Stable Postulation and Stable Ideal Generation: Conjectures for Fat Points in the Plane*

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Abstract

It is an open problem to determine the Hilbert function and graded Betti numbers for the ideal of a fat point subscheme supported at general points of the projective plane. In fact, there is not yet even a general explicit conjecture for the graded Betti numbers. Here we formulate explicit asymptotic conjectures for both problems. We work over an algebraically closed field K of arbitrary characteristic.

1 Introduction

We are interested here in studying the problem of computing $h^0(X, \mathcal{O}_X(tF))$ when $t \gg 0$, where F is a divisor on the blow up $\pi : X \to \mathbf{P}^2$ at a finite set of distinct generic points $P_1, \ldots, P_n$ of $\mathbf{P}^2$. We also consider the problem of determining the dimension of the cokernel of the map $\mu_{tF} : H^0(X, \mathcal{O}_X(tF)) \otimes H^0(X, \mathcal{O}_X(L)) \to H^0(X, \mathcal{O}_X(L + tF))$ for $t \gg 0$, where μ_{tF} is given by multiplication and L is the pullback to X of a general line in $\mathbf{P}^2$.

One motivation for computing $h^0(X, \mathcal{O}_X(F))$ for arbitrary F on X comes from fat points. If $I(P_i)$ is the ideal in the homogeneous coordinate ring $R = K[\mathbf{P}^2]$ generated by all forms vanishing at P_i, and if each m_i is a nonnegative integer, then the subscheme Z of $\mathbf{P}^2$ defined by the homogeneous ideal $\cap I(P_i)^{m_i}$ is known as a *fat point* subscheme of $\mathbf{P}^2$. We will denote the ideal by $I(Z)$ and Z by $Z = m_1P_1 + \cdots + m_nP_n$. The Hilbert function of $I(Z)$ is defined to be the function

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giving the K-vector space dimension $h(t, I(Z)) = \dim I(Z)_t$ of the homogeneous component $I(Z)_t$ of $I(Z)$ as a function of the degree t. For each t we can associate to Z the divisor $F_t(Z) = tL - m_1E_1 - \cdots - m_nE_n$, where $E_i = \pi^{-1}(P_i)$. Then it is well known that $h^0(X, \mathcal{O}_X(F_t(Z))) = h(t, I(Z))$. Moreover, the dimension of the cokernel of $\mu_{F_t(Z)}$ is the number of generators of degree $t+1$ in any minimal set of homogeneous generators of $I(Z)$. In fact, computing $h^0(X, \mathcal{O}_X(F_t(Z)))$ and the dimension of the cokernel of $\mu_{F_t(Z)}$ for each t is equivalent to computing the graded Betti numbers of a graded minimal free resolution of $I(Z)$ over R (see, for example, [GHI]).

Both problems are still open, whether approached from the point of view of fat points or from the point of view of complete linear systems on X. Here we consider stable (i.e., asymptotic) versions of these problems. From the perspective of fat points, given any s and Z supported at the points P_i, the stable version of the postulation problem is to find $h(ts, I(tZ))$ for all $t \gg 0$ (i.e., for all but finitely many t). The stable version of the ideal generation problem is to find the minimum number of homogeneous generators of $I(tZ)$ in degree $ts+1$ for all $t \gg 0$ (i.e., to find the dimension of the cokernel of $I(tZ)_{st} \otimes R_1 \to I(tZ)_{st+1}$ for $t \gg 0$). From what is to us the more convenient perspective of divisors on X, the stable versions of the postulation and ideal generation problems, given an arbitrary F, are to determine $h^0(X, \mathcal{O}_X(tF))$ and the dimension of the cokernel of μ_{tF} for all $t \gg 0$. We find that these stable versions can be cast in a way that is more purely geometric than the full problem. Indeed, we show that the well known SHGH Conjecture (see Conjecture 3.2), which gives a complete conjectural solution to the postulation problem, implies that to solve the Stable Postulation Problem it is enough to determine the integral curves C on X with $C^2 \leq 0$, and it implies that to solve the Stable Ideal Generation Problem it is enough to determine the dimension of the cokernel of μ_F in the case that $F = L + iE$ where E is a smooth rational curve with $E^2 = -1$ and where $i = L \cdot E$. We also include explicit conjectures for the complete solution to both stable problems; see Conjectures 3.6 and 3.8.

2 Background

The divisor classes $l = [L], e_1 = [E_1], \ldots, e_n = [E_n]$ give a free $\mathbf{Z}$-basis for the divisor class group $\mathrm{Cl}(X)$ of X. The intersection form is a bilinear form on $\mathrm{Cl}(X)$ compatible with a bilinear form on the group of divisors defined by having $L, E_1, \ldots,$ E_n be orthogonal with $L^2 = 1$ and $E_i^2 = -1$.

We now recall the definition of the Weyl group $W = W(X)$ of X; it is a subgroup of the orthogonal group acting on $\mathrm{Cl}(X)$. To avoid special cases, we will hereafter assume that $n \geq 3$. This is harmless, since blowing up additional points just embeds $\mathrm{Cl}(X)$ in a larger divisor class group but the dimension of the space of sections of a divisor F and the dimension of the cokernel of μ_F is the same whether one regards F on X or on the surface obtained after additional points are blown up.

The subgroup W is generated by the operators s_x for $x \in \{r_0, \ldots, r_{n-1}\}$, where $s_x(F) = F + (x \cdot F)x$ for any $F \in \mathrm{Cl}(X)$, with $r_0 = l - e_1 - e_2 - e_3$ and $r_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$. Given $F = dl - m_1e_1 - \cdots - m_ne_n$, note when $i > 0$ that $s_{r_i}(F)$

merely transposes m_i and m_{i+1}. Thus by the action of W we may always reduce to the case that $m_1 \geq m_2 \geq \cdots \geq m_n$. Moreover, if $F = dl - m_1e_1 - \cdots - m_ne_n$ with $m_1 \geq m_2 \geq \cdots \geq m_n$, then either $d \geq m_1 + m_2 + m_3$ or $L \cdot s_{r_0}F < L \cdot F = d$. In particular, letting Δ' denote the submonoid of $\mathrm{Cl}(X)$ of all classes F satisfying $F \cdot r_i \geq 0$ for all i, then given any class F, it is clear that either $wF \in \Delta'$ for some $w \in W$ or there is an element $w \in W$ such that $wF \cdot L < 0$.

It is easy to check that $F \cdot G = wF \cdot wG$ for any classes F and G and any $w \in W$. It is also easy to check that $K_X = wK_X$ for all $w \in W$, where K_X is the canonical class of X (which takes the form $K_X = -3l + e_1 + \cdots + e_n$).

We refer to [GHI] for general facts about W. We recall that since the points P_i are generic we have $h^0(X, F) = h^0(X, wF)$ for all F and $w \in W$ (Lemma A1.1.1(c) of [GHI]), where for convenience we write $h^0(X, F)$ in place of $h^0(X, \mathcal{O}_X(F))$. In view of our remark above regarding $wF \cdot L$, this means that $h^0(X, F) = 0$ unless there is some $w \in W$ such that $wF \in \Delta'$. This raises the question of what $h^0(X, F) = h^0(X, wF)$ is equal to when $wF \in \Delta'$.

In this regard, the submonoid $\Delta = \{F \in \Delta' : F \cdot e_n \geq 0\}$ of $\mathrm{Cl}(X)$ is of particular interest. Note that $dl - m_1e_1 - \cdots - m_ne_n \in \Delta$ if and only if $d \geq m_1 + m_2 + m_3$ and $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. Suppose $wF \in \Delta'$. If $wF \cdot L < 0$ or if $wF \cdot (L - E_1) < 0$, then $h^0(X, F) = h^0(X, wF) = 0$, since L and $L - E_1$ are nef (where we recall that a nef divisor is one which meets every effective divisor nonnegatively). On the other hand, if $wF \cdot L \geq 0$ and $wF \cdot (L - E_1) \geq 0$, we can apply the following lemma (which is essentially Lemma A1.1.1(e) of [GHI]). We recall that an *exceptional curve* is a smooth rational curve C such that $C^2 = -1$.

Lemma 2.1. *Let $F = dl - m_1e_1 - \cdots - m_ne_n$ where $F \in \Delta'$ with $L \cdot F \geq 0$ and $F \cdot (L - E_1) \geq 0$. Then there are classes $H \in \Delta$ and $N = c_1C_1 + \cdots + c_rC_r$ such that $F = H + N$, where each C_i is the class of an exceptional curve and $c_i \geq 0$ for all i, $H \cdot N = 0$ and $C_i \cdot C_j = 0$ for all $i \neq j$, and hence $h^0(X, F) = h^0(X, H)$.*

Proof. If $m_1 \leq 0$, then take $H = dL$ and $N = -m_1e_1 - \cdots - m_ne_n$, with $c_i = -m_i$ and $C_i = e_i$. If $m_1 > 0$, then $F \cdot (L - E_1) \geq 0$ implies $d \geq m_1$. If in addition $m_2 \leq 0$, then take $H = dL - m_1e_1$ and $N = -m_2e_2 - \cdots - m_ne_n$, with $c_1 = -m_2$, $C_1 = e_2$, etc. If however $m_2 > 0$ but $m_3 \leq 0$, there are two cases. If $F \cdot (L - E_1 - E_2) < 0$, then take $H = (dL - m_1e_1 - m_2e_2) + (F \cdot (L - E_1 - E_2))(l - e_1 - e_2)$ and $N = -(F \cdot (L - E_1 - E_2))(l - e_1 - e_2) - m_3e_3 - \cdots - m_ne_n$, with $c_1 = -(F \cdot (L - E_1 - E_2))$, $C_1 = l - e_1 - e_2$, $c_2 = -m_3$, $C_2 = e_3$, etc. If $F \cdot (L - E_1 - E_2) \geq 0$, then take $H = dL - m_1e_1 - m_2e_2$ and $N = -m_3e_3 - \cdots - m_ne_n$. Finally, if $m_1 > 0$, $m_2 > 0$ and $m_3 > 0$, then let j be the greatest index such that $m_j \geq 0$ and take $H = dL - m_1e_1 - \cdots - m_je_j$ with $N = \sum_{i>j} -m_ie_i$.

The fact that $h^0(X, F) = h^0(X, H)$ is now clear. If $h^0(X, F) > 0$, then N is in the base locus of $|F|$ and hence $h^0(X, F) = h^0(X, H)$. If $h^0(X, F) = 0$, then $h^0(X, H) = 0$ too, since otherwise F would be the sum $H + N$ with both H and N being classes of effective divisors. ∎

3 Problems and Conjectures

For any given class $F = dl - m_1e_1 - \cdots - m_ne_n$, it is thus easy (using the approach of the discussion above) to determine if $wF \in \Delta'$ for some $w \in W$, and if so to find an element w such that $wF \in \Delta'$ and thence to find the class $H \in \Delta$ corresponding to wF. It is therefore clear that to compute $h^0(X, F)$ for an arbitrary class F it is enough to do so for classes in Δ. The question remains as to what is the value of $h^0(X, H)$, and for this we have Conjecture 3.2 below.

The monoid Δ also plays a role for the problem of computing $\dim \operatorname{cok} \mu_F$. If $h^0(X, F) = 0$, then $\dim \operatorname{cok} \mu_F = h^0(X, L + F)$. If $h^0(X, F) > 0$, then for some $w \in W$ we have $wF \in \Delta'$ and hence $wF = H + N$ as above, in which case it is not hard to see that $\dim \operatorname{cok} \mu_F = (h^0(X, L + F) - h^0(X, L + w^{-1}H)) + \dim \operatorname{cok} \mu_{w^{-1}H}$ (viz. Lemma 2.1.1 of [GHI]). Thus, to be able to determine $\dim \operatorname{cok} \mu_F$ for an arbitrary F, it is enough to be able in general to compute h^0 and to be able to compute $\dim \operatorname{cok} \mu_{wH}$ for any $w \in W$ and $H \in \Delta$.

This motivates the following problem:

Problem 3.1. *Given $F \in \Delta$ and $w \in W$:*

(a) determine $h^0(X, F)$; and

(b) determine the dimension of the cokernel of $\mu_{wF} : H^0(X, wF) \otimes H^0(X, L) \to H^0(X, L + wF)$.

Although Problem 3.1 is open, there is a conjecture for the values of $h^0(X, F)$ for arbitrary F. Equivalent versions of this conjecture have been given by Segre [S], Harbourne [Ha2], Gimigliano [G] and Hirschowitz [Hi], and so we refer to them collectively as the SHGH Conjecture. In terms of our preceding discussion, the SHGH Conjecture is as follows:

Conjecture 3.2. *If $F \in \Delta$, then $h^0(X, F) = max(0, 1 + (F^2 - K_X \cdot F)/2)$.*

Although there are conjectures in special cases (see [GHI] for statements and discussion), there is as yet no general explicit conjecture for the dimension of the cokernel of μ_F. However, we now formulate a stable version of both parts of Problem 3.1, for both of which we will offer conjectures.

Problem 3.3. *Given $F \in \Delta$ and $w \in W$, for $t \gg 0$ (i.e., for all but finitely many $t > 0$):*

(a) determine $h^0(X, tF)$; and

(b) determine the dimension of the cokernel of $\mu_{twF} : H^0(X, twF) \otimes H^0(X, L) \to H^0(X, L + twF)$.

We are interested in developing conjectural solutions of Problem 3.3. We begin with part (a). We could just replace F in Conjecture 3.2 by tF, but in order to emphasize the stable aspect of Problem 3.3 (which will lead in Conjecture 3.6 to a more geometric statement), we propose:

Conjecture 3.4. *Let $F \in \Delta$. If $h^0(X, tF) > 0$ for some $t > 0$, then $h^0(X, tF) = 1 + ((tF)^2 - tK_X \cdot F)/2$ for all t sufficiently large and either $F^2 > 0$, or $F^2 = 0$ and F is a nonnegative multiple of either $3l - e_1 - \cdots - e_9$ or $l - e_1$.*

We now have:

Lemma 3.5. *Conjecture 3.2 implies Conjecture 3.4.*

Proof. Let $F \in \Delta$ with $h^0(X, tF) > 0$ for some $t > 0$, and hence $h^0(X, stF) > 0$ for all $s > 0$. Then by Conjecture 3.2 we must have $F^2 \geq 0$, since $F^2 < 0$ implies that $1 + ((stF)^2 - K_X \cdot (stF))/2 < 0$ for $s \gg 0$. If $F^2 > 0$, we are done, so assume $F^2 = 0$. By Conjecture 3.2 and $h^0(X, stF) > 0$ for $s \gg 0$, it follows that $-K_X \cdot F \geq 0$.

Since $F \in \Delta$, as in A1.1 of [GHI], it is not hard to check that F is a nonnegative integer linear combination $F = \sum_i a_i J_i$ of the classes $J_0 = l$, $J_1 = l - e_1$, $J_2 = 2l - e_1 - e_2$, $J_3 = 3l - e_1 - e_2 - e_3, \ldots, J_n = 3l - e_1 - \cdots - e_n = -K_X$. Since $0 \leq -K_X \cdot F = F \cdot J_n \leq F \cdot J_i$ for $i \geq 3$, while $F \cdot J_i \geq 0$ for $i < 3$ (since $F \in \Delta$ and, by direct check, $J_i \cdot J_k \geq 0$ for all k when $i \leq 3$), we see that $F^2 \geq a_i F \cdot J_i \geq 0$ for each i. Thus, in order to have $F^2 = 0$, it follows that $a_i = 0$ unless either $i = 1$ or $i \geq 9$ (since $J_k \cdot J_i > 0$ for all k if $1 \neq i < 9$, and hence $F \cdot J_i \geq a_k J_k \cdot J_i > 0$ if $1 \neq i < 9$). Now, if $a_1 > 0$, then $a_i = 0$ for all $i \neq 1$, since $J_i \cdot J_1 > 0$ for all $i \neq 1$. If $a_1 = 0$ but $a_i > 0$ for some $i > 9$, then $F \cdot J_i \leq a_i J_i^2 < 0$, since $J_i^2 < 0$ and $J_i \cdot J_k \leq 0$ for all $k \geq 9$. Thus $F^2 = 0$ implies either $F = a_1 J_1$ or $F = a_9 J_9$, as claimed. ∎

In fact, Conjecture 3.4 is equivalent to the following conjecture:

Conjecture 3.6. *Let C be the class of a reduced irreducible divisor on X. Then $C^2 \leq 0$ if and only if C is the class of an exceptional curve, or $C = w(l - e_1)$ or $C = w(3l - e_1 - \cdots - e_9)$, for some $w \in W$.*

Lemma 3.7. *Conjectures 3.4 and 3.6 are equivalent.*

Proof. Assume Conjecture 3.4, and consider the class C of a reduced irreducible divisor on X. First say $C^2 < 0$. If C is not exceptional, then $C \cdot E \geq 0$ for all exceptional E so (by A1.1.1(b) [GHI]) we may assume $wC \in \Delta$ for some $w \in W$. Let $F = dL + swC$ for some choices of $d > 0$ and $s > 0$ such that $F^2 > 0$ but $C \cdot F < -1$. Then Conjecture 3.4 implies that $h^0(X, tF) = 1 + ((tF)^2 - tK_X \cdot F)/2$ for large t (and hence $h^1(X, tF) = 0$ by Riemann-Roch, since $L \cdot (K_X - tF) < 0$ implies $h^2(X, tF) = h^0(X, K_X - tF) = 0$), but taking cohomology of $0 \to \mathcal{O}_X(tF - C) \to \mathcal{O}_X(tF) \to \mathcal{O}_C(tF) \to 0$ and keeping in mind that $h^2(X, tF - C) = 0$ as before for $t \gg 0$, while $h^1(C, tF) > 0$ by Riemann-Roch (since $F \cdot C < -1$), we see that $h^1(X, tF) > 0$, which is a contradiction. Thus C is exceptional if $C^2 < 0$.

Now say $C^2 = 0$; then $C = wF$ for some $w \in W$ and some $F \in \Delta$, again by A1.1.1(b) [GHI], and by Conjecture 3.4 F is a nonnegative multiple of either $3l - e_1 - \cdots - e_9$ or $l - e_1$. Since C is reduced and irreducible, the multiple must be 1.

Now assume Conjecture 3.6. Let $F \in \Delta$. If $h^0(X, tF) > 0$ for some $t > 0$, then F is nef. (If not, $|tF|$ has a fixed component C of negative self-intersection with $F \cdot C < 0$. Since $F \in \Delta$, by A1.1.1(b) of [GHI] we know $F \cdot E \geq 0$ for all exceptional E, thus C is not exceptional, which contradicts Conjecture 3.6.) Thus $F^2 \geq 0$.

First say $F^2 = 0$; this and nefness implies all irreducible components C of any section of tF have $C^2 \leq 0$ and $F \cdot C = 0$. If $C^2 < 0$ for some component C, then C is exceptional and $C \cdot C' > 0$ for some other component C' of tF. If C' is not exceptional or if $C \cdot C' > 1$, then $C + C'$ is nef and has positive self-intersection, but this contradicts $F \cdot (C + C') = 0$. If C' is exceptional and $C \cdot C' = 1$, then a general section D of $|C + C'|$ is reduced and irreducible of self-intersection 0. (To see that D is reduced and irreducible, note that $wC' = E_n$ for some $w \in W$, since C' is exceptional. Thus $wC = dL - m_1E_1 - \cdots - m_{n-1}E_{n-1} - E_n$, since $C \cdot C' = 1$. This means that $wD = dL - m_1E_1 - \cdots - m_{n-1}E_{n-1}$. If Y is the blow up of $\mathbf{P}^2$ at $P_1, \dots, P_{n-1}$ and if $X \to Y$ is the blow up of P_n, then $|wD|$ has a smooth integral section, regarded as a divisor on Y, since C, and hence wC, is smooth and integral. Thus the general section of D on Y is smooth and integral, hence also on X. Hence C is not a component of a general section of $C + C'$ and so not of tF either. Thus C cannot be exceptional, and we conclude $C^2 = 0$.)

Thus any component C of a general section of tF is reduced and irreducible with $C^2 = 0$, so by Conjecture 3.6 it is either $w(l - e_1)$ or $w(3l - e_1 - \cdots - e_9)$ for some $w \in W$. But $C \cdot F = 0$, and, applying A1.1.1(a) of [GHI], the only class in Δ orthogonal to $w(l - e_1)$, is a multiple of $l - e_1$. If $C = w(3l - e_1 - \cdots - e_9)$, a similar argument shows F is a multiple of $3l - e_1 - \cdots - e_9$. Thus F must itself be a multiple of either $l - e_1$ or $3l - e_1 - \cdots - e_9$, and for any nonnegative multiple tF of either $3l - e_1 - \cdots - e_9$ or $l - e_1$, it is not hard to check that $h^0(X, tF) = 1 + ((tF)^2 - tK_X \cdot F)/2$ for all t.

Finally, suppose $F^2 > 0$. Then, for t large enough, we have $(tF - K_X)^2 > 0$ and hence by Riemann-Roch $tF - K_X$ is effective for $t \gg 0$. But we also have $tF - K_X \in \Delta$ for $t \gg 0$, hence, as above, $tF - K_X$ is nef. By the Ramanujam vanishing theorem (see Theorem 2.8 of [Ha1]) and duality, we now have $h^1(X, tF) = h^1(X, -tF + K_X) = 0$, hence $h^0(X, tF) = 1 + ((tF)^2 - tK_X \cdot F)/2$ for all t sufficiently large. ∎

As mentioned above, there are conjectures for the dimension of the cokernel of μ_F only in special cases. We recall one such now (Conjecture 3.4 of [GHI]). To state it, let E be an exceptional curve. Pulling back and restricting the twisted cotangent bundle $\Omega_{\mathbf{P}^2}(1)$ gives a rank two bundle $(\pi^*(\Omega_{\mathbf{P}^2}(1)))|_E$ on E, which thus splits as $\mathcal{O}_E(-a_E) \oplus \mathcal{O}_E(-b_E)$ for some integers $a_E \le b_E$. We call (a_E, b_E) the *splitting type* of E.

Conjecture 3.8. *Let $F = L + iE$, where E is an exceptional curve and $0 \le i \le L \cdot E$. Then* $\dim \operatorname{cok} \mu_F = \binom{i-b_E}{2} + \binom{i-a_E}{2}$.

Actually part of the conjecture is known; note that the inequality $\dim \operatorname{cok} \mu_F \le \binom{i-b_E}{2} + \binom{i-a_E}{2}$ is proved in [GHI], Theorem 3.3, along with the equality in a range of cases.

Now we relate Conjecture 3.8 to Problem 3.3(b).

Proposition 3.9. *Conjecture 3.6 and Conjecture 3.8, if true, give a complete solution to Problem 3.3(b).*

Proof. Consider wF for some $w \in W$ and $F \in \Delta$. To determine the dimension of the cokernel of $\mu_{twF} : H^0(X, twF) \otimes H^0(X, L) \to H^0(X, L + twF)$ for large t, we may as well assume that twF is effective. Thus (assuming Conjecture 3.6 and hence Conjecture 3.4) F either has positive self-intersection or it is a nonnegative multiple of either $3L - E_1 - \cdots - E_9$ or $L - E_1$. If F is a nonnegative multiple of either $3L - E_1 - \cdots - E_9$ or $L - E_1$, it is not hard by induction (using Mumford's snake lemma, Lemma 2.3.1 [GHI]) on t to show that μ_{twF} has maximal rank (in fact, it is injective unless $wF \cdot L = 1$, in which case it is surjective), so we may as well assume that $F^2 > 0$. But then for all t large enough, $twF - L$ is effective, hence can be written as a sum $H_t + N_t$, where H_t is an effective nef divisor and N_t is the sum of the fixed components of $|twF - L|$ of negative self-intersection which meet $twF - L$ negatively. By Conjecture 3.6, N_t is a sum of exceptional curves which therefore must be disjoint and such that $H_t \cdot N_t = 0$. For all t large enough, we claim that $N_t = N_{t'}$ for all $t' \geq t$ and $wF \cdot N_t = 0$. If for some t we have $N_t = 0$, then clearly $N_{t'} = 0$ for all $t' > t$ (since F is nef), so say $N_t \neq 0$ for all large t. By definition, any component C of N_t has $C \cdot (twF - L) = C \cdot N_t < 0$. If C' is a component of $N_{t'}$ for some $t' > t$, then $0 > C' \cdot N_{t'} = C' \cdot (t'wF - L) \geq C' \cdot (twF - L) = C' \cdot N_t$, so all components of $N_{t'}$ are components of N_t. For t large enough, we may therefore assume that N_t stays the same as t increases. Thus for any component C of N_t for t large enough, we have $C \cdot (t'wF - L) < 0$ for all $t' > t$, hence $C \cdot wF = 0$, so $wF \cdot N_t = 0$ and in addition $-C \cdot N_t = C \cdot L$. We also see that $H_{t'} = (t' - t)wF + H_t$ for all $t' \geq t$, and hence that $H_t^2 > 0$ for $t \gg 0$. As above, $(t' - t)wF - K_X + H_t$ is nef and big for $t' \gg 0$, so duality and Ramanujam vanishing imply $h^1(X, H_{t'}) = h^1(X, K_X - ((t' - t)wF + H_t)) = h^1(X, -((t' - t)wF - K_X + H_t)) = 0$ for $t' \gg 0$.

Note that $h^1(X, H_t) = 0$ implies that $\mu_{twF} = \mu_{H_t+L}$ is surjective (by the usual fact that fat point ideals are generated in degrees less than the regularity). Thus $\mu_{twF} = \mu_{H_t+L}$ is surjective if $N_t = 0$. If $N \neq 0$ (suppressing the subscript t), by considering the exact sequences $0 \to \mathcal{O}_X(H + L) \to \mathcal{O}_X(H + L + N) \to \mathcal{O}_N(H + L + N) \to 0$ and $0 \to \mathcal{O}_X(L) \to \mathcal{O}_X(L + N) \to \mathcal{O}_N(L + N) \to 0$, keeping in mind that $\mathcal{O}_N(H + L + N)$ and $\mathcal{O}_N(L + N)$ are isomorphic, it follows (by Mumford's snake lemma, Lemma 2.3.1 [GHI]) that $\mu_{twF} : H^0(X, H + L + N) \otimes H^0(X, L) \to H^0(X, L + H + L + N)$ and $\mu_{L+N} : H^0(X, L + N) \otimes H^0(X, L) \to H^0(X, 2L + N)$ both have cokernels isomorphic to the cokernel of $\mu_{L+N,N} : H^0(N, L + N) \otimes H^0(X, L) \to H^0(N, 2L + N)$, and hence to each other. Writing $N = d_1C_1 + \cdots + d_rC_r$ as a sum of positive multiples of disjoint exceptional curves C_i (where $d_i = -C_i \cdot N = C \cdot L$), it follows that $\mu_{L+N,N} : H^0(N, L + N) \otimes H^0(X, L) \to H^0(N, 2L + N)$ is the direct sum of the maps $\mu_{L+d_iC_i,d_iC_i} : H^0(d_iC_i, L + d_iC_i) \otimes H^0(X, L) \to H^0(d_iC_i, 2L + d_iC_i)$, so the cokernel of μ_{L+N} (or equivalently, of μ_{twF}) is isomorphic to the direct sum of the cokernels of $\mu_{L+d_iC_i}$. Thus to solve Problem 3.3(b) it is enough to consider $\mu_F : H^0(X, F) \otimes H^0(X, L) \to H^0(X, L + F)$ in case $F = L + dC$ where C is exceptional and $d = C \cdot L$, and this is precisely the situation of Conjecture 3.8. ∎

Remark 3.10. When $F = L + (C \cdot L)C$ and C is an exceptional curve, as an aside we note that determining the dimension of the cokernel of $\mu_{tF} : H^0(X, tF) \otimes H^0(X, L) \to H^0(X, L + tF)$ for large t, is equivalent to doing so for $t = 1$:

For convenience, let $c = C \cdot L$, so $F = L + cC$. It is not hard to show that $h^1(X, (t - 1)F) = 0$, hence $\mu_{tF-cC} : H^0(X, tF - cC) \otimes H^0(X, L) \to H^0(X, L + tF - cC)$ is surjective by regularity considerations. But since $C \cdot F = 0$, we see $\mathcal{O}_{cC}(tF)$ is isomorphic to $\mathcal{O}_{cC}$, so we have an exact sequence $0 \to \mathcal{O}_X(tF - cC) \to \mathcal{O}_X(tF) \to \mathcal{O}_{cC} \to 0$, from which it now follows for $t > 0$ that μ_{tF} :

$H^0(X, tF) \otimes H^0(X, L) \to H^0(X, L + tF)$ and $\mu_{cC,cC} : H^0(cC, \mathcal{O}_{cC}) \otimes H^0(X, L) \to H^0(cC, \mathcal{O}_{cC}(L))$ have isomorphic cokernels, as in the argument above. Since the latter is independent of t, we see that the dimension of the cokernel of μ_{tF} : $H^0(X, tF) \otimes H^0(X, L) \to H^0(X, L + tF)$ is the same for all $t > 0$.

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