Weighted composition operators on some function spaces of entire functions

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Abstract

We characterize the boundedness and compactness of the weighted composition operators acting between some Fock-type spaces. Motivated by some recent ideas by S. Stević we also calculate the operator norm of some of these operators. Also we determine the form of the symbol which induces a bounded or compact composition operator between some Fock-type spaces.

1 Introduction

Throughout this paper, let dm denote the usual Lebesgue measure on $\mathbb{C}$. For each p, $0 < p < \infty$ and $\alpha > 0$, the *Bargmann–Fock space* $\mathcal{F}^p_\alpha$ is the space of all entire functions f in $\mathbb{C}$ for which

$$\|f\|^p_{p,\alpha} = \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dm(z) < \infty,$$

where $e^x = \exp(x)$ denotes the exponential function and the number $\frac{p\alpha}{2\pi}$ is a normalization constant, that is, it is chosen such that $\|1\|_{p,\alpha} = 1$. When $1 \leq p < \infty$, the space $\mathcal{F}^p_\alpha$ is a Banach space with norm $\|f\|_{p,\alpha}$. In particular, the space $\mathcal{F}^2_\alpha$ is a functional Hilbert space with the inner product

$$\langle f, g\rangle = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-\alpha|z|^2} dm(z).$$

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The space $\mathcal{F}^2_\alpha$ is called the *Fock space*. The reproducing kernel function for $\mathcal{F}^2_\alpha$ is given by $K(z,w) = \exp\{\alpha z\overline{w}\}$ (see [5]). We also use the normalized kernel function $k_w(z) = \exp\{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2\}$. Note that $\|k_w\|^p_{p,\alpha} = \|k_w\|^2_{2,(p\alpha)/2} = 1$ for each $p > 0$. Also the space $\mathcal{F}^\infty_\alpha$ is defined by

$$\mathcal{F}^\infty_\alpha = \left\{ f \text{ is an entire function } : \|f\|_{\infty,\alpha} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{\alpha}{2}|z|^2} < \infty \right\}.$$

Some problems related to the Bargmann-Fock space have been studied by many authors [4, 5, 9, 10], *etc*. In this paper, our object of investigation is a *weighted composition operator* uC_φ which is defined by $uC_\varphi f = u \cdot (f \circ \varphi)$ where u and φ are entire functions. When $u(z) = 1$, C_φ is called a *composition operator*. These type of operators on various analytic function spaces of typical bounded domains have been studied by many authors. S. Stević [11, 12, 13, 14, 16, 17] have studied these operators acting on weighted analytic function spaces, the *Bloch-type space* and *mixed-norm space*, which correspond to $\mathcal{F}^\infty_\alpha$ or $\mathcal{F}^p_\alpha$. Recently, the boundedness and compactness of C_φ on $\mathcal{F}^p_\alpha$ have been characterized by B. Carswell, B. MacCluer and A. Schuster [1]. Furthermore the boundedness and compactness of uC_φ on $\mathcal{F}^p_\alpha$ have been studied by the present author [22, 23]. However the case C_φ or uC_φ acting on $\mathcal{F}^\infty_\alpha$ still remains to be considered. Our first aim is to characterize the boundedness and compactness of $uC_\varphi : \mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta$ and $uC_\varphi : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$. Moreover, motivated by some recent ideas by S. Stević and his collaborator (see, [6, 8, 15, 16, 18, 19, 20, 21]) we also calculate the operator norm of the operator $uC_\varphi : \mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta$. For other works of related topics, we can refer to [2, 7]. Another aim is to determine the form of the symbol φ which induces a bounded or compact composition operator from $\mathcal{F}^\infty_\alpha$ into $\mathcal{F}^\infty_\beta$. This result is analogous to the result due to B. Carswell, B. MacCluer and A. Schuster [1].

2 Preliminaries

This section is devoted to collecting some lemmas which we will need in the proof of our results.

Lemma 1. *Let* $0 < p \leq \infty$ *and* $\alpha > 0$. *For each entire function* f *and* $z \in \mathbb{C}$,

$$|f(z)| \leq e^{\frac{\alpha}{2}|z|^2}\|f\|_{p,\alpha}.$$

Proof. It is enough to prove the case $0 < p < \infty$ since the case $p = \infty$ is verified by the definition of the norm $\|\cdot\|_{\infty,\alpha}$. Since $|f|^p$ is subharmonic if $0 < p < \infty$, we have

$$|f(0)|^p \leq \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi},$$

for all $r \geq 0$. Multiplying both sides of the last inequality by $2re^{-\frac{p\alpha}{2}r^2}dr$ and then integrating from 0 to ∞ with respect to r we get

$$|f(0)|^p \leq \frac{p\alpha}{2\pi}\int_{\mathbb{C}} |f(w)|^p e^{-\frac{p\alpha}{2}|w|^2}\, dm(w). \tag{1}$$

For fixed $z \in \mathbb{C}$, we set $I_z f(w) = k_z(w) f(w-z)$. A simple calculation shows that I_z is a surjective isometry of $\mathcal{F}^p_\alpha$ and its inverse $I_z^{-1} = I_{-z}$ (see [4], Proposition 2). Since $|I_{-z}f(0)|^p = |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2}$, the inequality (1) gives

$$|f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} = |I_{-z}f(0)|^p \le \|I_{-z}f\|^p_{p,\alpha} = \|f\|^p_{p,\alpha},$$

which completes the proof. ■

For an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, define $R_n f(z) = \sum_{k=n}^{\infty} a_k z^k$, acting on $\mathcal{F}^p_\alpha$ and $K_n = I - R_n$ where I denotes the identity operator on $\mathcal{F}^p_\alpha$. Note that K_n is compact on $\mathcal{F}^p_\alpha$, and so R_n is bounded on $\mathcal{F}^p_\alpha$. D. Garling and P. Wojtaszczyk [4] have proved that for each $f \in \mathcal{F}^p_\alpha$ $(1 < p < \infty)$, the partial sums of the Taylor expansion of f converges to f in the norm topology of $\mathcal{F}^p_\alpha$. Under the notation R_n, we can express this result in other words with $\|R_n f\|_{p,\alpha} \to 0$ as $n \to \infty$ for each $f \in \mathcal{F}^p_\alpha$. By the principle of uniform boundedness, we see that $\sup_{n\ge 1} \|R_n\| < \infty$.

Moreover the operator R_n has a close relation to the essential norm of uC_φ, which is the theme of the next lemma.

Lemma 2. *If $uC_\varphi : \mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta$ is a bounded operator, then*

$$\|uC_\varphi\|_{e,\mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta} \le \liminf_{n\to\infty} \|uC_\varphi R_n\|,$$

where $\|uC_\varphi\|_{e,\mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta}$ denotes the essential norm of $uC_\varphi : \mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta$.

Proof. Take a compact operator $\mathcal{K}$ on $\mathcal{F}^p_\alpha$. Since $uC_\varphi = uC_\varphi(R_n + K_n)$ where $K_n = \sum_{k=0}^{n-1} a_k z^k$, we have

$$\|uC_\varphi - \mathcal{K}\| \le \|uC_\varphi R_n\| + \|uC_\varphi K_n - \mathcal{K}\|, \tag{2}$$

for all $n \ge 1$. Since K_n is compact on $\mathcal{F}^p_\alpha$, we see that $uC_\varphi K_n : \mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta$ is compact. Hence we have $\|uC_\varphi K_n\|_{e,\mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta} = 0$ for all $n \ge 1$. Taking the infimum over compact operators $\mathcal{K}$ and letting $n \to \infty$ in (2), we obtain the desired inequality. ■

Let $L^p_\alpha(\mathbb{C})$ denote the space of all measurable functions f such that $f(z)e^{-\frac{\alpha}{2}|z|^2}$ is in $L^p(\mathbb{C}, dm)$. We define P_α as follows

$$P_\alpha f(z) = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(w) e^{\alpha z \overline{w}} e^{-\alpha|w|^2} dm(w). \tag{3}$$

Then P_α is a bounded self-adjoint projection from $L^p_\alpha(\mathbb{C})$ onto $\mathcal{F}^p_\alpha(\mathbb{C})$ $(1 \le p < \infty)$. For these results, see [5]. The property of P_α verifies the following lemma.

Lemma 3. *Let $1 < p < \infty$ and q be the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. For each $f \in \mathcal{F}^p_\alpha$, there is a positive constant C such that*

$$|R_n f(w)| \le C\|f\|_{p,\alpha} \sum_{k=n}^{\infty} \frac{\alpha^k |w|^k}{k!} \left\{ \left(\frac{2}{q\alpha}\right)^{\frac{qk}{2}+1} \Gamma\left(\frac{qk}{2}+1\right) \right\}^{\frac{1}{q}},$$

for all $w \in \mathbb{C}$ and positive integers n. Here $\Gamma(x)$ is the Gamma function.

Proof. The equation (3) and the orthogonality of monomials z^k show that

$$R_n f(w) = P_\alpha R_n f(w) = \frac{\alpha}{\pi}\int_{\mathbb{C}} R_n f(z) e^{\alpha w\overline{z}} e^{-\alpha|z|^2} dm(z)$$
$$= \frac{\alpha}{\pi}\int_{\mathbb{C}} f(z) R_n e^{\alpha w\overline{z}} e^{-\alpha|z|^2} dm(z).$$

Hence, by using the expansion $e^{\alpha w\overline{z}} = \sum \frac{\alpha^k}{k!} w\overline{z}^k$ and Hölder's inequality, we obtain

$$|R_n f(w)| \le \frac{\alpha}{\pi}\int_{\mathbb{C}} |f(z)||R_n e^{\alpha w\overline{z}}| e^{-\alpha|z|^2} dm(z)$$
$$\le \frac{\alpha}{\pi}\sum_{k=n}^{\infty} \frac{\alpha^k}{k!}|w|^k \int_{\mathbb{C}} |f(z)||z|^k e^{-\alpha|z|^2} dm(z)$$
$$\le \frac{\alpha}{\pi}\sum_{k=n}^{\infty} \frac{\alpha^k}{k!}|w|^k \left[\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dm(z)\right]^{\frac{1}{p}} \left[\int_{\mathbb{C}} |z|^{qk} e^{-\frac{q\alpha}{2}|z|^2} dm(z)\right]^{\frac{1}{q}}.$$

Here, integration in polar coordinates gives

$$\int_{\mathbb{C}} |z|^{qk} e^{-\frac{q\alpha}{2}|z|^2} dm(z) = \pi\left(\frac{2}{q\alpha}\right)^{\frac{qk}{2}+1} \Gamma\left(\frac{qk}{2}+1\right).$$

This completes the proof of the lemma. ∎

3 Weighted Composition Operators

In this section, we give a characterization for the boundedness and compactness of $uC_\varphi : \mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta$. Our results involve the estimate for the operator norm and the essential norm of uC_φ. As a corollary, we also get a characterization for the compactness of the weighted composition operator $uC_\varphi : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$.

Theorem 1. *Let $0 < p \le \infty$, α, $\beta > 0$, u and φ be entire functions. Then $uC_\varphi : \mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta$ is a bounded operator if and only if u and φ satisfy*

$$\sup_{z\in\mathbb{C}} |u(z)| \exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\} < \infty.$$

Moreover we have that its operator norm $\|uC_\varphi\|_{\mathcal{F}^p_\alpha\to\mathcal{F}^\infty_\beta}$ is equal to the above supremum.

Proof. For every $f \in \mathcal{F}^p_\alpha$, Lemma 1 gives

$$\|uC_\varphi f\|_{\infty,\beta} = \sup_{z\in\mathbb{C}} |u(z)f(\varphi(z))| e^{-\frac{\beta}{2}|z|^2} \le \|f\|_{p,\alpha} \sup_{z\in\mathbb{C}} |u(z)| e^{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2},$$

and so we have

$$\|uC_\varphi\|_{\mathcal{F}^p_\alpha\to\mathcal{F}^\infty_\beta} \le \sup_{z\in\mathbb{C}} |u(z)| \exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\}.$$

On the other hand, the test function k_w gives

$$\|uC_\varphi k_w\|_{\infty,\beta} \geq |u(z)e^{\alpha\varphi(z)\overline{w}}|e^{-\frac{\alpha}{2}|w|^2-\frac{\beta}{2}|z|^2},$$

for all w, $z \in \mathbb{C}$. Since $\|k_w\|_{p,\alpha} = 1$ for all $w \in \mathbb{C}$, we obtain

$$\|uC_\varphi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta} \geq \|uC_\varphi k_{\varphi(z)}\|_{\infty,\beta} = |u(z)|e^{\frac{\alpha}{2}|\varphi(z)|^2-\frac{\beta}{2}|z|^2},$$

for all $z \in \mathbb{C}$. Hence we have

$$\|uC_\varphi\|_{\mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta} \geq \sup_{z\in\mathbb{C}} |u(z)| \exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\}.$$

This completes the proof. ∎

Theorem 2. *Let $1 < p < \infty$ and α, $\beta > 0$. Suppose that u is an entire function and φ is a non-constant entire function which induce the bounded operator $uC_\varphi : \mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta$. Then its essential norm $\|uC_\varphi\|_{e,\mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta}$ is comparable to*

$$\limsup_{|\varphi(z)|\to\infty} |u(z)| \exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\},$$

and so $uC_\varphi : \mathcal{F}^p_\alpha \to \mathcal{F}^\infty_\beta$ is a compact operator if and only if u and φ satisfy

$$\lim_{|\varphi(z)|\to\infty} |u(z)| \exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\} = 0.$$

Proof. First we will prove that a constant multiple of $\limsup_{|\varphi(z)|\to\infty} |u(z)|e^{\frac{\alpha}{2}|\varphi(z)|^2-\frac{\beta}{2}|z|^2}$ is an upper bound for the essential norm of uC_φ. Fix $r > 0$ and take $f \in \mathcal{F}^p_\alpha$ with $\|f\|_{p,\alpha} \leq 1$. Since $R_n f \in \mathcal{F}^p_\alpha$ for any positive integer n, Lemma 1 gives

$$\begin{aligned}|u(z)R_n f(\varphi(z))|e^{-\frac{\beta}{2}|z|^2} &\leq e^{\frac{\alpha}{2}|\varphi(z)|^2}\|R_n f\|_{p,\alpha}|u(z)|e^{-\frac{\beta}{2}|z|^2} \\ &\leq \sup_{n\geq 1}\|R_n\||u(z)|e^{\frac{\alpha}{2}|\varphi(z)|^2-\frac{\beta}{2}|z|^2}.\end{aligned}$$

Thus we have

$$\sup_{|\varphi(z)|>r} |uC_\varphi R_n f(z)|e^{-\frac{\beta}{2}|z|^2} \leq C \sup_{|\varphi(z)|>r} |u(z)|e^{\frac{\alpha}{2}|\varphi(z)|^2-\frac{\beta}{2}|z|^2}. \tag{4}$$

Lemma 3 gives

$$\sup_{|\varphi(z)|\leq r} |uC_\varphi R_n f(z)|e^{-\frac{\beta}{2}|z|^2} \leq \|u\|_{\infty,\beta} \sum_{k=n}^{\infty} \frac{\alpha^k r^k}{k!} \left\{\left(\frac{2}{q\alpha}\right)^{\frac{qk}{2}+1} \Gamma\left(\frac{qk}{2}+1\right)\right\}^{\frac{1}{q}}. \tag{5}$$

Stirling's formula shows that

$$\frac{\alpha^k r^k}{k!}\left\{\left(\frac{2}{q\alpha}\right)^{\frac{qk}{2}+1}\Gamma\left(\frac{qk}{2}+1\right)\right\}^{\frac{1}{q}} \asymp \frac{\alpha^k r^k}{k!}\left(\frac{2}{q\alpha}\right)^{\frac{k}{2}}\left(\frac{qk}{2}\right)^{\frac{k}{2}+\frac{1}{q}-\frac{1}{2q}} e^{-\frac{k}{2}},$$

as $k \to \infty$. Since the convergence of the series

$$\sum_{k=0}^{\infty}\frac{\alpha^k r^k}{k!}\left(\frac{2}{q\alpha}\right)^{\frac{k}{2}}\left(\frac{qk}{2}\right)^{\frac{k}{2}+\frac{1}{q}-\frac{1}{2q}} e^{-\frac{k}{2}}$$

follows easily by using d'Alembert criterion, the series

$$\sum_{k=0}^{\infty}\frac{\alpha^k r^k}{k!}\left\{\left(\frac{2}{q\alpha}\right)^{\frac{qk}{2}+1}\Gamma\left(\frac{qk}{2}+1\right)\right\}^{\frac{1}{q}}$$

also converges. Hence we have

$$\sum_{k=n}^{\infty}\frac{\alpha^k r^k}{k!}\left\{\left(\frac{2}{q\alpha}\right)^{\frac{qk}{2}+1}\Gamma\left(\frac{qk}{2}+1\right)\right\}^{\frac{1}{q}} \to 0,$$

as $n \to \infty$. Combining this with inequality (5), we have

$$\liminf_{n\to\infty}\sup_{\|f\|_{p,\alpha}\le 1}\sup_{|\varphi(z)|\le r}|uC_\varphi R_n f(z)|e^{-\frac{\beta}{2}|z|^2} = 0. \tag{6}$$

By (4), (6) and Lemma 2, we obtain

$$\|uC_\varphi\|_{e,\mathcal{F}_\alpha^p\to\mathcal{F}_\beta^\infty} \le C\sup_{|\varphi(z)|>r}|u(z)|e^{\frac{\alpha}{2}|\varphi(z)|^2-\frac{\beta}{2}|z|^2}.$$

Letting $r \to +\infty$, then we have

$$\|uC_\varphi\|_{e,\mathcal{F}_\alpha^p\to\mathcal{F}_\beta^\infty} \le C\limsup_{|\varphi(z)|\to\infty}|u(z)|e^{\frac{\alpha}{2}|\varphi(z)|^2-\frac{\beta}{2}|z|^2}.$$

Next we prove a lower estimate for the essential norm. Take a compact operator $\mathcal{K} : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^\infty$. Since $\{k_w\}_{w\in\mathbb{C}}$ is a bounded family in $\mathcal{F}_\alpha^p$ and converges to 0 as $|w| \to \infty$ uniformly on compact subsets of $\mathbb{C}$, [23, Proposition 2.5] implies $k_w \to 0$ weakly in $\mathcal{F}_\alpha^p$ as $|w| \to \infty$. Hence the compactness of $\mathcal{K}$ yields $\|\mathcal{K}k_{w_j}\|_{\infty,\beta} \to 0$ as $j \to \infty$ for an arbitrary sequence $\{w_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that $|w_j| \to \infty$ as $j \to \infty$.

Take a sequence $\{z_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that $|\varphi(z_j)| \to \infty$ as $j \to \infty$. Hence we have $\|\mathcal{K}k_{\varphi(z_j)}\|_{\infty,\beta} \to 0$ as $j \to \infty$. Therefore, we obtain

$$\|uC_\varphi - \mathcal{K}\|_{\mathcal{F}_\alpha^p\to\mathcal{F}_\beta^\infty} \ge \limsup_{j\to\infty}\|(uC_\varphi - \mathcal{K})k_{\varphi(z_j)}\|_{\infty,\beta}$$

$$\geq \limsup_{j\to\infty} \|uC_\varphi k_{\varphi(z_j)}\|_{\infty,\beta}$$
$$\geq \limsup_{j\to\infty} |u(z_j)| e^{\frac{\alpha}{2}|\varphi(z_j)|^2 - \frac{\beta}{2}|z_j|^2},$$

which completes the proof. ∎

Next we give a characterization for the compactness of $uC_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$. Its proof depends on the following proposition.

Proposition 1. *Let α, $\beta > 0$, u and φ be entire functions. Suppose that $uC_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ is bounded. Then $uC_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ is compact if and only if for every bounded sequence $\{f_j\} \subset \mathcal{F}_\alpha^\infty$ which converges to 0 as $j \to \infty$ uniformly on compact subsets of $\mathbb{C}$, $\{uC_\varphi f_j\}$ converges to 0 in the norm topology in $\mathcal{F}_\beta^\infty$.*

Proof. The proof of this proposition can be obtained by adapting the proof of Proposition 3.11 in [3]. For a detailed proof see, e.g., Lemma 3 in [11]. Hence we omit the detail. ∎

Corollary 1. *Let α, $\beta > 0$. Suppose that u is an entire function and φ is a non-constant entire function which induce the bounded operator $uC_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$. Then $uC_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ is a compact operator if and only if u and φ satisfy*

$$\lim_{|\varphi(z)|\to\infty} |u(z)| \exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\} = 0. \tag{7}$$

Proof. Since the compactness of $uC_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ derives that $uC_\varphi : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^\infty$ is compact, Theorem 2 shows that the condition (7) is a necessary condition for the compactness of $uC_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$.

Now we assume (7) and take a bounded sequence $\{f_j\} \subset \mathcal{F}_\alpha^\infty$ which converges to 0 as $j \to \infty$ uniformly on compact subsets of $\mathbb{C}$. Fix $\varepsilon > 0$. We can choose $r > 0$ such that

$$|u(z)| \exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\} < \varepsilon,$$

for all $z \in \mathbb{C}$ with $|\varphi(z)| > r$. Since $\{f_j\}$ converges to 0 uniformly on compact subsets, we can also choose a positive integer j_0 such that

$$\max_{|\varphi(z)|\leq r} |f_j(\varphi(z))| < \varepsilon,$$

for all $j \geq j_0$. Hence we obtain

$$|uC_\varphi f_j(z)| e^{-\frac{\beta}{2}|z|^2} \leq |u(z)| e^{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2} \|f_j\|_{\infty,\alpha} < \sup_{j\geq 1} \|f_j\|_{\infty,\alpha} \cdot \varepsilon \quad (|\varphi(z)| > r),$$
$$|uC_\varphi f_j(z)| e^{-\frac{\beta}{2}|z|^2} \leq \|u\|_{\infty,\beta} \max_{|\varphi(z)|\leq r} |f_j(\varphi(z))| < \|u\|_{\infty,\beta} \cdot \varepsilon \quad (|\varphi(z)| \leq r),$$

for all $j \geq j_0$. Moreover the constant function $f(z) \equiv 1$ and the boundedness of $uC_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ imply that $\|u\|_{\infty,\beta} < \infty$. So we have $\|uC_\varphi f_j\|_{\infty,\beta} \to 0$ as $j \to \infty$. Proposition 1 shows that $uC_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ is compact. ∎

4 Composition Operators

In this section we consider the case $u(z) \equiv 1$. As an immediate consequence of Theorem 1 and Corollary 1, we see that $C_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ is bounded if and only if

$$\sup_{z\in\mathbb{C}} \exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\} < \infty, \tag{8}$$

and $C_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ is compact if and only if

$$\lim_{|\varphi(z)|\to\infty} \exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\} = 0. \tag{9}$$

We will prove that these conditions determine the form of the symbol φ which induces the bounded or compact composition operators $C_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$.

Lemma 4. *Let $\alpha > 0$ and $\tau_b(z) = z + b$ where $b \in \mathbb{C}$ with $\operatorname{Re} b > 0$. Then C_{τ_b} is not bounded on $\mathcal{F}_\alpha^\infty$.*

Proof. For each positive integer n, we put $f_n(z) = e^{\alpha n z} e^{-\frac{\alpha}{2}n^2} (= k_n(z))$. Then $\{f_n\} \subset \mathcal{F}_\alpha^\infty$ and $\|f_n\|_{\infty,\alpha} \le 1$. Moreover we see

$$\|C_{\tau_b} f_n\|_{\infty,\alpha} \ge |f_n(\tau_b(n))| e^{-\frac{\alpha}{2}n^2} = e^{\alpha n \operatorname{Re} b} \to \infty,$$

as $n \to \infty$. This implies that C_{τ_b} is unbounded on $\mathcal{F}_\alpha^\infty$. ∎

Theorem 3. *Let α, $\beta > 0$. Suppose that φ is a non-constant entire function.*

(a) *If $C_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ is bounded, then $\varphi(z) = az + b$ where $a \in \mathbb{C}$ and $b \in \mathbb{C}$. Furthermore, $|a| \le \sqrt{\frac{\beta}{\alpha}}$, and if $|a| = \sqrt{\frac{\beta}{\alpha}}$, then $b = 0$.*

(b) *If $C_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ is compact, then $\varphi(z) = az + b$ where $a \in \mathbb{C}$ with $|a| < \sqrt{\frac{\beta}{\alpha}}$ and $b \in \mathbb{C}$.*

Proof. Suppose that $C_\varphi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\beta^\infty$ is bounded. Let $\varphi(z) = b + zh(z)$ where $b \in \mathbb{C}$ and h is an entire function. We must show that h is a constant function. Assume that h is not constant. Then there exists a sequence $\{z_k\}$ with $|z_k| \to \infty$ such that $|h(z_k)| \to \infty$ as $k \to \infty$. Moreover we can assume $|h(z_k) + \frac{b}{z_k}|^2 - \frac{\beta}{\alpha} \ge 1$ for sufficiently large k. Hence we have

$$\begin{aligned}
&\exp\left\{\frac{\alpha}{2}|\varphi(z_k)|^2 - \frac{\beta}{2}|z_k|^2\right\} = \sum_{n=0}^{\infty} \frac{1}{n!}\left\{\frac{\alpha}{2}|\varphi(z_k)|^2 - \frac{\beta}{2}|z_k|^2\right\}^n \\
&= \sum_{n=0}^{\infty} \frac{\alpha^n |z_k|^{2n}}{2^n n!}\left\{\frac{|\varphi(z_k)|^2}{|z_k|^2} - \frac{\beta}{\alpha}\right\}^n \ge \frac{\alpha^n |z_k|^{2n}}{2^n n!}\left\{\frac{|\varphi(z_k)|^2}{|z_k|^2} - \frac{\beta}{\alpha}\right\}^n \\
&= \frac{\alpha^n |z_k|^{2n}}{2^n n!}\left\{\left|h(z_k) + \frac{b}{z_k}\right|^2 - \frac{\beta}{\alpha}\right\}^n \ge \frac{\alpha^n |z_k|^{2n}}{2^n n!} \to \infty,
\end{aligned}$$

as $k \to \infty$. This contradicts the condition (8). Thus h is a constant function, so we can write $\varphi(z) = az + b$.

Next we prove $|a| \le \sqrt{\frac{\beta}{\alpha}}$. In order to prove this we assume $|a| > \sqrt{\frac{\beta}{\alpha}}$ and fix ε, $0 < \varepsilon < |a|^2 - \frac{\beta}{\alpha}$. Since $|a + \frac{b}{z}|^2 \to |a|^2$ as $|z| \to \infty$, we see that $||a + \frac{b}{z}|^2 - |a|^2| < \varepsilon$ for sufficiently large $|z|$. Hence we have

$$\exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\} \ge \frac{\alpha^n|z|^{2n}}{2^n n!}\left\{\left|a + \frac{b}{z}\right|^2 - \frac{\beta}{\alpha}\right\}^n \ge \frac{\alpha^n|z|^{2n}}{2^n n!}\left\{|a|^2 - \varepsilon - \frac{\beta}{\alpha}\right\}^n,$$

for sufficiently large $|z|$. This also contradicts the condition in (8). Hence we obtain $|a| \le \sqrt{\frac{\beta}{\alpha}}$. To prove that if $|a| = \sqrt{\frac{\beta}{\alpha}}$, then $b = 0$, we assume $\mathrm{Re}\, b > 0$ first. Put $\psi_a(z) = az$ and $\tau_b(z) = z + b$, then $\varphi = \tau_b \circ \psi_a$ and $C_\varphi = C_{\psi_a}C_{\tau_b}$. Since $|a| = \sqrt{\frac{\beta}{\alpha}}$ implies that $C_{\psi_a} : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$ is a surjective isometry, we see that $C_{\tau_b} = C^{-1}_{\psi_a}C_\varphi$ is bounded on $\mathcal{F}^\infty_\alpha$. However Lemma 4 shows that C_{τ_b} is unbounded on $\mathcal{F}^\infty_\alpha$, so we obtain a contradiction. If $\mathrm{Re}\, b < 0$, then we can choose $\eta \in \mathbb{C}$ with $|\eta| = 1$ such that $\mathrm{Re}\,\eta b > 0$. Put $\Phi(z) = \eta\varphi(z)$, then $\Phi = \psi_\eta \circ \varphi$ and $C_\Phi = C_\varphi C_{\psi_\eta} : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$ is bounded. On the other hand, the form $\Phi = \tau_{\eta b} \circ \psi_{\eta a}$ and the fact $C_{\psi_{\eta a}} : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$ is a surjective isometry imply that $C_{\tau_{\eta b}} = C^{-1}_{\psi_{\eta a}}C_\Phi$ is bounded on $\mathcal{F}^\infty_\alpha$. This also contradicts Lemma 4. Hence we see that if $|a| = \sqrt{\frac{\beta}{\alpha}}$, then $b = 0$.

To prove (b) we assume $|a| = \sqrt{\frac{\beta}{\alpha}}$. Since $b = 0$ by (a), we have $\varphi(z) = az$. Hence we obtain

$$\exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\} = 1,$$

for all $z \in \mathbb{C}$. This contradicts the condition (9). Hence we obtain $|a| < \sqrt{\frac{\beta}{\alpha}}$. ■

Theorem 4. *Let $\alpha, \beta > 0$. Suppose that $\varphi(z) = az + b$ where $a \in \mathbb{C}$ with $|a| \le \sqrt{\frac{\beta}{\alpha}}$ and $b \in \mathbb{C}$.*

(a) *If $b = 0$ whenever $|a| = \sqrt{\frac{\beta}{\alpha}}$, then $C_\varphi : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$ is bounded.*

(b) *If $|a| < \sqrt{\frac{\beta}{\alpha}}$, then $C_\varphi : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$ is compact.*

Proof. By a straight forward calculation, we see that $C_\varphi : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$ is a surjective isometry if $b = 0$ whenever $|a| = \sqrt{\frac{\beta}{\alpha}}$. To prove (b) we assume $|a| < \sqrt{\frac{\beta}{\alpha}}$. We put $G(x) = \frac{1}{2}(\alpha|a|^2 - \beta)x^2 + \alpha|a||b|x + \frac{\alpha}{2}|b|^2$ $(x \ge 0)$. Then $G(x)$ has the maximum $\frac{(\alpha|a||b|)^2}{2(\beta - \alpha|a|^2)} + \frac{\alpha}{2}|b|^2$ and $G(|z|) \to -\infty$ as $|z| \to \infty$. Since

$$\begin{aligned}\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2 &= \frac{1}{2}(\alpha|a|^2 - \beta)|z|^2 + \alpha\mathrm{Re}\,\langle az, b\rangle + \frac{\alpha}{2}|b|^2 \\ &\le \frac{1}{2}(\alpha|a|^2 - \beta)|z|^2 + \alpha|az||b| + \frac{\alpha}{2}|b|^2 = G(|z|),\end{aligned}$$

we have

$$\exp\left\{\frac{\alpha}{2}|\varphi(z)|^2 - \frac{\beta}{2}|z|^2\right\} \le \exp G(|z|), \tag{10}$$

for all $z \in \mathbb{C}$. Assume that $a \neq 0$. Since the formula $\varphi(z) = az + b$ implies that $|z| \to \infty$ if $|\varphi(z)| \to \infty$, it follows from (9) and (10) that $C_\varphi : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$ is compact. When $a = 0$, it is clear that $C_\varphi : \mathcal{F}^\infty_\alpha \to \mathcal{F}^\infty_\beta$ is compact. ■

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