Seshadri constants and surfaces of minimal degree

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Abstract

In [11] we showed that if the multiple point Seshadri constants of an ample line bundle on a smooth projective surface in very general points satisfy certain inequality then the surface is fibred by curves computing these constants. Here we characterize the border case of polarized surfaces whose Seshadri constants in general points fulfill the equality instead of inequality and which are not fibred by Seshadri curves. It turns out that these surfaces are the projective plane and surfaces of minimal degree.

Introduction and the main result

Given a smooth projective variety X and a nef line bundle L on X, Demailly defines the Seshadri constant of L at a point $P \in X$ as the real number

$$\varepsilon(L;P) := \inf_{C} \frac{L.C}{\mathrm{mult}_P C},$$

where the infimum is taken over all reduced and irreducible curves passing through P (see [3] and [7, Chapt. 5]).

This concept was extended by Xu [13] to finite subsets of a given variety. Let r be an integer and $P_1, \ldots, P_r$ points in X. Then the r-tuple Seshadri constant of L at the set $P_1, \ldots, P_r$ is the real number

$$\varepsilon(L;P_1,\ldots,P_r) := \inf_{C \cap \{P_1,\ldots,P_r\} \neq \varnothing} \frac{L.C}{\sum \mathrm{mult}_{P_i} C},$$

where the infimum is taken over all irreducible curves passing through at least one of the points $P_1, \ldots, P_r$.

There is an alternative and useful description of Seshadri constants in terms of the nef cone of a blown up variety. Specifically, let $f : Y \longrightarrow X$ be the blowing up of $P_1, \ldots, P_r \in X$ with exceptional divisors $E_1, \ldots, E_r$. Then the Seshadri constant can be computed as

$$\varepsilon(L; P_1, \ldots, P_r) = \sup \left\{ \lambda > 0 : \; f^*L - \lambda \cdot \sum_{i=1}^{r} E_i \text{ is nef} \right\}.$$

The Kleiman criterion of ampleness implies then that the multiple point Seshadri constants are subject to the following upper bound which depends only on the degree of L and the number of points

$$\varepsilon(L; P_1, \ldots, P_r) \leqslant \sqrt[\dim X]{\frac{L^{\dim X}}{r}} =: \alpha(L; r).$$

Whenever there is a strong inequality

$$\varepsilon(L; P_1, \ldots, P_r) < \alpha(L; r) \tag{1}$$

then the Seshadri constant is actually computed by a curve and not approximated by a sequence of curves. For Seshadri constants at a single point this follows from [1, Lemma 5.2] and the argument easily modifies to the multiple point case. We call any curve C with

$$\varepsilon(L; P_1, \ldots, P_r) = \frac{L.C}{\sum \operatorname{mult}_{P_i} C}$$

a *Seshadri curve for L at the r-tuple $P_1, \ldots, P_r$.*

Oguiso (see [9]) studied the behavior of Seshadri constants $\varepsilon(L; P)$ under the variation of the point P. He showed that the Seshadri function

$$\varepsilon_1 : X \ni P \longrightarrow \varepsilon(L; P) \in \mathbb{R}$$

is semi-continuous and that it attains its maximal value at a set which is a complement of an at most countable union of Zariski closed proper subsets of X i.e. for a very general point P. Oguiso arguments can be easily adapted to finite subsets. By $\varepsilon(L; r)$ we will abbreviate the maximal value of the function

$$\varepsilon_r : X^r \ni (P_1, \ldots, P_r) \longrightarrow \varepsilon(L; P_1, \ldots, P_r) \in \mathbb{R}$$

i.e. $\varepsilon(L; r) := \max \varepsilon_r$.

Nakamaye (see [8, Corollary 3]) observed that in case of surfaces an inequality of type

$$\varepsilon(L; 1) < \lambda \cdot \alpha(L; 1)$$

with a small factor λ has strong consequences for the geometry of the surface. Namely there exists a non-trivial fibration of X over a curve B whose fibers are Seshadri curves for L. On surfaces this was studied in more detail by Tutaj-Gasińska and the second author [12]. Hwang and Keum passed from surfaces to varieties of arbitrary dimension (see [5]). In [11] we started research along the same lines for multiple point Seshadri constants. In particular we proved the following theorem.

Theorem on fibrations. *Let X be a smooth projective surface, L a nef and big line bundle on X and $r \geqslant 2$ a fixed integer. If*

$$\varepsilon(L;r) < \sqrt{\frac{r-1}{r}} \cdot \alpha(L;r) \tag{2}$$

then there exists a fibration $f : X \longrightarrow B$ over a curve B such that given $P_1, \ldots, P_r \in X$ very general, for arbitrary $i = 1, \ldots, r$ the fiber $f^{-1}(f(P_i))$ computes $\varepsilon(L; P_1, \ldots, P_r)$ i.e. the fiber is a Seshadri curve of L.

Furthermore we showed that the bound in the Theorem is sharp in the sense that for every integer r there exists a surface X together with an ample line bundle L such that one has equality in (2) and X is not fibred by Seshadri curves of L.

The purpose of this note is to characterize the pairs (X, L) for which one has an equality in (2) and X is not fibred by Seshadri curves. The description of such pairs is provided in the next theorem which is our main result.

Theorem 1. *Let $r \geqslant 2$ be a given integer, X a smooth projective surface and L a nef and big line bundle on X such that*

$$\varepsilon(L;r) = \sqrt{\frac{r-1}{r}} \cdot \alpha(L;r)\ . \tag{3}$$

If X is not fibred by Seshadri curves for L, then

a) either $r = 2$, $X = \mathbb{P}^2$ and $L = \mathcal{O}(1)$,

b) or X is a surface of minimal degree in $\mathbb{P}^r$ and $L = \mathcal{O}_X(1)$.

Remarks.

(i) A similar theorem for $r = 1$ was already obtained by us in [11, Theorem 3.2] but the result and the methods are somewhat different.

(ii) A smooth surface is of minimal degree if and only if it is the Veronese surface in $\mathbb{P}^5$ or a rational normal scroll. This was proved by Del Pezzo (see [2]).

(iii) The converse of the Theorem holds: for any surface X of minimal degree and $L = \mathcal{O}_X(1)$, the equality (3) holds. This is easy to see taking the hyperplane section through r given points.

1 Useful Lemmas

Here we recall two Lemmas which are essential for the proof of the main result.

The first Lemma goes back to Xu [13, Lemma 1].

Lemma 1.1. *Let X be a smooth projective surface, let $(C_t, (P_1)_t, \ldots, (P_r)_t)_{t\in\Delta}$ be a non-trivial one parameter family of pointed reduced and irreducible curves on X and let m_i be positive integers such that $\mathrm{mult}_{(P_i)_t} C_t \geqslant m_i$ for all $i = 1, \ldots, r$. Then*

for $r = 1$ and $m_1 \geqslant 2$ $\quad C_t^2 \geqslant m_1(m_1 - 1) + 1$ *and*

for $r \geqslant 2$ $\quad C_t^2 \geqslant \sum_{i=1}^r m_i^2 - \min\{m_1, \ldots, m_r\}$.

The second lemma was obtained by Küchle in [6] and has purely arithmetical character.

Lemma 1.2. *Let $r \geqslant 2$ and $m_1, \ldots, m_r \in \mathbb{Z}$ be integers with $m_1 \geqslant \ldots \geqslant m_r \geqslant 1$ and $m_1 \geqslant 2$. Then we have*

$$(r+1)\sum_{i=1}^{r} m_i^2 > \left(\sum_{i=1}^{r} m_i\right)^2 + m_r(r+1).$$

2 Proof of the Theorem

In this section we prove Theorem 1. First we give a short overview of the proof. Since the Seshadri constants of the line bundle in Theorem 1 are strictly less than the upper bound, they must be computed by Seshadri curves.

We investigate properties of these curves in three steps. First we show that under assumptions of Theorem 1 the multiplicities of Seshadri curves in points $P_1, \ldots, P_r$ must all be equal to 1. This is an arithmetical part of the proof.

In the second step which is more analytical, we show that Seshadri curves must be rational.

The third step is geometrical and realizes Seshadri curves as hyperplane sections of X embedded in a projective space as a surface of minimal degree.

Let us now turn to the details.

2.1 Multiplicities of Seshadri curves

By assumptions of the Theorem 1 inequality (1) is satisfied so for every r-tuple $P_1, \ldots, P_r$ there exists a Seshadri curve $(C; P_1, \ldots, P_r)$. By [10, Proposition 1.3] there are finitely many such curves for every r-tuple. For a very general r-tuple we have the equality

$$\frac{L.(C; P_1, \ldots, P_r)}{\sum_{i=1}^{r} \mathrm{mult}_{P_i} C} = \frac{1}{r} \cdot \sqrt{(r-1)L^2}. \tag{4}$$

The number of algebraic families of curves satisfying this equality is at most countable. So at least one of these families must not be discrete. From now on we are interested in Seshadri curves $(C_t; (P_1)_t, \ldots, (P_r)_t)$ for L moving in a non-trivial family over some algebraic set Δ. Let m_i be the biggest integers such that

$$\mathrm{mult}_{(P_i)_t} C_t \geqslant m_i$$

for all $t \in \Delta$. Making Δ a little bit smaller if necessary we may assume that actually $m_i = \mathrm{mult}_{(P_i)_t} C_t$ for all t.

Renumbering the points if necessary we may also assume that

$$m_1 \geqslant \ldots \geqslant m_r.$$

There are the following three cases possible:

(A) $m_r \geqslant 1$ and $m_1 \geqslant 2$;

(B) $m_1 = \ldots m_r = 1$;

(C) $m_r = 0$.

In this step we want to exclude (A) and (C).

In case (A) we are in the position to apply Lemma 1.2. Thus

$$\frac{1}{r+1}\left(\sum_{i=1}^{r} m_i\right)^2 < \sum_{i=1}^{r} m_i^2 - m_r \leqslant C_t^2,$$

where the second inequality is assured by Lemma 1.1. Multiplying the above inequality by L^2 and applying the index theorem on the right hand side we arrive to the following inequality

$$\frac{1}{r+1}\left(\sum_{i=1}^{r} m_i\right)^2 \cdot L^2 < (L.C_t)^2.$$

Dividing by the sum of multiplicities and revoking (4) we obtain

$$\frac{1}{r+1} \cdot L^2 < \frac{r-1}{r^2} \cdot L^2$$

which is not possible.

In case (C) if $r \geqslant 3$, then we have

$$\frac{L.(C; P_1, \ldots, P_{r-1})}{\sum_{i=1}^{r-1} m_i} = \frac{L.(C; P_1, \ldots, P_r)}{\sum_{i=1}^{r} m_i} = \sqrt{\frac{r-1}{r}} \cdot \alpha(L; r) < \sqrt{\frac{r-2}{r-1}} \cdot \alpha(L; r-1).$$

Hence our Theorem on fibrations shows that X is covered by Seshadri curves for L contradicting the assumption of Theorem 1.

If $r = 2$, then by assumption we have

$$\varepsilon(L; 1) = \sqrt{\frac{1}{4} L^2}$$

and in this case we get the same contradiction by [12, Theorem].

Thus we showed that for $P_1, \ldots, P_r$ very general the Seshadri curve for L has multiplicities equal 1 at all these points. In particular we conclude from (4) that

$$L.(C; P_1, \ldots, P_r) = \sqrt{(r-1)L^2}\,. \tag{5}$$

Together with the index theorem we get

$$C^2 \leqslant r - 1. \tag{6}$$

2.2 Rationality of Seshadri curves

In this part we follow basically the deformation argument of [4] with necessary modifications. First we observe that one can fix the points $P_1, \ldots, P_{r-1}$ and consider Seshadri curves for the r-tuples $P_1, \cdots, P_{r-1}, P$ with the last point moving. Among these curves one can find again a non-trivial family $(C_t; P_1, \ldots, P_{r-1}, P_t)$ over some smooth base Δ. For t general the corresponding Kodaira-Spencer map

$$T_t\Delta \longrightarrow H^0(C_t, N_{C_t/X})$$

factorizes in fact over $H^0(C_t, N_{C_t/X}(-P_1 - \cdots - P_{r-1}))$.

Lemma 1.1 implies that $C_t^2 \geqslant r-1$. In view of (6) we obtain that in fact $\deg N_{C_t/X} = C_t^2 = r-1$. Since the image of the Kodaira-Spencer map is non-zero we conclude that the line bundle $N_{C_t/X}(-P_1 - \cdots - P_{r-1})$ is trivial. Equivalently, there is a section s_r in $H^0(C_t, N_{C_t/X})$ whose zero locus is exactly the divisor $P_1 + \cdots + P_{r-1}$. Fixing P_r and moving instead another point in the tuple we get in the same manner sections $s_1, s_2 \ldots, s_r$ in $H^0(C_t, N_{C_t/X})$ whose zero loci are $P_2 + \cdots + P_r$, $P_1 + P_3 + \cdots + P_r$, $\ldots$, $P_1 + \cdots + P_{r-1}$ respectively. They are obviously independent. This shows that $N_{C_t/X}$ is a line bundle of degree $r-1$ with at least r sections. This can happen only in the case when C_t is a rational curve. Thus we showed that under assumptions of Theorem the Seshadri curves are rational.

2.3 Embedding X as a surface of minimal degree

It follows from the last part that X is rationally connected hence it is a rational surface. Since $C^2 = r-1$ for Seshadri curves, it follows from the index theorem and (5) that the Seshadri curves are numerically equivalent. On rational surfaces this implies the linear equivalence, so Seshadri curves move in a single linear system. We call this system $|M|$ and we show that M is in fact very ample.

First we show that M separates points. Let P and Q be two distinct points on X. Let C be a Seshadri curve for L lying in $|M|$ and passing through P. It might happen that Q lies also on C. Taking $P_2, \ldots, P_{r-1}$ general on C we have that $(C; P, P_2, \ldots, P_{r-1}, Q)$ is a Seshadri curve for L. Taking Q' very general away of C there exists also a Seshadri curve $(C'; P, P_2, \ldots, P_{r-1}, Q')$. Since $C.C' = r-1$ this new curve cannot pass through Q and thus we separated P and Q.

Next we show that M separates tangent vectors. To this end for a fixed point P it is enough to find two Seshadri curves intersecting transversally at P. Again, this is the case for the curves C and C' from the argument above as they have $r - 1 = C.C'$ points in common, so must intersect at every of these points transversally.

If $r = 2$ then M has degree 1. This shows that X is $\mathbb{P}^2$. For $r \geqslant 3$ and a smooth curve $C \in |M|$ we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(M) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0.$$

Since $H^1(\mathcal{O}_X) = 0$ and $h^0(C, \mathcal{O}_C(C)) = r$ as already established in the previous part, we conclude from the long cohomology sequence that M has $r+1$ sections. Hence the image of X under the mapping given by $|M|$ must be a surface of minimal degree.

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