0,T]$$

where, $a, b > 0, \theta > 1, a < (1 - \frac{1}{\theta})\frac{1}{(\theta b)^{\frac{1}{\theta}}}$ and $X(0) \leq \frac{1}{(\theta b)^{\frac{1}{\theta - 1}}}$. Then

$$X \leq \frac{\theta}{\theta - 1} a, \text{ on } [0,T].$$

Finally we recall the following Sobolev embedding

Proposition 1.11. (Sobolev embedding) *We have*

$$W^{s,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$$

whenever

$$1 < p < q < \infty, s > 0 \quad \text{and} \quad \frac{1}{p} \leq \frac{1}{q} + \frac{s}{d}.$$

The rest of the paper is organized as follows. First, we show local well-posedness of (1.1) using a standard fixed point argument. Second, we show global well-posedness in the defocusing case. In the last section we prove the scattering Theorem 1.4.
We mention that C is an absolute positive constant which may vary from line to line.If A and B are nonnegative real numbers, $A \lesssim B$ (resp$A \simeq B$) means that $A \leq CB$ (resp $B \lesssim A \lesssim B$).

2 Local well-posedness

This section is devoted to the proof of Theorem 1.2. First, we prove the local existence by a fixed point argument.

2.1 Local Existence

We start with the following technical lemma which is crucial in the proof of Theorem 1.2.

Lemma 2.1. *For any positive real number ε there exists a positive constant C_ε such that*

$$|f(z_1) - f(z_2)| \leq C_\varepsilon |z_1 - z_2| \sum_{i=1,2} \left(e^{\varepsilon |z_i|^2} - 1 \right). \tag{2.17}$$

Proof of Lemma 2.1. Let us identify f with the $\mathcal{C}^\infty$ function defined on $\mathbb{R}^2$ and denote by Df the $\mathbb{R}^2$ derivative of the identified function. Then using the mean value theorem and the convexity of the exponential function, we derive the following property

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq |z_1 - z_2| \sup_{[z_1, z_2]} |Df(z)| \\ &\lesssim |z_1 - z_2| \sum_{i=1,2} \left((1 + |z_i|^2)(e^{(1+4\pi|z_i|^2)^{\frac{\alpha}{2}}} - e) + |z_i|^2 e^{(1+4\pi|z_i|^2)^{\frac{\alpha}{2}}} \right), \end{aligned}$$

and the conclusion follows. ∎

Remark 2.2. *Of course we have (using an easy computation)*

$$C_\varepsilon \to +\infty \quad as \quad \varepsilon \to 0,$$

but this fact not matter in the rest of the proof. The reason is that in the proof we shall choose a small (but fixed) ε in order to use Trudinger-Moser type inequality. Actually, the choice of

$$\varepsilon = \frac{\pi}{3\Big(1 + \|u_0\|_{H^1(\mathbb{R}^2)}\Big)^2},$$

is enough to carry out the proof.

Remark 2.3. *In what follows, the symbol C_ε stands for a constant depending on ε but may vary from line to line.*

Let T be a positive real time and w the solution of the following free Schrödinger problem

$$i\partial_t w + \Delta w = 0, \qquad w(0,.) = u_0. \tag{2.18}$$

We recall that the space $\mathcal{E}_T := C([0,T], H^1(\mathbb{R}^2)) \cap L^4([0,T], W^{1,4}(\mathbb{R}^2))$ is complete under the norm

$$\|h\|_T := \sup_{t\in[0,T]} \|h(t,.)\|_{H^1(\mathbb{R}^2)} + \|h\|_{L^4_T(W^{1,4}(\mathbb{R}^2))}.$$

Let $\mathcal{E}_T(1)$ be the ball in $\mathcal{E}_T$ with center zero and radius 1. We consider the map ϕ on $\mathcal{E}_T(1)$ given by $\phi(v) = \tilde{v}$, where $\tilde{v}$ solves

$$i\partial_t \tilde{v} + \Delta \tilde{v} = \sigma f(v + w), \qquad \tilde{v}(0,.) = 0. \tag{2.19}$$

We prove that the map ϕ leaves $\mathcal{E}_T(1)$ stable and is a contraction for T sufficiently small. Applying the energy and Strichartz estimate (1.13)-(1.14) to $v_1, v_2 \in \mathcal{E}_T(1)$, we get

$$\|\tilde{v_1} - \tilde{v_2}\|_T \lesssim \|f(v_1 + w) - f(v_2 + w)\|_{L^1([0,T],H^1(\mathbb{R}^2))} := \|f(u_1) - f(u_2)\|_{L^1([0,T],H^1(\mathbb{R}^2))}.$$

Using Sobolev embedding and (2.17), we deduce for any $\varepsilon > 0$,

$$\begin{aligned} \|f(u_1) - f(u_2)\|^2_{L^2(\mathbb{R}^2)} &\lesssim C_\varepsilon \Big\| |u_1 - u_2|^2 \sum_{i=1,2} \Big(e^{\varepsilon|u_i|^2} - 1\Big) \Big\|_{L^1(\mathbb{R}^2)} \\ &\lesssim C_\varepsilon \sum_{i=1,2} \Big\| |u_1 - u_2|^2 \Big(e^{\varepsilon|u_i|^2} - 1\Big) \Big\|_{L^1(\mathbb{R}^2)} \\ &\lesssim C_\varepsilon \|u_1 - u_2\|^2_{L^4(\mathbb{R}^2)} \sum_{i=1,2} \|e^{2\varepsilon|u_i|^2} - 1\|^{\frac{1}{2}}_{L^1(\mathbb{R}^2)}. \end{aligned} \tag{2.20}$$

On other hand, using the energy conservation, we get

$$\|v_i + w\|^2_{H^1(\mathbb{R}^2)} \leq \Big(1 + \|u_0\|_{H^1(\mathbb{R}^2)}\Big)^2.$$

Now, let ε be a real number satisfying

$$0 < \varepsilon \leq \frac{\pi}{3\Big(1 + \|u_0\|_{H^1(\mathbb{R}^2)}\Big)^2}. \tag{2.21}$$

Using Moser-Trudinger inequality (1.16) we have

$$\begin{aligned}\|\mathrm{e}^{2\varepsilon|u_i|^2}-1\|_{L^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2}\Big(\mathrm{e}^{4\pi(\sqrt{\frac{\varepsilon}{2\pi}}|u_i|)^2}-1\Big)dx\\ &\leq \mathcal{K}.\end{aligned}$$

Thus by (2.20)

$$\|f(u_1)-f(u_2)\|_{L^1_TL^2(\mathbb{R}^2)} \lesssim T^{\frac{3}{4}}\|u_1-u_2\|_T. \tag{2.22}$$

It remains to estimate $\|\nabla(f(u_1)-f(u_2))\|_{L^1_TL^2(\mathbb{R}^2)}$. Write

$$\begin{aligned}&\|\nabla(f(u_1)-f(u_2))\|_{L^2(\mathbb{R}^2)}\\ &\quad\leq \|\nabla u_1(Df(u_1)-Df(u_2))\|_{L^2(\mathbb{R}^2)}+\|Df(u_2)(\nabla u_1-\nabla u_2)\|_{L^2(\mathbb{R}^2)}\\ &\quad= (\mathcal{I}_1)+(\mathcal{I}_2).\end{aligned}$$

With a convexity argument, we get, for $z_1,z_2\in\mathbb{C}$,

$$|Df(z_1)-Df(z_2)| \lesssim |z_1-z_2|\sum_{i=1,2}|z_i|(1+|z_i|^2)\mathrm{e}^{(1+|z_i|^2)^{\frac{\alpha}{2}}}. \tag{2.23}$$

Now, taking for $0\leq\alpha<2$ and $\varepsilon>0$,

$$C_{\alpha,\varepsilon} := \sup_{\mathbb{R}}\frac{|x|(1+x^2)\mathrm{e}^{(1+x^2)^{\frac{\alpha}{2}}}}{|x|+\mathrm{e}^{\varepsilon x^2}-1} \tag{2.24}$$

and using Moser-Trudinger inequality (1.16), we have

$$\begin{aligned}\mathcal{I}_1 &\lesssim \sum_{i=1,2}\|\nabla u_1(u_1-u_2)(|u_i|+\mathrm{e}^{\varepsilon|u_i|^2}-1)\|_{L^2(\mathbb{R}^2)}\\ &\lesssim \sum_{i=1,2}\|\nabla u_1(u_1-u_2)u_i\|_{L^2(\mathbb{R}^2)}+\|\nabla u_1(u_1-u_2)(\mathrm{e}^{\varepsilon|u_i|^2}-1)\|_{L^2(\mathbb{R}^2)}\\ &\lesssim \sum_{i=1,2}\|\nabla u_1\|_{L^4(\mathbb{R}^2)}\|u_1-u_2\|_{L^8(\mathbb{R}^2)}\|u_i\|_{L^8}+\\ &\qquad\qquad \|\nabla u_1\|_{L^4(\mathbb{R}^2)}\|u_1-u_2\|_{L^8}\|\mathrm{e}^{\varepsilon|u_i|^2}-1\|_{L^8(\mathbb{R}^2)}\\ &\lesssim \|u_1-u_2\|_T\|\nabla u_1\|_{L^4(\mathbb{R}^2)}\sum_{i=1,2}\Big(\|u_i\|_{L^8(\mathbb{R}^2)}+\|\mathrm{e}^{\varepsilon|u_i|^2}-1\|_{L^8(\mathbb{R}^2)}\Big)\\ &\lesssim \|u_1-u_2\|_T\|\nabla u_1\|_{L^4(\mathbb{R}^2)}(1+\|u_0\|_{H^1}).\end{aligned}$$

Moreover

$$\begin{aligned}\mathcal{I}_2 &\leq C_\varepsilon\|\nabla(u_1-u_2)(\mathrm{e}^{\varepsilon|u_2|^2}-1)\|_{L^2(\mathbb{R}^2)}\\ &\leq C_\varepsilon\|\nabla(u_1-u_2)\|_{L^4(\mathbb{R}^2)}\|\mathrm{e}^{\varepsilon|u_2|^2}-1\|_{L^4(\mathbb{R}^2)}\\ &\lesssim \|\nabla(u_1-u_2)\|_{L^4(\mathbb{R}^2)}.\end{aligned}$$

Integrating with respect to time, we obtain

$$\begin{aligned}\|\nabla(f(u_1)-f(u_2))\|_{L^1_TL^2(\mathbb{R}^2)} &\lesssim \|u_1-u_2\|_T\|\nabla u_1\|_{L^1_TL^4(\mathbb{R}^2)}(1+\|u_0\|_{H^1(\mathbb{R}^2)})\\ &\quad +\|\nabla(u_1-u_2)\|_{L^1_TL^4(\mathbb{R}^2)}\\ &\lesssim T^{\frac34}\Big(\|u_1-u_2\|_T\|\nabla u_1\|_{L^4_TL^4(\mathbb{R}^2)}(1+\|u_0\|_{H^1(\mathbb{R}^2)})\\ &\quad +\|\nabla(u_1-u_2)\|_{L^4_TL^4(\mathbb{R}^2)}\Big)\\ &\lesssim T^{\frac34}\|u_1-u_2\|_T\Big(\|\nabla u_1\|_{L^4_TL^4(\mathbb{R}^2)}(1+\|u_0\|_{H^1(\mathbb{R}^2)})+1\Big)\\ &\lesssim T^{\frac34}\|u_1-u_2\|_T(\|u_0\|^2_{H^1(\mathbb{R}^2)}+1).\end{aligned}$$

Thus

$$\|\tilde{v_1}-\tilde{v_2}\|_T\lesssim T^{\frac34}\Big(\|u_0\|^2_{H^1(\mathbb{R}^2)}+1\Big)\|v_1-v_2\|_T. \tag{2.25}$$

For $v_2=-w$, $\tilde{v_2}=0$, so

$$\begin{aligned}\|\tilde{v_1}\|_T &\lesssim T^{\frac34}\Big(\|u_0\|^2_{H^1(\mathbb{R}^2)}+1\Big)\|v_1+w\|_T\\ &\lesssim T^{\frac34}\Big(\|u_0\|^2_{H^1(\mathbb{R}^2)}+1\Big)\Big(1+2\|u_0\|_{H^1(\mathbb{R}^2)}\Big).\end{aligned} \tag{2.26}$$

We conclude by (2.25)-(2.26) that for small T, ϕ is a contraction which maps $\mathcal{E}_T(1)$ into itself, which prove existence of local solution to (1.1)-(1.2).

2.2 Uniqueness in the energy space

In what follows, we prove the uniqueness of solution to the Cauchy problem (1.1)-(1.2) in the energy space.[2] Let T be a positive time, u_1 and u_2 two solutions to (1.1)-(1.2) in $\mathcal{C}_T(H^1(\mathbb{R}^2))$. Then, setting $w:=u_1-u_2$

$$i\partial_t w+\Delta w=f(u_1)-f(u_2) \qquad w(0,.)=0. \tag{2.27}$$

By energy estimate (1.14),

$$\|w\|_{L^\infty_TH^1(\mathbb{R}^2)}\lesssim\|f(u_1)-f(u_2)\|_{L^1([0,T],H^1(\mathbb{R}^2))}.$$

Using the precedent computation and the fact that there exists $T>0$ such that $\|u_i\|_{L^\infty_TH^1(\mathbb{R}^2)}\le 1+\|u_0\|_{H^1(\mathbb{R}^2)}$, $i\in\{1,2\}$, we have

$$\|f(u_1)-f(u_2)\|_{L^1_TL^2(\mathbb{R}^2)}\lesssim T\|w\|_{L^\infty_TH^1(\mathbb{R}^2)},$$

and $\|\nabla(f(u_1)-f(u_2))\|_{L^1_TL^2(\mathbb{R}^2)}$

$$\begin{aligned}&\lesssim \|w\|_{L^\infty_TH^1(\mathbb{R}^2)}\|\nabla u_1\|_{L^1_TL^4(\mathbb{R}^2)}(1+\|u_0\|_{H^1(\mathbb{R}^2)})+\|\nabla w\|_{L^1_TL^4(\mathbb{R}^2)}\\ &\lesssim T^{\frac43}(\|w\|_{L^\infty_TH^1(\mathbb{R}^2)}\|\nabla u_1\|_{L^4_TL^4(\mathbb{R}^2)}(1+\|u_0\|_{H^1(\mathbb{R}^2)})+\|\nabla w\|_{L^4_TL^4(\mathbb{R}^2)}).\end{aligned} \tag{2.28}$$

The following Lemma concludes the uniqueness proof.

[2]Note that the uniqueness is in the energy space.

Lemma 2.4. *We have*

1. $\|\nabla w\|_{L^{4}_{T}L^{4}(\mathbb{R}^2)} \lesssim \|w\|_{L^{\infty}_{T}H^1(\mathbb{R}^2)}(1+\|u_0\|_{H^1(\mathbb{R}^2)})^2 T^{\frac{5}{6}}$

2. $\|\nabla u_1\|_{L^{4}_{T}L^{4}(\mathbb{R}^2)} \lesssim 1+(1+\|u_0\|_{H^1(\mathbb{R}^2)})T^{\frac{5}{6}}.$

Proof. By Strichartz estimate (1.13)

$$\|\nabla w\|_{L^{4}_{T}L^{4}(\mathbb{R}^2)} \lesssim \|\nabla(f(u_1)-f(u_2))\|_{L^{\frac{6}{5}}([0,T],L^{\frac{3}{2}}(\mathbb{R}^2))}.$$

Moreover $\|\nabla(f(u_1)-f(u_2))\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}$

$$\begin{aligned} &\leq \|\nabla u_1(Df(u_1)-Df(u_2))\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} + \|\nabla w Df(u_2)\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \\ &= \mathcal{J}_1+\mathcal{J}_2. \end{aligned}$$

Using (2.23) – (2.24) and Moser-Trudinger inequality (1.16) for ε satisfying (2.21), we get

$$\begin{aligned} \mathcal{J}_1 &\lesssim \sum_{i=1,2} \|\nabla u_1(u_1-u_2)(|u_i|+\mathrm{e}^{\varepsilon|u_i|^2}-1)\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \\ &\lesssim \sum_{i=1,2}\Big(\|\nabla u_1 w u_i\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} + \|\nabla u_1 w(\mathrm{e}^{\varepsilon|u_i|^2}-1)\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}\Big) \\ &\lesssim \sum_{i=1,2}\Big(\|\nabla u_1\|_{L^2(\mathbb{R}^2)}\|w\|_{L^{12}(\mathbb{R}^2)}\|u_i\|_{L^{12}(\mathbb{R}^2)} \\ &\qquad\qquad + \|\nabla u_1\|_{L^2(\mathbb{R}^2)}\|w\|_{L^{12}(\mathbb{R}^2)}\|\mathrm{e}^{\varepsilon|u_i|^2}-1\|_{L^{12}(\mathbb{R}^2)}\Big) \\ &\lesssim \|w\|_{L^{\infty}_{T}H^1(\mathbb{R}^2)}(1+\|u_0\|_{H^1(\mathbb{R}^2)})^2, \end{aligned}$$

and for $\varepsilon>0$,

$$\begin{aligned} \mathcal{J}_2 &\lesssim \|\nabla w(\mathrm{e}^{\varepsilon|u_2|^2}-1)\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \\ &\lesssim \|\nabla w\|_{L^2(\mathbb{R}^2)}\|\mathrm{e}^{\varepsilon|u_2|^2}-1\|_{L^6(\mathbb{R}^2)} \\ &\lesssim \|\nabla w\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \|w\|_{L^{\infty}_{T}H^1(\mathbb{R}^2)}. \end{aligned}$$

So

$$\|\nabla(f(u_1)-f(u_2))\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \lesssim \|w\|_{L^{\infty}_{T}H^1(\mathbb{R}^2)}(1+\|u_0\|_{H^1(\mathbb{R}^2)})^2.$$

Thus for the Strichartz couple $(\alpha',\beta')=(\frac{6}{5},\frac{3}{2})$

$$\begin{aligned} \|\nabla w\|_{L^{4}_{T}L^{4}(\mathbb{R}^2)} &\lesssim \|\nabla(f(u_1)-f(u_2))\|_{L^{\alpha'}([0,T],L^{\beta'})(\mathbb{R}^2)} \\ &\lesssim T^{\frac{5}{6}}\|w\|_{L^{\infty}_{T}H^1(\mathbb{R}^2)}(1+\|u_0\|_{H^1(\mathbb{R}^2)})^2. \end{aligned}$$

Similarly we have

$$\begin{aligned}
\|\nabla u_1\|_{L^4_T L^4(\mathbb{R}^2)} &\lesssim \|u_1(0)\|_{H^1(\mathbb{R}^2)} + \|\nabla f(u_1)\|_{L^{\frac{6}{5}}([0,T],L^{\frac{3}{2}}(\mathbb{R}^2))} \\
&\lesssim 1 + \|\nabla u_1 (e^{\varepsilon|u_1|^2} - 1)\|_{L^{\frac{6}{5}}([0,T],L^{\frac{3}{2}}(\mathbb{R}^2))} \\
&\lesssim 1 + \|u_1\|_{L^\infty_T H^1(\mathbb{R}^2)} T^{\frac{5}{6}} \\
&\lesssim 1 + (1 + \|u_0\|_{H^1(\mathbb{R}^2)}) T^{\frac{5}{6}}.
\end{aligned}$$

■

Thus, by Lemma 2.4 and (2.28), for small time T

$$\begin{aligned}
&\|w\|_{L^\infty_T H^1(\mathbb{R}^2)} \\
&\quad\lesssim \|f(u_1) - f(u_2)\|_{L^1([0,T],H^1(\mathbb{R}^2))} \\
&\quad\lesssim T\|w\|_{L^\infty_T H^1(\mathbb{R}^2)} + T^{\frac{4}{3}}\Big(\|w\|_{L^\infty_T H^1(\mathbb{R}^2)}\|\nabla u_1\|_{L^4_T L^4(\mathbb{R}^2)}(1 + \|u_0\|_{H^1(\mathbb{R}^2)}) \\
&\qquad\qquad + \|\nabla w\|_{L^4_T L^4(\mathbb{R}^2)}\Big) \\
&\quad\lesssim \|w\|_{L^\infty_T H^1(\mathbb{R}^2)}\Big(T + T^{\frac{4}{3}}(1 + \|u_0\|_{H^1(\mathbb{R}^2)})\Big).
\end{aligned}$$

So for small time T

$$\|w\|_{L^\infty_T H^1(\mathbb{R}^2)} = 0.$$

Which prove the uniqueness for small time and so for all time.

3 Global well-posedness in the defocusing case

This section is devoted to prove Theorem1.3 in the case $\sigma = 1$. We recall an important fact that is the time of local existence depends only on the quantity $\|u_0\|_{H^1(\mathbb{R}^2)}$. Let u be the unique maximal solution of (1.1) in the space $\mathcal{E}_{T^*}$ with initial data u_0, where $0 < T^* \leq +\infty$ is the lifespan of u. We shall prove that u is global. By contradiction, suppose that $T^* < +\infty$, we consider for $0 < s < T^*$, the following problem

$$(\mathcal{P}_s)\begin{cases} i\partial_t v + \Delta v &= f(v) \\ v(s,.) &= u(s,.). \end{cases}$$

Using the same arguments used in the local existence, and taking

$$\varepsilon \leq \frac{\pi}{1 + 2E(u(0))},$$

we can find a real $\tau > 0$ and a solution v to $(\mathcal{P}_s)$ on $[s, s+\tau]$. According to the section of local existence, and using the conservation of energy, τ does not depend on s. Thus, if we let s be close to T^* such that $s + \tau > T^*$, we can extend v for times higher than T^*. This fact contradicts the maximality of T^*. We obtain the result claimed in Theorem 1.3.

4 Scattering

In this section we prove, as claimed in Theorem1.4, the scattering of the following nonlinear Schrödinger problem

$$(\mathcal{P})\left\{\begin{array}{lcl} i\partial_t u+\Delta u &=& f(u)\\ u(0,.) &=& u_0\in H^1(\mathbb{R}^2). \end{array}\right.$$

Where

$$f(u):=\lambda u(1+4\pi|u|^2)^{\frac{\alpha}{2}-1}\Big(\mathrm{e}^{(1+4\pi|u|^2)^{\frac{\alpha}{2}}}-e(1+4\pi|u|^2)^{\frac{\alpha}{2}}\Big),\alpha\in[0,2[,\lambda\geq 0. \tag{4.29}$$

For any global solution $u\in C(\mathbb{R},H^1)$ of (1.1) and any time slab $I\subset\mathbb{R}$, we denote

$$\|u\|_{S^1(I)}=\|u\|_{L^\infty(I,H^1(\mathbb{R}^2))}+\|u\|_{L^4(I,W^{1,4}(\mathbb{R}^2))}.$$

In order to prove of Theorem 1.4 we will use the following Lemma

Lemma 4.1. *For any global solution $u\in C(\mathbb{R},H^1)$ of* (1.1)*, any time slab I*

$$\|u\|_{S^1(I)}\lesssim\|u(T)\|_{H^1(\mathbb{R}^2)}+\|u\|_{L^4(I,L^8(\mathbb{R}^2))}\|u\|^2_{S^1(I)}\quad ,T\in I.$$

Proof. By Strichartz-estimate (1.13) we have

$$\forall\eta\in]0,\frac{1}{2}],\quad \|u\|_{S^1(I)}\lesssim\|u(T)\|_{H^1(\mathbb{R}^2)}+\|f(u)\|_{L^{\frac{2}{1+2\eta}}(I,W^{1,\frac{1}{1-\eta}}(\mathbb{R}^2))}. \tag{4.30}$$

Take $\eta=\frac{1}{4}$, by Moser-Trudinger (1.16),

$$\begin{aligned}\|f(u)\|_{L^{\frac{1}{1-\eta}}(\mathbb{R}^2)} &= \lambda\|u(1+4\pi|u|^2)^{\frac{\alpha}{2}-1}\Big(\mathrm{e}^{(1+4\pi|u|^2)^{\frac{\alpha}{2}}}-e(1+4\pi|u|^2)^{\frac{\alpha}{2}}\Big)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\\ &\lesssim \||u|^3(\mathrm{e}^{\varepsilon|u|^2}-1)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)},\quad\text{for}\quad\varepsilon>0\\ &\lesssim \|u\|^3_{L^{\frac{24}{5}}(\mathbb{R}^2)}.\end{aligned}$$

With the interpolation inequality $\|\cdot\|^3_{L^{\frac{24}{5}}}\leq\|\cdot\|^2_{L^4}\|\cdot\|_{L^8}$, we have

$$\begin{aligned}\|f(u)\|_{L^{\frac{4}{3}}(I,L^{\frac{4}{3}}(\mathbb{R}^2))} &\lesssim \|\|u(t)\|^2_{L^4(\mathbb{R}^2)}\|u(t)\|_{L^8(\mathbb{R}^2)}\|_{L^{\frac{4}{3}}(I)}\\ &\lesssim \|u\|^2_{L^4(I,L^4(\mathbb{R}^2))}\|u\|_{L^4(I,L^8(\mathbb{R}^2))} \qquad (4.31)\\ &\lesssim \|u\|^2_{S^1(I)}\|u\|_{L^4(I,L^8(\mathbb{R}^2))}. \qquad (4.32)\end{aligned}$$

It remains to control $\|\nabla f(u)\|_{L^{\frac{4}{3}}(I,L^{\frac{4}{3}}(\mathbb{R}^2))}$. We recall that

$$\forall\varepsilon>0,\quad |\nabla f(u)|\lesssim|\nabla u||u|^2(\mathrm{e}^{\varepsilon|u|^2}-1).$$

Using Hölder and Moser-Trudinger inequalities, coupled with the interpolation inequality $\|\cdot\|^2_{L^{\frac{16}{3}}} \leq \|\cdot\|_{L^8}\|\cdot\|_{L^4}$ we get

$$\begin{aligned}
\|\nabla f(u)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} &\lesssim \|\nabla u u^2(e^{\varepsilon|u|^2}-1)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \\
&\lesssim \|\nabla u\|_{L^4(\mathbb{R}^2)}\|u\|^2_{L^{\frac{16}{3}}(\mathbb{R}^2)}\|e^{\varepsilon|u|^2}-1\|_{L^8(\mathbb{R}^2)} \\
&\lesssim \|\nabla u\|_{L^4(\mathbb{R}^2)}\|u\|^2_{L^{\frac{16}{3}}(\mathbb{R}^2)} \\
&\lesssim \|\nabla u\|_{L^4(\mathbb{R}^2)}\|u\|_{L^8(\mathbb{R}^2)}\|u\|_{L^4(\mathbb{R}^2)}.
\end{aligned}$$

With Hölder inequality, we get

$$\begin{aligned}
\|\nabla f(u)\|_{L^{\frac{4}{3}}(I,L^{\frac{4}{3}}(\mathbb{R}^2))} &\lesssim \big\|\|\nabla u(t)\|_{L^4(\mathbb{R}^2)}\|u(t)\|_{L^8(\mathbb{R}^2)}\|u(t)\|_{L^4(\mathbb{R}^2)}\big\|_{L^{\frac{4}{3}}(I)} \\
&\lesssim \|\nabla u\|_{L^4(I,L^4(\mathbb{R}^2))}\|u\|_{L^4(I,L^8(\mathbb{R}^2))}\|u\|_{L^4(I,L^4(\mathbb{R}^2))} \quad (4.33) \\
&\lesssim \|u\|_{L^4(I,L^8(\mathbb{R}^2))}\|u\|^2_{S^1(I)}. \quad (4.34)
\end{aligned}$$

Thus, by (4.30)-(4.32)-(4.34),

$$\|u\|_{S^1(I)} \lesssim \|u(T)\|_{H^1(\mathbb{R}^2)} + \|u\|_{L^4(I,L^8(\mathbb{R}^2))}\|u\|^2_{S^1(I)}.$$

Which close the proof of the Lemma 4.1. ∎

We will also use a global a priori bound, proved by Colliander et al. [14] and Planchon-Vega [31],

Lemma 4.2. *Let u be a global solution of* (1.1) *in $H^1(\mathbb{R}^2)$. Then*

$$\|u\|_{L^4(\mathbb{R},L^8(\mathbb{R}^2))} \lesssim \|u\|^{\frac{3}{4}}_{L^\infty(\mathbb{R},L^2(\mathbb{R}^2))}\|\nabla u\|^{\frac{1}{4}}_{L^\infty(\mathbb{R},L^2(\mathbb{R}^2))} \lesssim M(u)^{\frac{3}{4}}H(u)^{\frac{1}{4}}.$$

Using the preceding Lemma we can decompose $\mathbb{R}$ to a finite number of intervals I where $\|u\|_{L^4(I,L^8(\mathbb{R}^2))}$ is small enough. Then with the absorbing Lemma 1.10 we obtain

$$\|u\|_{S^1(\mathbb{R})} < \infty.$$

So $u \in L^\infty(\mathbb{R},H^1(\mathbb{R}^2)) \cap L^4(\mathbb{R},W^{1,4}(\mathbb{R}^2))$ and by (4.31)-(4.33),$f(u) \in L^{\frac{4}{3}}(\mathbb{R},W^{1,\frac{4}{3}}(\mathbb{R}^2))$.

Let the operator

$$T:\mathbb{R}\to L(H^1(\mathbb{R}^2)), \quad T(t)\phi := K_t * \phi.$$

Where

$$K_t(x) := \frac{1}{4i\pi t}e^{i\frac{|x|^2}{4t}}.$$

Remark 4.3. *We recall that*

1. *$T(t)$ is an isometry of $H^1(\mathbb{R}^2)$ which satisfies $T(t+s) = T(t)T(s)$.*
2. *$T(t)\phi$ is the solution in $C(\mathbb{R},H^1(\mathbb{R}^2))$ to the free Schrödinger equation* (1.10) *with data $\phi \in H^1(\mathbb{R}^2)$.*

3. $u(t) = T(t)u_0 - i\int_0^t T(t-s)f(u(s))ds.$

Let $v(t) := T(-t)u(t) = u_0 - i\int_0^t T(-s)f(u(s))ds$. By Strichartz estimate (1.13)

$$\begin{aligned}\|v(t)-v(\tau)\|_{H^1(\mathbb{R}^2)} &= \|T(t)(v(t)-v(\tau))\|_{H^1(\mathbb{R}^2)}\\ &= \|\int_\tau^t T(t-s)f(u(s))ds\|_{H^1(\mathbb{R}^2)}\\ &\lesssim \|f(u)\|_{L^{\frac{4}{3}}((t,\tau),W^{1,\frac{4}{3}}(\mathbb{R}^2))} \overset{t,\tau\to\infty}{\longrightarrow} 0.\end{aligned}$$

Denoting $v_\pm$ the limit of $v(t)$ as $t \to \pm\infty$ in $H^1(\mathbb{R}^2)$, we have

$$\|v(t)-v_\pm\|_{H^1(\mathbb{R}^2)} \overset{t\to\infty}{\longrightarrow} 0, \quad v_\pm = u_0 - i\int_0^{\pm\infty} T(-s)f(u(s))ds.$$

Moreover, $u_\pm(t) := T(t)v_\pm$ is solution to the free Schrödinger equation (1.10) with data $v_\pm$ and satisfies

$$\|(u-u_\pm)(t)\|_{H^1(\mathbb{R}^2)} \overset{t\to\pm\infty}{\longrightarrow} 0.$$

Which close the proof of the first part of Theorem 1.4.

Remark 4.4. *We recall that*

1. $u_\pm(t) = T(t)u_0 - i\int_0^{\pm\infty} T(t-s)f(u(s))ds.$
2. $u(t) = u_\pm(t) + i\int_t^{\pm\infty} T(t-s)f(u(s))ds.$

It remains to show that the map $\psi : u_0 \longmapsto v_\pm = \lim_{t\to\pm\infty} T(-t)u(t)$ in $H^1(\mathbb{R}^2)$ is an homeomorphism of $H^1(\mathbb{R}^2)$. As a first step we show in what follows that ψ is bijective.

Lemma 4.5. $\psi : H^1(\mathbb{R}^2) \to H^1(\mathbb{R}^2) \quad u_0 \longmapsto v_\pm = \lim_{t\to\pm\infty} T(-t)u(t)$ *in* $H^1(\mathbb{R}^2)$ *is bijective, u being the global solution to* $(\mathcal{P})$ *in* $C(\mathbb{R}, H^1(\mathbb{R}^2))$ *with data* u_0.

Proof. We treat the case $t > 0$, the case $t < 0$ is similar.
Let $v_+ \in H^1(\mathbb{R}^2), u_+(t) := T(t)v_+$ we will show that there exists a unique $u \in C(\mathbb{R}, H^1(\mathbb{R}^2))$ solution to $(\mathcal{P})$ and satisfying

$$\|(u-u_+)(t)\|_{H^1(\mathbb{R}^2)} \overset{t\to+\infty}{\longrightarrow} 0.$$

We proceed with fixed point argument. Let $S > 0$ and the map

$$g : S^1(S,\infty) \to S^1(S,\infty)$$

$$u \longmapsto u_+(t) + i\int_t^{+\infty} T(t-s)f(u(s))ds.$$

Using Strichartz estimate for $\alpha^{'} = \beta^{'} = \frac{4}{3}$ and (4.31)-(4.33), we see that g is well defined and satisfies

$$\|g(u)-g(v)\|_{S^1(t,\infty)} \lesssim \|f(u)-f(v)\|_{L^{\frac{4}{3}}((t,\infty),W^{1,\frac{4}{3}}(\mathbb{R}^2))}.$$

Using Moser-Trudinger and Hölder inequalities, we obtain (for some $\varepsilon > 0$ small enough)

$$\begin{aligned}\|f(u)-f(v)\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))} &\lesssim \|w|u|^2(\mathrm{e}^{\varepsilon|u|^2}-1)\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))}\\ &+ \|w|v|^2(\mathrm{e}^{\varepsilon|v|^2}-1)\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))}\\ &\lesssim \|\|w(t)\|_{L^4(\mathbb{R}^2)}\|u(t)\|^2_{L^6(\mathbb{R}^2)}\|_{L^{\frac{4}{3}}(t,\infty)}\\ &+ \|\|w(t)\|_{L^4(\mathbb{R}^2)}\|v(t)\|^2_{L^6(\mathbb{R}^2)}\|_{L^{\frac{4}{3}}(t,\infty)},\end{aligned}$$

where $w := u - v$.

By the classical interpolation inequality $\|\cdot\|^2_{L^6} \le \|\cdot\|^{\frac{2}{3}}_{L^4}\|\cdot\|^{\frac{4}{3}}_{L^8}$ and Proposition 1.11, we get

$$\begin{aligned}&\|\|w(t)\|_{L^4(\mathbb{R}^2)}\|u(t)\|^2_{L^6(\mathbb{R}^2)}\|_{L^{\frac{4}{3}}(t,\infty)}\\ &\quad\lesssim \|\|w(t)\|_{L^4(\mathbb{R}^2)}\|u(t)\|^{\frac{2}{3}}_{L^4(\mathbb{R}^2)}\|u(t)\|^{\frac{4}{3}}_{L^8(\mathbb{R}^2)}\|_{L^{\frac{4}{3}}(t,\infty)}\\ &\quad\lesssim \|w\|_{L^4((t,\infty),L^4(\mathbb{R}^2))}\|u\|^{\frac{2}{3}}_{L^4((t,\infty),L^4(\mathbb{R}^2))}\|u\|^{\frac{4}{3}}_{L^4((t,\infty),L^8(\mathbb{R}^2))}\\ &\quad\lesssim \|w\|_{S^1(t,\infty)}\|u\|^{\frac{2}{3}}_{L^4((t,\infty),L^4(\mathbb{R}^2))}\|u\|^{\frac{4}{3}}_{L^4((t,\infty),L^8(\mathbb{R}^2))}\\ &\quad\lesssim \|w\|_{S^1(t,\infty)}\|u\|^2_{L^4((t,\infty),W^{1,4}(\mathbb{R}^2))}.\end{aligned}$$

Now, write

$$\begin{aligned}\|\nabla(f(u)-f(v))\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))} &\le \|\nabla u(Df(u)-Df(v))\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))}\\ &+ \|\nabla w Df(v)\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))}\\ &\le (\mathcal{J}_1)+(\mathcal{J}_2).\end{aligned}$$

Arguing as previously, we have

$$\begin{aligned}(\mathcal{J}_2) &\lesssim \|\nabla w|v|^2(\mathrm{e}^{\varepsilon|v|^2}-1)\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))}\\ &\lesssim \|w\|_{S^1(t,\infty)}\|v\|^2_{L^4((t,\infty),W^{1,4}(\mathbb{R}^2))},\end{aligned}$$

and

$$(\mathcal{J}_1) \lesssim \|\nabla uuw(\mathrm{e}^{\varepsilon|u|^2}-1)\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))} + \|\nabla uvw(\mathrm{e}^{\varepsilon|v|^2}-1)\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))}.$$

Moreover

$$\begin{aligned}&\|\nabla uuw(\mathrm{e}^{\varepsilon|u|^2}-1)\|_{L^{\frac{4}{3}}((t,\infty),L^{\frac{4}{3}}(\mathbb{R}^2))}\\ &\quad\lesssim \|\|w(s)\|_{L^4(\mathbb{R}^2)}\|\nabla u(s)\|_{L^4(\mathbb{R}^2)}\|u(s)\|_{L^8(\mathbb{R}^2)}\|_{L^{\frac{4}{3}}(t,\infty)}\\ &\quad\lesssim \|w\|_{L^4((t,\infty),L^4(\mathbb{R}^2))}\|\nabla u\|_{L^4((t,\infty),L^4(\mathbb{R}^2))}\|u\|_{L^4((t,\infty),L^8(\mathbb{R}^2))}\\ &\quad\lesssim \|w\|_{S^1(t,\infty)}\|u\|^2_{L^4((t,\infty),W^{1,4}(\mathbb{R}^2))}.\end{aligned}$$

Since $\|u\|_{L^4((t,\infty),W^{1,4}(\mathbb{R}^2))} + \|v\|_{L^4((t,\infty),W^{1,4}(\mathbb{R}^2))} \overset{t\to+\infty}{\longrightarrow} 0$, we obtain (for t large enough)

$$\begin{aligned} \|g(u)-g(v)\|_{S^1(t,\infty)} &\lesssim (\|u\|^2_{L^4((t,\infty),W^{1,4}(\mathbb{R}^2))} + \|v\|^2_{L^4((t,\infty),W^{1,4}(\mathbb{R}^2))})\|u-v\|_{S^1(t,\infty)} \\ &\leq k\|u-v\|_{S^1(t,\infty)}, \quad k\in]0,1[. \end{aligned} \tag{4.35}$$

Thus, for S large enough, g has a unique fixed point $u \in S^1(S,\infty)\cap C((S,\infty),H^1(\mathbb{R}^2))$, satisfying

$$u(t) = u_+(t) + i\int_t^{+\infty} T(t-s)f(u(s))ds, \quad t > S. \tag{4.36}$$

Now take $\psi := T(-S)u(S) \in H^1(\mathbb{R}^2)$, note that u is solution to the problem

$$(\mathcal{P}_S)\begin{cases} i\partial_t u + \Delta u &= f(u), \quad t > S \\ u(S,.) &= \psi. \end{cases}$$

By Theorem 1.3 u is global, so $u(0)$ is well defined in $H^1(\mathbb{R}^2)$. Moreover, by Strichartz estimate and (4.31)-(4.33),

$$\begin{aligned} \|(u-u_+)(t)\|_{H^1(\mathbb{R}^2)} &\lesssim \|u\|_{L^4((t,\infty),L^8(\mathbb{R}^2))}\|u\|_{L^4((t,\infty),W^{1,4}(\mathbb{R}^2))}\|u\|_{S^1(t,\infty)} \\ &\lesssim \|u\|^2_{L^4((t,\infty),W^{1,4}(\mathbb{R}^2))}\|u\|_{S^1(t,\infty)} \overset{t\to+\infty}{\longrightarrow} 0. \end{aligned}$$

It remains to prove uniqueness of such u.
Let $u_1, u_2 \in C(\mathbb{R}, H^1(\mathbb{R}^2))$ solution to (1.1) such that

$$\|(u_i - u_+)(t)\|_{H^1(\mathbb{R}^2)} \overset{t\to+\infty}{\longrightarrow} 0, i\in\{1,2\}.$$

By Remark 4.4 we know that $u_i = g(u_i)$, so using (4.35), for large t

$$\begin{aligned} \|u_1-u_2\|_{S^1(t,+\infty)} &= \|g(u_1)-g(u_2)\|_{S^1(t,+\infty)} \\ &\leq \frac{1}{2}\|u_1-u_2\|_{S^1(t,+\infty)}. \end{aligned}$$

Thus $u_1 = u_2$ and u is unique. This end the proof of the Lemma 4.5. ∎

The second step is to prove the continuity of ψ.

Lemma 4.6. *The map* $\psi : H^1(\mathbb{R}^2) \to H^1(\mathbb{R}^2) \quad u_0 \longmapsto v_\pm = \lim_{t\to\pm\infty} T(-t)u(t)$ *in* $H^1(\mathbb{R}^2)$ *is continuous,* $u \in C(\mathbb{R}, H^1(\mathbb{R}^2))$ *being the solution to* $(\mathcal{P})$*.*

Proof. Let $u_0 \in H^1(\mathbb{R}^2)$ (resp $(u_0^n)_n \in H^1(\mathbb{R}^2)^{\mathbb{N}}$), u (resp u^n) denotes the global solution to $(\mathcal{P})$ in $C(\mathbb{R}, H^1(\mathbb{R}^2))$ with data u_0(resp u_0^n), $v(t) = T(-t)u(t)$(resp $v^n(t) = T(-t)u^n(t)$) and $v_+ = \lim_{t\to+\infty} v(t)$(resp $v_+^n = \lim_{t\to+\infty} v^n(t)$) in $H^1(\mathbb{R}^2)$.
Assume that $\lim_{n\to\infty} \|u_0^n - u_0\|_{H^1(\mathbb{R}^2)} = 0$.
By Lemma 4.2, we have $\sup_n \|u^n\|_{L^4((0,\infty),L^8(\mathbb{R}^2))} < \infty$, so

$$\lim_{t\to\infty}\Big[\sup_n \|u^n\|_{L^4((t,\infty),L^8(\mathbb{R}^2))}\Big] = 0.$$

Thus, with Lemma 4.1, we have $\sup_n \|u^n\|_{S^1(0,\infty)} < \infty$.
Let $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that for $w^n := u^n - u$,

$$\sup_n \Big[\|w^n\|_{S^1(T_\varepsilon,\infty)}\Big(\|u\|^2_{L^4((T_\varepsilon,\infty),L^8(\mathbb{R}^2))} + \|u^n\|^2_{L^4((T_\varepsilon,\infty),L^8(\mathbb{R}^2))}\Big)\Big] \leq \frac{\varepsilon}{3}. \tag{4.37}$$

Arguing as previously and using computations shown in the course of the proof of Lemma 4.5, we have

$$\begin{aligned}
&\|v^n_+ - v_+\|_{H^1(\mathbb{R}^2)} \\
&\quad = \|u^n_0 - u_0 - i\int_0^\infty T(-s)(f(u^n) - f(u))ds\|_{H^1(\mathbb{R}^2)} \\
&\quad \lesssim \|u^n_0 - u_0\|_{H^1(\mathbb{R}^2)} + \|f(u^n) - f(u)\|_{L^{\alpha'}((0,\infty),W^{1,\beta'}(\mathbb{R}^2))} \\
&\quad \lesssim \|u^n_0 - u_0\|_{H^1(\mathbb{R}^2)} + \|w^n\|_{S^1(T_\varepsilon,\infty)}\Big(\|u\|^2_{L^4((T_\varepsilon,\infty),L^8(\mathbb{R}^2))} + \|u^n\|^2_{L^4((T_\varepsilon,\infty),L^8(\mathbb{R}^2))}\Big) \\
&\qquad\qquad + \|w^n\|_{S^1(0,T_\varepsilon)}\Big(\|u\|^2_{L^4((0,T_\varepsilon),L^8(\mathbb{R}^2))} + \|u^n\|^2_{L^4((0,T_\varepsilon),L^8(\mathbb{R}^2))}\Big) \\
&\quad \lesssim \|u^n_0 - u_0\|_{H^1(\mathbb{R}^2)} + \|w_n\|_{S^1(0,T_\varepsilon)} + \frac{\varepsilon}{3}.
\end{aligned} \tag{4.38}$$

Using global well-posedness of $(\mathcal{P})$, we have

$$\lim_{n\to\infty} \|w^n\|_{S^1(0,T_\varepsilon)} = 0. \tag{4.39}$$

Since $\lim_{n\to\infty} \|u^n_0 - u_0\|_{H^1(\mathbb{R}^2)} = 0$, using (4.38)-(4.39), there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|v^n_+ - v_+\|_{H^1(\mathbb{R}^2)} \leq \varepsilon, \quad \forall n \geq n_\varepsilon.$$

The proof of the Lemma 4.6 is achieved. ■

In the last step we prove the continuity of ψ^{-1}.

Lemma 4.7. *The map* $\psi^{-1} : H^1(\mathbb{R}^2) \to H^1(\mathbb{R}^2) \quad v_\pm \longmapsto u_0$ *is continuous.*

Proof. Let $v_+ \in H^1(\mathbb{R}^2), (v^n_+)_n \in (H^1(\mathbb{R}^2))^{\mathbb{N}}$, $u_0 := \psi^{-1}(v_+), u^n_0 := \psi^{-1}(v^n_+)$. Let u (resp u^n) the global solution to $(\mathcal{P})$ in $C(\mathbb{R}, H^1(\mathbb{R}^2))$ with data u_0 (resp u^n_0), $v(t) = T(-t)u(t)$(resp $v^n(t) = T(-t)u^n(t)$) we recall that $v_+ = \lim_{t\to+\infty} v(t)$(resp $v^n_+ = \lim_{t\to+\infty} v^n(t)$) in $H^1(\mathbb{R}^2)$.
Assume that $\lim_{n\to\infty} \|v^n_+ - v_+\|_{H^1(\mathbb{R}^2)} = 0$.
With conservation of the mass

$$\|u_0\|_{L^2(\mathbb{R}^2)} = \lim_{t\to\infty} \|u(t)\|_{L^2(\mathbb{R}^2)} = \|v_+\|_{L^2(\mathbb{R}^2)},$$

and

$$\|u^n_0\|_{L^2(\mathbb{R}^2)} = \lim_{t\to\infty} \|u^n(t)\|_{L^2(\mathbb{R}^2)} = \|v^n_+\|_{L^2(\mathbb{R}^2)}.$$

So

$$\lim_{n\to\infty} \|u^n_0\|_{L^2(\mathbb{R}^2)} = \|u_0\|_{L^2(\mathbb{R}^2)}. \tag{4.40}$$

We recall the hamiltonian

$$H(u,t) = \|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} F(u(t))dx,$$

where $F(u) := \frac{\lambda}{2\pi\alpha}\Big(\mathrm{e}^{(1+4\pi|u|^2)^{\frac{\alpha}{2}}} - \frac{\varepsilon}{2}[(1+4\pi|u|^2)^{\alpha}+1]\Big)$.
Since $\lim_{t\to+\infty}\|T(t)v_+\|_{L^r(\mathbb{R}^2)} = 0$ for all $r > 2$, we have $\lim_{t\to+\infty}\|u(t)\|_{L^r(\mathbb{R}^2)} = 0$ for $r > 2$. Using Moser-Trudinger inequality (for $\varepsilon > 0$ small enough), we get

$$\begin{aligned}\|F(u(t))\|_{L^1(\mathbb{R}^2)} &\leq C_\varepsilon\|u(t)(\mathrm{e}^{\varepsilon|u(t)|^2}-1)\|_{L^1(\mathbb{R}^2)}\\ &\leq C_\varepsilon\|u(t)\|_{L^5(\mathbb{R}^2)}\|\mathrm{e}^{\varepsilon|u(t)|^2}-1\|_{L^{\frac{5}{4}}(\mathbb{R}^2)}\\ &\lesssim \|u(t)\|_{L^5(\mathbb{R}^2)}.\end{aligned}$$

So, we have $\lim_{t\to+\infty}\int_{\mathbb{R}^2} F(u(t))dx = 0$. Thus, since $\lim_{t\to+\infty}\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = \|\nabla v_+\|_{L^2(\mathbb{R}^2)}$,

$$H(u,0) = \lim_{t\to+\infty} H(u,t) = \|\nabla v_+\|^2_{L^2(\mathbb{R}^2)}.$$

With the same way $H(u^n,0) = \|\nabla v_+^n\|^2_{L^2(\mathbb{R}^2)}$. So $\lim_{n\to+\infty} H(u^n,0) = H(u,0)$. Thus

$$\sup_n \|u_0^n\|_{H^1(\mathbb{R}^2)} < \infty.$$

By Lemma 4.2, we have $\sup_n \|u^n\|_{L^4((0,\infty),L^8(\mathbb{R}^2))} < \infty$, so

$$\lim_{t\to\infty}\Big[\sup_n \|u^n\|_{L^4((t,\infty),L^8(\mathbb{R}^2))}\Big] = 0.$$

Thus, with Lemma 4.1, we have $\sup_n \|u^n\|_{S^1(0,\infty)} < \infty$.
Let $\varepsilon > 0$, there existe $T_\varepsilon > 0$ such that for $w^n := u^n - u$,

$$\sup_n \Big[\|w^n\|_{S^1(T_\varepsilon,\infty)}\Big(\|u\|^2_{L^4((T_\varepsilon,\infty),L^8(\mathbb{R}^2))} + \|u^n\|^2_{L^4((T_\varepsilon,\infty),L^8(\mathbb{R}^2))}\Big)\Big] \leq \frac{\varepsilon}{2}. \tag{4.41}$$

Arguing as previously, we have

$$\begin{aligned}&\|w^n(T_\varepsilon)\|_{H^1(\mathbb{R}^2)}\\ &\lesssim \|v_+^n - v_+\|_{H^1(\mathbb{R}^2)} + \|f(u^n)-f(u)\|_{L^{\alpha'}((T_\varepsilon,\infty),W^{1,\beta'}(\mathbb{R}^2))}\\ &\lesssim \|v_+^n - v_+\|_{H^1(\mathbb{R}^2)} + \|w^n\|_{S^1(T_\varepsilon,\infty)}\Big(\|u\|^2_{L^4((T_\varepsilon,\infty),L^8(\mathbb{R}^2))} + \|u^n\|^2_{L^4((T_\varepsilon,\infty),L^8(\mathbb{R}^2))}\Big)\\ &\lesssim \|v_+^n - v_+\|_{H^1(\mathbb{R}^2)} + \frac{\varepsilon}{2}.\end{aligned}$$

Thus, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|w^n(T_\varepsilon)\|_{H^1(\mathbb{R}^2)} \leq \varepsilon, \quad \forall n > n_\varepsilon.$$

We conclude the proof of the Lemma 4.7 via a time translation and well-posedness of $(\mathcal{P})$ in the energy space. ∎

The proof of Theorem 1.4 is achieved via Lemmas 4.6-4.5-4.7.

Acknowledgment. The author would to thank Professor Mohamed Majdoub for suggesting this problem and for his kind advice. He also thank the referee for providing constructive comments and help in improving the contents of this paper.

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