Heat equation approach to index theorems on odd dimensional manifolds

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Abstract

The subject of this paper is the index theorem on odd-dimensional manifolds with boundary. Such a theorem has been formulated and proved by D. Freed and his proof is based on analysis by Calderon and Seeley. In this paper we prove this theorem using the heat kernel methods for boundary conditions of Dirichlet and Neumann type. Moreover, we also consider the Atiyah-Patodi-Singer spectral boundary condition which is not studied in Freed's paper. As a direct consequence of the method, we obtain some information about isospectral invariants of the boundary conditions. This proof does not use the cobordism invariance of the index and is generalized easily to the family case.

1 Introduction

Dirac type operators on a closed odd dimensional manifold are formally self-adjoint, so their index vanishes and have no interest. In contrast, if the boundary of the underlying manifold is non-empty, one gets non-self-adjoint operators with non-zero index by putting suitable boundary conditions. In [6], D. Freed, inspired by the work of physicians, formulated such an index theorem and proved it by means of symbol calculus of elliptic boundary problems. In this paper we give a heat equation proof of this theorem and consider, in addition, the case of the Atiyah-Patodi-Singer(APS) spectral boundary condition. To compute the contribution of the APS condition we use the perfect symmetry between positive and

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negative parts of the Dirac operators spectrum in even dimension rather than using calculus which are established to handle this problem in general (see, e.g. [8] and [4]). In section 2 we state the theorem and prove it for one Dirac operator. At the end of that section we give a necessary condition for the isospectrality of these boundary problems with different sets of boundary condition on different connected components of the boundary. In section 3 we formulate and prove the index theorem for a family of Dirac operators by studying the Chern character of a superconnection adopted to the family of Dirac operators. Although our notation refers to the Dirac operator acting on standard complex spinor fields, the method can be applied directly to the case of Dirac type operators and a family of them.

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2 Index theorem for a Dirac type operator

Let M be a compact spin manifold of dimension $2n+1$ with a riemannian metric taking the product form $(du)^2 + g$ in a collar neighborhood $U = [0, 1^+) \times \partial M$ of the boundary, where 1^+ stands for a fixed real number greater than 1. In the sequel we denote the coordinate of the interval $[0, 1^+)$ by u and v and those of ∂M by y and z. A typical point of M will be denoted by x. We fix a spin structure on M giving rise to the complex spin vector bundle S on M. The orientation and the spin structure on M induce an orientation and a spin structure on ∂M. So the restriction of spin bundle S to the collar neighborhood of the boundary splits into the positive and negative parts

$$S_{|U} \simeq S^+ \oplus S^- ,$$

where $S^+ \oplus S^-$ is the spin bundle of ∂M lifted to the collar neighborhood U in the obvious manner. According to this splitting, each spinor field $\phi \in C^\infty(M, S)$ has a decomposition $\phi_{|U} = \phi^+ \oplus \phi^-$, where $\phi^\pm$ are u-dependent smooth sections of $S^\pm$. With respect to this decomposition the Dirac operator D takes the following form where A denotes the Dirac operator of ∂M (see, e.g. [5, chapter 9])

$$\begin{pmatrix} i\partial_u & iA^- \\ -iA^+ & -i\partial_u \end{pmatrix} .$$

Let $\partial M = \sqcup_i N_i$ where N_i's run over the connected components of the boundary. For each i let ϵ_i be 0, $+$ or $-$ arbitrarily and fix them. For $\epsilon_i = \pm$ let P^{ϵ_i} denote the following local boundary condition

$$(\phi_{|N_i})^{\epsilon_i} = 0. \tag{2.1}$$

The boundary condition P^0, corresponding to $\epsilon_i = 0$, denotes the Atiyah-Patodi-Singer boundary condition whose definition is as follows. The Dirac operator A_i of each connected component N_i has a discrete resolution $\{\phi_\lambda, \lambda\}$. Since N_i is even dimensional, there is a symmetry between the positive and the negative parts of the A_i spectrum. This symmetry is given by the following unitary isomorphism $\mathcal{U}$ between eigenspaces with opposite eigenvalues

$$\mathcal{U}(\phi_\lambda^+ + \phi_\lambda^-) = \phi_\lambda^+ - \phi_\lambda^- \ . \tag{2.2}$$

Let $P^0 \in \mathrm{End}(L^2(N_i\ , S))$ be the orthogonal projection on the subspace generated by ϕ_λ's with $\lambda \geq 0$. This projection defines the APS boundary condition. By P^ϵ we mean this set of boundary conditions.
For smooth spinor fields ϕ and ψ on M one has

$$\langle D\,\phi, \psi\rangle_{L^2} - \langle \phi, D\,\psi\rangle_{L^2} = -\int_{\partial M} \langle \mathrm{cl}(n)\phi, \psi\rangle\, dy\ , \tag{2.3}$$

which implies that the operators (D, P^+) and (D, P^-) are adjoint to each other. It also implies that the adjoint operator for (D, P^0) is $(D, Id - P^0)$. We denote the formal adjoint of the boundary problem $D := (D, P^\epsilon)$ by $D^* := (D, P^{\bar{\epsilon}})$. As a consequence of the above discussion we have $\ker D^*D = \ker D$ and $\ker DD^* = \ker D^*$. These boundary problems are elliptic so they have finite dimensional kernels consisting of smooth spinor fields. Therefore one can define the index of this problem by

$$\mathrm{ind}(D, P^\epsilon) = \dim\ker\ D^*D - \dim\ker\ DD^*. \tag{2.4}$$

Theorem 1 (See theorem B of [6]). *The following formula holds*

$$\mathrm{ind}(D, P^\epsilon) = \frac{1}{2}\sum_{\epsilon_i=-} \mathrm{ind}\ A_i - \frac{1}{2}\sum_{\epsilon_i=+} \mathrm{ind}\ A_i - \frac{1}{2}\sum_{\epsilon_i=0} \dim\ker A_i.$$

To prove this theorem, at first, we consider the case of the half cylinder $\mathbb{R}^{\geq 0} \times N$ with the product spin structure and the product Riemannian metric. In the sequel, N stands for each one of N_i's and A stands for its Dirac operator.

2.1 Index density of the half cylinder with local conditions

We consider only the boundary condition P^+ since the case of P^- is similar. On the half cylinder the Dirac operator D, acting on compactly supported spinor fields subjected to condition P^+, takes the following form

$$\begin{pmatrix} i\partial_u & iA^- \\ -iA^+ & -i\partial_u \end{pmatrix}. \tag{2.5}$$

The associated second degree elliptic operators D^*D and DD^*, acting on compactly supported spinor fields $\phi(u, y)$, take the form $-\partial_u^2 + A^2$. The induced boundary condition for D^*D is

$$\phi^+(0, y) = 0 \ \text{ and } \ (\partial_u\phi^- + A^+\phi^+)_{|u=0} = 0.$$

Since A is a tangential operator, these conditions reduce to Dirichlet condition for ϕ^+ and Neumann condition for ϕ^-

$$\phi^+(0, y) = 0\ , \tag{2.6}$$

$$\frac{\partial\phi^-}{\partial u}(0, y) = 0\ . \tag{2.7}$$

A similar argument shows that the induced boundary conditions for DD^* are the following

$$\frac{\partial}{\partial u}\psi^+(0,y) = 0 \text{ and } \psi^-(0,y) = 0. \tag{2.8}$$

For $t > 0$ consider the heat operators e^{-tD^*D}. The kernel $\bar{K}_1(t,u,v,y,z)$ of this operator with respect to the boundary condition (2.6) has the following explicite form, cf. [7]

$$\frac{1}{\sqrt{4\pi t}}\{\exp(\frac{-(u-v)^2}{4t}) - \exp(\frac{-(u+v)^2}{4t})\}e^{-tA^-A^+}(t,y,z), \tag{2.9}$$

while the heat kernel for boundary condition (2.7) is

$$\frac{1}{\sqrt{4\pi t}}\{\exp(\frac{-(u-v)^2}{4t}) + \exp(\frac{-(u+v)^2}{4t})\}e^{-tA^+A^-}(t,y,z). \tag{2.10}$$

Therefore the trace density of the heat operator e^{-tD^*D}, as a function of t and u, is

$$\begin{aligned}\bar{K}_1(t,u) :&= \int_N \mathrm{tr}_y\, \bar{K}_1(t,u,v,y,y)\,dy \\ &= \frac{\mathrm{Tr}\ e^{-tA^-A^+}}{\sqrt{4\pi t}}\{1-\exp(\frac{-u^2}{t})\} + \frac{\mathrm{Tr}\ e^{-tA^+A^-}}{\sqrt{4\pi t}}\{1+\exp(\frac{-u^2}{t})\}. \end{aligned} \tag{2.11}$$

Denote the kernel of the heat operator e^{-tDD^*} by $\bar{K}_2(t,u,v,y,z)$. A similar discussion as in above gives the following expression for the trace density of this operator

$$\bar{K}_2(t,u) = \frac{\mathrm{Tr}\ e^{-tA^-A^+}}{\sqrt{4\pi t}}\{1+\exp(\frac{-u^2}{t})\} + \frac{\mathrm{Tr}\ e^{-tA^+A^-}}{\sqrt{4\pi t}}\{1-\exp(\frac{-u^2}{t})\}. \tag{2.12}$$

So we obtain the following formula for trace density $\bar{K}_+(t,u) = \bar{K}_1(t,u) - \bar{K}_2(t,u)$ of the operator $e^{-tD^*D} - e^{-tDD^*}$ with the boundary condition P^+

$$\bar{K}_+(t,u) = \frac{e^{-\frac{u^2}{t}}}{\sqrt{\pi t}}\{\mathrm{Tr}\ e^{-tA^+A^-} - \mathrm{Tr}\ e^{-tA^-A^+}\}.$$

Integrating with respect to $u \in \mathbb{R}^{\geq 0}$ we get

$$\int_0^\infty \bar{K}_+(t,u)\,du = -\frac{1}{2}\,\mathrm{ind}\,A. \tag{2.13}$$

Here we have used the McKean-Singer formula $\mathrm{ind}\,A = Tr\,e^{-tA^-A^+} - Tr\,e^{-tA^+A^-}$. Similarly, if $\bar{K}_-(t,u)$ denotes the trace density of the operator $e^{-tD^*D} - e^{-tDD^*}$ with the boundary condition P^-, then

$$\int_0^\infty \bar{K}_-(t,u)\,du = \frac{1}{2}\,\mathrm{ind}\,A.$$

If in above we had integrated on the finite interval $[0, \frac{1}{2}]$ instead of $[0, \infty)$, the difference would have been of exponential decay when t goes toward 0. So

$$\int_0^{\frac{1}{2}} \bar{K}_{\pm}(t,u)\,du = \mp\frac{1}{2}\,\mathrm{ind}\,A + O(e^{-\frac{1}{4t}}) \tag{2.14}$$

The following relation is a result of the above discussion and will be used later

$$\bar{K}_{\pm}(t,u) \sim 0 \ \text{ exponentially at } t = 0 \text{ for } u \neq 0. \tag{2.15}$$

Remark 1. *If $y \neq z$, then it is well known that the heat kernels $e^{-tA^-A^+}(t,y,z)$, $e^{-tA^+A^-}(t,y,z)$ and their derivatives with respect to y vanish exponentially at $t = 0$. If $y = z$ and $u \neq v$ the expression given in formulas* (2.9) *and* (2.10) *have this property. In this case the differentiation may be taken with respect to t.*

2.2 Index density of the half cylinder with the APS condition

Now let $\epsilon_i = 0$ and consider the half cylinder $\mathbb{R}^{\geq 0} \times N$ with boundary problems $D = (D, P^0)$ and $D^* = (D, Id - P^0)$. The induced boundary condition for D^*D is

$$P^0(\phi) = 0 \ \text{ and } \ (Id - P^0)D\phi = 0.$$

By exchanging the roles of P and $(Id - P)$ we obtain the adjoint induced boundary condition for D^*D. Denote the trivial vector bundle on $\mathbb{R}^{\geq 0}$ by E_λ whose fibers are the λ-eigenspace of A and put $A_\lambda := A_{|E_\lambda}$. The operators D^*D and DD^* take the form $-\partial_u^2 + \lambda^2$ on the smooth sections of E_λ. For a section $\phi(u,y)$ of E_λ the induced boundary conditions for D^*D are as follows

$$\phi(0,y) = 0 \qquad \text{for } \lambda \geq 0\,; \tag{2.16}$$

$$(\frac{\partial}{\partial u} + \lambda)_{|u=0}\,\phi(u,y) = 0 \qquad \text{for } \lambda < 0, \tag{2.17}$$

while the induced boundary conditions for DD^* are the following

$$\psi(0,y) = 0 \qquad \text{for } \lambda < 0\,; \tag{2.18}$$

$$(\frac{\partial}{\partial u} + \lambda)_{|u=0}\,\psi(u,y) = 0 \qquad \text{for } \lambda \geq 0. \tag{2.19}$$

Let K_1^λ and K_2^λ respectively denote the heat kernels of D^*D and DD^*. Let $\bar{K}(t,u)$ denote the supertrace density $\mathrm{Tr}_N\, e^{-tD^*D}(t,u) - \mathrm{Tr}_N\, e^{-tDD^*}(t,u)$. We are interested in the following quantity which is in fact the index density of D, acting on spinor fields subjected to the APS condition at $\{0\} \times N$

$$\int_0^{\frac{1}{2}} \bar{K}(t,u) = \sum_\lambda \int_0^{\frac{1}{2}} K_1^\lambda(t,u) - \int_0^{\frac{1}{2}} K_2^{-\lambda}(t,u). \tag{2.20}$$

For $\lambda \neq 0$ the operator $\mathcal{U}$ provides the following unitary isomorphism

$$\mathcal{U}_\lambda := \mathcal{U} : C^\infty(\mathbb{R}^{\geq 0}, E_\lambda) \to C^\infty(\mathbb{R}^{\geq 0}, E_{-\lambda})\,.$$

It is easy to show that $-A_{-\lambda} = \mathcal{U}_\lambda A_\lambda \mathcal{U}_\lambda^{-1}$, which implies $DD^*_{|E_{-\lambda}} = \mathcal{U}_\lambda D^* D_{|E_\lambda} \mathcal{U}_\lambda^{-1}$. Moreover the boundary condition (2.16) will be replaced by (2.18) under the action of $\mathcal{U}_\lambda$. Therefore in the expression (2.20) the terms indexed by $\lambda > 0$ vanish. The trace density of the fundamental solution of $\partial_t - \partial_u^2 + \lambda^2 = 0$ with boundary condition (2.17) is given by the following expression, c.f. [1, relation 2.17]

$$K_1^\lambda(t,u) = \left(\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{1 + \exp(\frac{-u^2}{4t})\} + \lambda e^{-2\lambda u} \mathrm{erfc}\{\frac{\mathrm{u}}{\sqrt{\mathrm{t}}} - \check{}\sqrt{\mathrm{t}}\} \right) . \dim E_\lambda \ ; \ \lambda < 0 \, ,$$

while the fundamental solution with respect to the boundary condition (2.19) is

$$K_2^{-\lambda}(t,u) = \left(\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{1 + \exp(\frac{-u^2}{4t})\} - \lambda e^{2\lambda u} \mathrm{erfc}\{\frac{\mathrm{u}}{\sqrt{\mathrm{t}}} + \check{}\sqrt{\mathrm{t}}\} \right) . \dim E_{-\lambda} \ ; \ \lambda < 0 \, .$$

Here the error function is defined by the following formula

$$\mathrm{erfc}(\mathrm{x}) = \frac{2}{\sqrt{ß}} \int_{\mathrm{x}}^{\infty} \mathrm{e}^{-\mathrm{s}^2} \, \mathrm{ds}.$$

Subtracting above expressions, we get the following relations at $t = 0$ for $\lambda < 0$

$$\begin{aligned} &K_1^\lambda(t,u) - K_2^{-\lambda}(t,u) \sim 0 \ \text{ exponentially for } u \neq 0, \\ &K_1^\lambda(t,0) - K_2^{-\lambda}(t,0) \sim \lambda + o(1) \, , \end{aligned}$$

consequently

$$\int_0^{\frac{1}{2}} K_1^\lambda(t,u) - K_2^{-\lambda}(t,u) \, du \sim 0 \ \text{ exponentially at } t = 0.$$

Therefore when t goes toward zero, only the terms indexed by $\lambda = 0$ have probably nonzero contribution in the sum (2.20). For $\lambda = 0$, the boundary conditions (2.16) and (2.19) are respectively the Dirichlet and Neumann boundary conditions. Regarding expressions (2.9) and (2.10) we get

$$K_1^0(t,u) - K_2^0(t,u) = -\frac{e^{\frac{-u^2}{t}}}{\sqrt{\pi t}} \dim \ker A. \tag{2.21}$$

Hence

$$\int_0^{\frac{1}{2}} (K_0^1(t,u) - K_0^2(t,u)) \, du \sim -\frac{1}{2} \dim \ker A \ \text{ exponentially at } t = 0.$$

We summarize the above discussion in the following proposition

Proposition 2. *Denote the heat kernels of the boundary problems D^*D and DD^* respectively by $\bar{K}_1$ and $\bar{K}_2$. Put $E^+ := \oplus_{\lambda>0} E_\lambda$ and $E^- := \oplus_{\lambda<0} E_\lambda$. The unitary operator $\mathcal{U}$ provides a natural isomorphism between the space of the L^2-sections of E^+ and E^- and satisfies*

$$\begin{aligned} &\bar{K}_{2|E^-} = \mathcal{U} \bar{K}_{1|E^+} \mathcal{U}^{-1} \\ &\bar{K}_{2|E^+} \sim \mathcal{U} \bar{K}_{1|E^-} \mathcal{U}^{-1} + o(1) \ \textit{ exponentially at } t = 0. \end{aligned}$$

Moreover with $\bar{K}(t,u) := \bar{K}_1(t,u) - \bar{K}_2(t,u)$ *we have*

$$\int_0^{\frac{1}{2}} \bar{K}(t,u)\,du \sim -\frac{1}{2}\dim\ker A \;\; \textit{exponentially at } t=0. \tag{2.22}$$

Remark 2. *For later uses we write the following result of the above discussion*

$$\bar{K}(t,u) \sim 0 \;\; \textit{exponentially for } u \neq 0. \tag{2.23}$$

2.3 Index theorem for one Dirac operator

The heat equation proof of the theorem 1 is based on the McKean-Singer formula. Let D denote the boundary problem (D, P^{ϵ}) and D^* denote its adjoint problem $(D, P^{\bar{\epsilon}})$. The second degree operators D^*D and DD^* with induced boundary conditions are elliptic and self-adjoint. So they provide spectral resolutions for $L^2(M,S)$. Moreover the non-zero part of their spectrums are identical. This implies the following McKean-Singer type formula

$$\mathrm{ind}(D, P^{\epsilon}) = \mathrm{Tr}\; e^{-tD^*D} - \mathrm{Tr}\; e^{-tDD^*}\;;\; t > 0.$$

Denote by $\bar{K}_{1\epsilon}$ and $\bar{K}_{2\bar{\epsilon}}$ respectively the fundamental solutions of the cylindrical heat operators e^{-tD^*D} and e^{-tDD^*}. Let K_ϵ and $K_{\bar{\epsilon}}$ be respectively the fundamental solutions of e^{-tD^*D} and e^{-tDD^*}. We are going to give asymptotic expressions, at $t=0$, for these fundamental solutions in terms of $\bar{K}_{1\epsilon}$, $\bar{K}_{2\bar{\epsilon}}$ and of the fundamental solution $\tilde{K}$ of e^{-tDD} on the double of M, i.e. $M \sqcup_{\partial M} M^-$. For this purpose and following [1, Page 54] let $\rho(a,b)$ be a smooth increasing function on $\mathbb{R}^{\geq 0}$ such that

$$\rho(u) = 0 \text{ for } u \leq a\;;\; \rho(u) = 1 \text{ for } u \geq b.$$

The collar neighborhood of the connected components of the boundary ∂M is assumed to be parameterized by $u \in [0, 1^+]$. So the following functions can and will be considered as smooth functions on M with constant extensions into M.

$$f_2 = \rho(\frac{1}{4}, \frac{1}{2}), \qquad g_2 = \rho(\frac{1}{2}, \frac{3}{4})$$
$$f_1 = 1 - \rho(\frac{3}{4}, 1), \qquad g_1 = 1 - g_2$$

Put

$$\mathcal{K}_\epsilon = f_1 \bar{K}_{1\epsilon} g_1 + f_2 \tilde{K} g_2 \;\text{ and }\; \mathcal{K}_{\bar{\epsilon}} = f_1 \bar{K}_{2\bar{\epsilon}} g_1 + f_2 \tilde{K} g_2. \tag{2.24}$$

Since $f_i = 1$ on the support of g_i, one concludes that $\mathcal{K}_\epsilon$ as an operator on $C^\infty(M,S)$ goes toward Id when $t \to 0$. Moreover the remark 1 shows that $(\frac{\partial}{\partial t} + D^*D)\mathcal{K}_\epsilon$ is exponentially small out of the diagonal, when $t \to 0$. These two conditions are sufficient for using $\mathcal{K}_\epsilon$ as the initial step in the construction of the heat kernel using Levi's sum, c.f. [7]. As a consequence, the difference between heat kernel K_ϵ and $\mathcal{K}_\epsilon$ is exponentially small when t goes toward 0. This argument applies as well to $K_{\bar\epsilon}$ and $\mathcal{K}_{\bar\epsilon}$ and gives rise to the following sequence of equalities

$$\begin{aligned}
\operatorname{ind}(D, P^{\epsilon}) &= \int_{diag(M)} \{tr\, K_{\epsilon}(t,x,x) - tr\, K_{\bar\epsilon}(t,x,x)\} \\
&= \lim_{t\to 0} \int_{diag(M)} \{tr\, \mathcal{K}_{\epsilon}(t,x,x) - tr\, \mathcal{K}_{\bar\epsilon}(t,x,x)\} \\
&= \lim_{t\to 0} \sum_i \int_0^{\frac{1}{2}} \bar{K}_{1\epsilon}(t,u) - \bar{K}_{2\bar\epsilon}(t,u) \\
&= \lim_{t\to 0} \sum_i \int_0^{\frac{1}{2}} \bar{K}_{\epsilon}(t,u)\, du.
\end{aligned}$$

To deduce the last equality we have used the fact that the contribution of the trace of the heat operator on double $M \sqcup M^{-}$ is the same in the expressions $tr\,\mathcal{K}_{\epsilon}$ and $tr\,\mathcal{K}_{\bar\epsilon}$, so they cancel out each other. We have used also $f_1(u) = g_1(u) = 1$ for $0 \leq u \leq \frac{1}{2}$. Now the relations (2.13), (2.14) and (2.22) together imply the desired formula

$$\operatorname{ind}(D, P^{\epsilon}) = \frac{1}{2} \sum_{\epsilon=-} \operatorname{ind} A_i - \frac{1}{2} \sum_{\epsilon=+} \operatorname{ind} A_i - \frac{1}{2} \dim\ker(A_i).$$

2.4 Isospectrality problem

In this subsection we denote by Q^{ϵ} the second degree boundary condition for D^*D induced from P^{ϵ}. For $t > 0$ the heat operator e^{-tD^*D} is a smoothing self-adjoint compact operator. So the self-adjoint unbounded operator (D^*D, Q^{ϵ}) has a real discrete spectrum. Let $P^{\epsilon'}$ be another set of boundary conditions inducing the second order condition $Q^{\epsilon'}$. For the simplicity we assume that ϵ_i and ϵ'_i take the value 0 on the same connected components of ∂M.

Theorem 3. *If two boundary problems (D^*D, Q^{ϵ}) and $(D^*D, Q^{\epsilon'})$ are isospectral then*

$$\sum_{\epsilon_i=-,\epsilon'_i=+} \operatorname{ind} A_i = \sum_{\epsilon_i=+,\epsilon'_i=-} \operatorname{ind} A_i \tag{2.25}$$

Moreover this necessary condition is the only one which can be deduced from the heat equation asymptotic formulas.

proof Let

$$\operatorname{Tr}\, e^{-tA_i^- A_i^+} \sim t^{-n} \sum_{k=0}^{\infty} a_{ik}^+ t^k\,,$$

$$\operatorname{Tr}\, e^{-tA_i^+ A_i^-} \sim t^{-n} \sum_{k=0}^{\infty} a_{ik}^- t^k$$

be the asymptotic expansions for the trace of the heat operators on N_i. Using the McKean-Singer formula we have

$$a_{ik}^+ = a_{ik}^- \text{ for } k \neq n \text{ and } a_{in}^+ = a_{in}^- + \operatorname{ind} A_i. \tag{2.26}$$

If the boundary problems (D^*D, Q^ϵ) and $(D^*D, Q^{\epsilon'})$ have the same spectrum then for $t > 0$ the difference $Tr(e^{-tD^*D}, Q^\epsilon) - Tr(e^{-tD^*D}, Q^{\epsilon'})$ vanishes. The relations (2.24) and the discussion following them show that this difference is asymptotic to the following expression at $t = 0$

$$\int_M \mathcal{K}_\epsilon(t,x,x) - \int_M \mathcal{K}_{\epsilon'}(t,x,x) = \int_0^{1/2}\int_{\partial M} \bar{K}_{1\epsilon}(t,y,u) - \bar{K}_{1\epsilon'}(t,y,u)\, dy\, du.$$

On the other hand, using (2.11) and (2.12) the last expression is asymptotic to

$$\left(\frac{1}{4\pi t}\right)^{\frac{n+1}{2}} \sum_{k,i}\Big\{\sum_{\epsilon_i=+} t^k a_{ik}^- + \sum_{\epsilon_i=-} t^k a_{ik}^+ - \sum_{\epsilon_i'=+} t^k a_{ik}^- - \sum_{\epsilon_i'=-} t^k a_{ik}^+\Big\},$$

which simplifies to the following expression by (2.26)

$$\sum_{\epsilon_i=-,\epsilon_i'=+} \operatorname{ind} A_i - \sum_{\epsilon_i=+,\epsilon_i'=-} \operatorname{ind} A_i\ .$$

Therefore the isospectrality of the operators (D^*D, Q^ϵ) and $(D^*D, Q^{\epsilon'})$ implies the vanishing of the above expression which prove the assertion of the theorem. ■

Remark 3. *Let ϵ_i's take only the values $+$ or $-$ and let $\epsilon_i' = \pm$ if $\epsilon_i = \mp$. In this case the cobordism invariance of the index implies the vanishing of the both sides of* (2.25). *So the necessary condition of the above theorem is satisfied for this special switching of boundary conditions. This leads to the following natural question: Does there exist a spin manifold with boundary such that (D^*D, Q^ϵ) and $(D^*D, Q^{\epsilon'})$ have the same spectrum?*

3 Index theorem for families

Theorem 1 can be extended to a family of Dirac type operators. We recall at first the geometric setting for the family index theorem. Let $M \hookrightarrow F \to B$ be a fibration of odd dimensional compact spin manifolds with boundary over a compact smooth manifold B. The boundaries of fibers form another fibration $\partial M \hookrightarrow F' \to B$. This fibration has a fibered collar neighborhood U of the form $[0,1^+) \times F' \subset F$ which restricts in each fiber to a collar neighborhood of the boundary of that fiber. By π we denote the projection on the second factor. Assume that the fibration F is endowed with a fiberwise riemannian metric which is of product form $d^2u + g$ in the collar neighborhood U. Here g is a fiberwise metric on the boundary fibration F'. We assume also that the fiberwise spin structure is of product form in U. By these assumptions a typical fiber M of the fibration F satisfies all conditions described in the previous section. So in the sequel we will use freely the previous notation in this family context. Let D and A denote, respectively, the fiberwise family of Dirac operators associated to fibrations F and F'. Put $F' = \sqcup_i N_i$, where each N_i is a connected fibration over B whose fibers N_{ib} are even dimensional closed spin manifolds for $b \in B$. For each i let ϵ_i be $0, +$ or $-$ arbitrarily and fix it. The family (D, P^ϵ) determines an analytic index $[\operatorname{ind}(D, P^\epsilon)]$ in $K_0(B)$. On the other hand, the boundary family $A := (A_b)_{b\in B}$ determines in its turn a class $[\operatorname{ind} A]$ in $K_0(B)$ (see, e.g. [2]). In this section we prove the following theorem using the heat equation methods applied to superconnections.

Theorem 4. *The following equality holds in $H^*_{dr}(B)$ provided that* $\dim(\ker A_{ib})$ *is constant on each component N_i of the boundary with $\epsilon_i = 0$*

$$\mathrm{Ch}[\mathrm{ind}(D, P^{\epsilon})] = \frac{1}{2}\sum_{\epsilon_i=-} \mathrm{Ch}[\mathrm{ind}\, A_i] - \frac{1}{2}\sum_{\epsilon_i=+} \mathrm{Ch}[\mathrm{ind}\, A_i] - \frac{1}{2}\sum_{\epsilon_i=0} \mathrm{Ch}(\ker A_i).$$

For being able to use the language of the theory of super graded differential modules, we consider the direct sum $S \oplus S$ as a smooth family of super spin bundles where the first and the second summands are respectively the even and the odd parts. The Clifford action of a vertical tangent vector $v \in TM_b$ on $S_b \oplus S_b$ is given by the following grading reversing matrix

$$\begin{pmatrix} 0 & \mathrm{cl}(v) \\ \mathrm{cl}(v) & 0 \end{pmatrix}.$$

With this Clifford action we obtain the following families of grading reversing Dirac operators on F and on F'

$$\mathsf{D} := \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \; ; \; \mathsf{A} := \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$$

Here D denotes the boundary problem (D, P^{ϵ}) while D^* denote $(D, P^{\bar{\epsilon}})$. These operators act on the vertical spinor fields $\phi \oplus \psi$ satisfying boundary conditions $P^{\epsilon}(\phi) = 0$ and $P^{\bar{\epsilon}}(\psi) = 0$. For each $b \in B$, D_b is a vertical self-adjoint differential operator and D_b^2 is the operator with the induced second order boundary conditions. In below we deal with the infinite dimensional bundle $\mathcal{E}$ on B whose fiber $\mathcal{E}_b$ over b consists of smooth spinor fields on M_b. The bundle $\mathcal{E} \oplus \mathcal{E}$ is $\mathbb{Z}_2$-grading in the obvious way. To prove theorem 4 we need a connection ∇ on the smooth sections of $\Lambda^* B \otimes \mathcal{E}$. In the following two subsections we will study the case of a fibration with half cylindrical fibers endowed with more special connections. Using a partition of unity on half line $\mathbb{R}^{\geq 0}$ we can and will assume that the connection ∇ coincides with theses special connections in the collar neighborhood $[0, 1^+) \times F'$. We need to extend the actions of D and A on the smooth section $\omega \otimes \xi$ of $\Lambda^* B \otimes \mathcal{E}$. These extensions are given as follows

$$D(\omega \otimes \xi) = (-1)^{\deg \omega} \omega \otimes D\xi \; ; \; A(\omega \otimes \xi) = (-1)^{\deg \omega} \omega \otimes A\xi.$$

Now $\nabla \oplus \nabla$, D and A are graded differential operators acting on smooth sections of graded bundle

$$\Lambda^* B \otimes \mathcal{E} \bigoplus \Lambda^* B \otimes \mathcal{E} \,. \tag{3.1}$$

In bellow we study the superconnection $\mathbb{B} = \mathsf{D} + \nabla \oplus \nabla$ adapted to the family D of Dirac operators and the connection $\mathbb{A} = A + \nabla$ which is adopted to the family A. We denote by $\mathbb{B}_t$ and $\mathbb{A}_t$ their rescaled versions. The rescaled curvature $\mathbb{F}_t := \mathbb{B}_t^2$ has the form $t\mathsf{D}^2 + \mathbb{F}_{t[+]}$ where $\mathbb{F}_{t[+]}$ is the following differential operator with differential form coefficients of positive degree

$$\mathbb{F}_{t[+]} =: \begin{pmatrix} \nabla^2 & t^{1/2}(D\nabla + \nabla D) \\ t^{1/2}(D\nabla + \nabla D) & \nabla^2 \end{pmatrix}. \tag{3.2}$$

We will be dealing with the structure of the heat operator of the rescaled curvature so we explain briefly how to construct it. Let $e^{-t\mathsf{D}^2} = e^{-tD^*D} \oplus e^{-tDD^*}$ be the heat operator of $(\mathsf{D}, P^\epsilon \oplus P^{\bar{\epsilon}})$ and let R be a family of smoothing operator. The heat kernel of the perturbed family $\mathsf{D}_s := \mathsf{D} + sR$, for $0 \leq s \leq 1$, is given by the Voltera formula:

$$e^{-t\mathsf{D}_s} = e^{-t\mathsf{D}^2} + \sum_{k=1}^{\infty} (-t)^k I_k(t\mathsf{D}_s, sR),$$
$$I_k(t\mathsf{D}, sR) := \int_{\triangle_k} e^{-s_0 t\mathsf{D}^2}\, sR\, e^{-s_1 t\mathsf{D}}\, sR \ldots e^{-s_{k-1}t\mathsf{D}^2}\, sR\, e^{-s_k t\mathsf{D}^2},$$
$$\triangle_k = \{(s_0, s_1, \ldots, s_k) \in \mathbb{R}^{k+1} | s_i \geq 0\, ;\, \sum_i s_i = 1\}\,.$$

Because R is smoothing, the operator $e^{-t\mathsf{D}}R$ has a smooth kernel for $t \geq 0$ and $\|e^{-t\mathsf{D}}R\|_\ell \leq C(\ell)\|R\|_\ell$ for a constant $C(\ell)$. So

$$\|I_k(t\mathsf{D}, R)\|_\ell \leq \frac{C(\ell)^{k+1}\|R\|_\ell^k}{k!},$$

which implies the convergence of the above sum in C^ℓ-norm. By the Voltera formula the following relation holds at $t = 0$

$$e^{-t\mathsf{D}_s} - e^{-t\mathsf{D}} = o(1)\,. \tag{3.3}$$

Consequently, the smoothing perturbation sR has no effect on the asymptotic behavior of the heat operator at $t = 0$. Let $\mathbb{B}_s := \mathbb{B} + sR$ be the superconnection adopted to perturbed family D_s. Its rescaled curvature has the form $\mathbb{F}_{s,t} = t\mathsf{D}_s + \mathbb{F}_{s,t[+]}$, where

$$\mathbb{F}_{s,t[+]} = \mathbb{F}_{t[+]} + O(t). \tag{3.4}$$

The heat operator of the supercurvature $\mathbb{F}_{s,t}$ is given again by the Voltera formula

$$e^{-\mathbb{F}_{s,t}} = e^{-t\mathsf{D}_s^2} + \sum_{k=1}^{\dim B} (-1)^k I_k(t\mathsf{D}s, \mathbb{F}_{s,t[+]}), \tag{3.5}$$

It should be clear from this construction that $e^{-\mathbb{F}_{s,t}}$ is a vertical family of smoothing operator with coefficients in $\Lambda^*(B)$. So its vertical supertrace is finite and defines an element in $\Omega^*(B)$. Although $I_k(t\mathsf{D}_s, \mathbb{F}_{s,t[+]})$ depends on the involved operator in a rather complicated way, its asymptotic behavior at $t = 0$ is simple to describe. At first it follows from (3.3) and (3.4) that $I_k(t\mathsf{D}_s, \mathbb{F}_{s,t[+]}) = I_k(t\mathsf{D}, \mathbb{F}_{t[+]}) + o(1)$. Moreover, in the expression $I_k(t\mathsf{D}, \mathbb{F}_{t[+]})$ the contribution of the off-diagonal operators in (3.2) can be neglected, , when t goes toward 0, because these operators have a multiplicative factor $t^{1/2}$. So $I_k(\mathsf{D}, \mathbb{F}_{t[+]}) = I_k(\mathsf{D}, \nabla^2 \oplus \nabla^2) + o(1)$ and we get the following relation in $\Omega^*(B)$ with C^ℓ-norms

$$e^{-\mathbb{F}_{s,t}} = e^{-t\mathsf{D}^2 + \nabla^2 \oplus \nabla^2} + o(1).$$

The Chern character of the rescaled perturbed superconnection is given by the relation $\mathrm{Ch}(\mathbb{B}_{s,t}) := \mathrm{STr}\, e^{-\mathbb{F}_{s,t}}$. To investigate the relationship between $\mathrm{Ch}(\mathbb{B}_{s,t})$

and $\mathrm{Ch}(\mathbb{B}_t)$ we recall the following fact (see e.g. [3, theorem 9.17]). Let B_τ be a smooth family of superconnections on a differential superbundle. The following formula holds

$$\frac{d}{d\tau}\mathrm{Ch}(\mathsf{B}_\tau) = -d\ \mathrm{STr}\,(\frac{d\,\mathsf{B}_\tau}{d\tau}e^{-\mathsf{B}_\tau^2}) \in \Omega^*(B). \tag{3.6}$$

If we apply this formula to the s-dependent family of superconnections $\mathbb{B}_{s,t}$, we get

$$\mathrm{Ch}(\mathbb{B}_{1,t}) - \mathrm{Ch}(\mathbb{B}_t) = -d\int_0^1 \mathrm{STr}(\frac{d\,\mathbb{B}_{s,t}}{ds}\,e^{-\mathbb{F}_{s,t}})\,ds$$

Therefore the perturbation R does not change the class of the Chern character in the de-Rham cohomology of B. A such perturbation is used to define the analytical index of a family of elliptic operator. In fact, there is a general method (see [2, Lemma 2.1]) to construct a self-adjoint perturbation R such that $\dim(\ker(D+R))$ be independent of $b \in B$. In this case $\ker(\mathsf{D}+R)$ is a smooth finite dimensional super vector bundle over B and determines a class in $K^0(B)$. This class, being independent of the perturbation, is denoted by $[\mathrm{ind}\ D]$ and is called the analytical index of the family D. We summarize the above discussions in the following proposition

Proposition 5. *We can and will assume that* $\dim(\ker \mathsf{D}_b)$ *is independent of* $b \in B$ *by perturbing by a family of self-adjoint smoothing operators. This perturbation does not affect neither the class of the Chern form* $\mathrm{Ch}(\mathbb{B}_t)$ *in* $H^*_{dr}(B)$ *nor its behavior when* t *goes toward* 0*. Moreover one has*

$$e^{-\mathbb{F}_t} = e^{-t\mathsf{D}^2+\nabla^2\oplus\nabla^2} + o(1). \tag{3.7}$$

With this assumption we have $[\mathrm{ind}\ D] = [\ker\ \mathsf{D}] \in K^0(B)$.

Our proof for the theorem 4 is based on a precise study of the behavior of $\mathrm{Ch}(\mathbb{B}_t)$ at $t=0$ and $t=\infty$ and comparing them. Following the above proposition, $\ker\mathsf{D}$ is a vector bundle on B and the formal difference $\ker(D, P^{\epsilon}) - \ker(D^*, P^{\bar\epsilon})$ of its even and odd parts represents the index class $[\mathrm{ind}\ D] \in K^0(B)$. Let Q_0 be the projection on $\ker\mathsf{D}$ which is a continuous family of vertical smoothing operators. It is clear that $\nabla_0 = Q_0\nabla Q_0$ is a connection on the vector bundle $\ker\mathsf{D}$. Therefore the differential form $\mathrm{Str}\,\mathrm{e}^{-\nabla_0^2}$ is closed and provides a representation for $\mathrm{Ch}(\mathrm{ind}\ D) \in H^*_{Dr}(B)$.

Proposition 6. 1. *For* $\ell \in \mathbb{N}$ *the following convergence occurs in* $\Omega^*(M)$ *with respect to the uniform* C^ℓ*-norm*

$$\lim_{t\to+\infty}\mathrm{Ch}(\mathbb{B}_t) = \mathrm{Ch}(\ker\,\mathsf{D},\nabla_0).$$

2. *For* $t>0$*, the Chern form* $\mathrm{Ch}(\mathbb{B}_t)$ *is closed and its class in the de-Rham cohomology* $H^*_{dr}(B)$ *is independent of* t*. In particular*

$$\mathrm{Ch}(\mathbb{B}_t) = \mathrm{Ch}(\nabla_0) \in H^*_{dr}(B)\ \textit{for}\ \ t>0.$$

proof Consider the following orthogonal decomposition of bundles

$$\mathcal{E} \oplus \mathcal{E} = \ker \mathsf{D} \oplus \operatorname{Im} \mathsf{D}. \tag{3.8}$$

The heat operator $e^{-t\mathsf{D}}$ is a family of smoothing non-negative operators parameterized by the compact set B. So there is a uniform gap around 0 in the spectrum of each element of this family. This simple observation and general properties of graded nilpotent algebras can be used to get the following relation in $\mathcal{C}^\ell$ with respect to the above direct sum decomposition. (see [3, page 290])

$$e^{-\mathbb{B}_t} \sim \begin{pmatrix} e^{-\nabla_0^2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} O(t^{-1/2}) & O(t^{-1/2}) \\ O(t^{-1/2}) & O(t^{-1}) \end{pmatrix}. \tag{3.9}$$

This relation proves the first part of the proposition. To prove the second part we apply (3.6) to the family $\mathbb{B}_t$ to get the following relation for $t_2 > t_1 > 0$

$$\mathrm{Ch}(\mathbb{B}_{t_2}) - \mathrm{Ch}(\mathbb{B}_{t_1}) = -d \int_{t_1}^{t_2} \mathrm{STr}\,(\frac{d\,\mathbb{B}_t}{dt} e^{-\mathbb{F}_t})\, dt.$$

If we regard this relation in the de-Rham cohomology group $H^*_{dr}(B)$ we obtain the first assertion of the second part of the proposition. Now let t_1 goes toward ∞ in the above relation. Using the first part of the proposition we get

$$\mathrm{Ch}(\mathbb{B}_{t_2}) - \mathrm{Ch}(\nabla_0) = d \int_{t_2}^{\infty} \mathrm{STr}\,(\frac{d\,\mathbb{B}_t}{dt} e^{-\mathbb{F}_t})\, dt \in \Omega^*(B), \tag{3.10}$$

provided that the right hand side is finite. To prove the convergence of this integral, we notice that

$$\frac{d\,\mathbb{B}_t}{dt} = \frac{1}{2t^{1/2}} \begin{pmatrix} 0 & 0 \\ \mathsf{D} & 0 \end{pmatrix},$$

with respect to the decomposition (3.8). This relation with (3.9) give rise to

$$\mathrm{STr}\,(\frac{d\,\mathbb{B}_t}{dt} e^{-\mathbb{F}_t}) = O(t^{-3/2})\,,$$

which implies the convergent at $t = \infty$. ■

Now we study the behavior of $\mathrm{Ch}(\mathbb{B}_t)$ when $t \to 0$. For this purpose, as in the previous section, we give an approximation of the heat operator $e^{-\mathbb{F}_t}$ in terms of the heat operator on double of F and the heat operator on half cylinder fibration $\mathbb{R}^{\geq 0} \times F'$ with typical fiber $\mathbb{R}^{\geq 0} \times \partial M$. All local structures in this cylindrical case are exactly the same of the collar neighborhood U, e.g. the Dirac operator takes the form (2.5).

3.1 Index density of a family of half cylinders with local conditions

Let $\mathcal{E}_0 = \mathcal{E}_0^+ \oplus \mathcal{E}_0^-$ be the bundle over B whit typical fiber $C^\infty(\partial M_b, S^+ \oplus S^-)$. Let $\nabla_0 = \nabla_0^+ \oplus \nabla_0^-$ be a connection on $\mathcal{E}_0$, then $\bar{\nabla} = \pi^*\nabla_0$ is a connection on $\mathcal{E} := \pi^*\mathcal{E}_0$. We denote by $\bar{\mathbb{B}}$ the superconnection $\mathsf{D} + \bar{\nabla} \oplus \bar{\nabla}$ acting on $\mathcal{E} \oplus \mathcal{E}$. We consider $\mathbb{R}^{\geq 0} \times N$ with local boundary condition P^+, the case of the boundary condition P^- is completely similar. Here N may denote any one of N_i's and for the simplicity we drop the index i. It follows from (3.7) that, as far as we are interested in the asymptotic behavior of the heat operators at $t = 0$, we can replace $\bar{\mathbb{F}}_t := \bar{\mathbb{B}}_t^2$ by the following operator

$$\begin{pmatrix} -t\partial_u^2 & 0 \\ 0 & -t\partial_u^2 \end{pmatrix} + \pi^* \begin{pmatrix} tA^2 + \nabla_0^2 & 0 \\ 0 & tA^2 + \nabla_0^2 \end{pmatrix} \tag{3.11}$$

The relation (3.7) can also be applied to the superconnection $\mathbb{A} = A + \nabla_0$, where $\nabla_0 = \pi^*\nabla_0^+ \oplus \pi^*\nabla_0^-$. Therefore $e^{-\mathbb{A}_t^2} = e^{-(tA^2+\nabla_0^2)} + o(1)$ which implies

$$e^{-\bar{\mathbb{F}}_t} = e^{-\bar{\mathsf{F}}_t} + o(1), \tag{3.12}$$

where

$$\bar{\mathsf{F}}_t := \begin{pmatrix} -t\partial_u^2 & 0 \\ 0 & -t\partial_u^2 \end{pmatrix} + \pi^* \begin{pmatrix} \mathbb{A}_t^2 & 0 \\ 0 & \mathbb{A}_t^2 \end{pmatrix}.$$

Notice that the operator ∂_u commutes with all other operators involved in the above expression, so the results of the previous section can be used to give explicite expressions for $e^{-\bar{\mathsf{F}}_t}$. For example its even part acting on $\phi^+(u,y) \oplus \phi^-(u,y)$ subjected to the boundary conditions (2.6) and (2.7) is

$$\frac{1}{\sqrt{4\pi t}} \begin{pmatrix} \exp(\frac{-(u-v)^2}{4t}) - \exp(\frac{-(u+v)^2}{4t}) & 0 \\ 0 & \exp(\frac{-(u-v)^2}{4t}) + \exp(\frac{-(u+v)^2}{4t}) \end{pmatrix} \otimes e^{-\mathbb{A}_t^2}$$

A similar formula, by exchanging the diagonal coefficients, gives the odd part of the heat kernel of $\bar{\mathsf{F}}_t$ with the boundary conditions (2.8). These expressions and the relation (3.12) provide together an explicite asymptotic formula for the heat kernel $\bar{K}_+(t,u,v,y,z)$ of $\bar{\mathbb{F}}_t$. Therefore we get the following formula, with respect to the C^ℓ-norm, for the supertrace density of $e^{-\bar{\mathbb{F}}_t}$ as a function of t and u

$$\begin{aligned} \bar{K}_+(t,u) &:= \int_N \operatorname{str} \bar{\mathrm{K}}_+(\mathrm{t},\mathrm{u},\mathrm{u},\mathrm{y},\mathrm{y})\,\mathrm{dy} \\ &= -\frac{\mathrm{STr}\; e^{-\mathbb{A}_t^2}}{\sqrt{\pi t}} e^{-\frac{u^2}{t}} + o(1) \in \Omega^*(B). \end{aligned}$$

The case $\epsilon = -$ produces the same expression with the opposite sign. So by integrating on $[0, 1/2]$, with respect to u, we get the following asymptotic formula at $t = 0$

$$\bar{K}_{\epsilon_i}(t) := \int_0^{1/2} \bar{K}_{\epsilon_i}(t,u)\,du \sim -\epsilon_i \frac{1}{2} \mathrm{STr}\; e^{-\mathbb{A}_t^2} + o(1) \in \Omega^*(B).$$

In other words, the following relation holds in $\Omega^*(B)$ with C^ℓ-topology

$$K_{\epsilon_i}(t) = -\epsilon_i \frac{1}{2} \mathrm{Ch}(\mathbb{A}_{it}) + o(1). \tag{3.13}$$

3.2 Index density of a family of half cylinders with the APS conditions

Consider now the family of half cylinders $\mathbb{R}^{\geq 0} \times N \to B$ with the APS boundary condition. We assume that $\dim \ker(A_b)$ does not depend on $b \in B$, where A_b indicates the Dirac operator on N_b for $b \in B$. So the B-parameterized vector spaces $\ker A_b$ form the vector bundle $\ker A$ on B. We denote by $\mathcal{E}_0$ the lifting of this bundle by π which is again a vector bundle over B. In the same way and with the notation of proposition 2, infinite dimensional vector spaces $E^{\pm}(b)$ together form an infinite dimensional bundles over B. These bundles can be lifted by π to bundles $\mathcal{E}^{\pm}$ over B. These bundles are isomorphism via the unitary operator $\mathcal{U}$. Let $\bar{\nabla}^{\pm}$ be two connections on the smooth sections of $\mathcal{E}^{\pm}$ such that $\mathcal{U}\bar{\nabla}^{\pm}\mathcal{U}^{-1} = \bar{\nabla}^{\mp}$. These connections are assumed to be constant along $\mathbb{R}^{\geq 0}$, i.e. they are lifting of two connections by π. We assume also a connection $\bar{\nabla}_0$ on $\mathcal{E}_0 = \pi^* \ker A$. Put $\bar{\nabla} = \bar{\nabla}^- \oplus \bar{\nabla}_0 \oplus \bar{\nabla}^+$ and consider the superconnection $\tilde{\mathbb{B}} = \mathsf{D} + \bar{\nabla} \oplus \bar{\nabla}$. By (3.7), the heat operator of the associated rescaled supercurvature $\tilde{\mathbb{F}}_t$, up to a term of order $o(1)$, is equal to the heat operator of

$$\mathsf{F}_t := \begin{pmatrix} -tD^*D & 0 \\ 0 & -tDD^* \end{pmatrix} + \begin{pmatrix} \bar{\nabla}^2 & 0 \\ 0 & \bar{\nabla}^2 \end{pmatrix} \tag{3.14}$$

We show that when t goes toward 0 the supertrace of the heat operator $e^{-\mathsf{F}_t}$ restricted to $\mathcal{E}^+ \oplus \mathcal{E}^-$ goes toward zero in $(\Omega^*(B), \| \, \|_\ell)$. For this purpose, using the Voltera formula, we have

$$(e^{-tD^*D+\bar{\nabla}^2})_{|\mathcal{E}^+} = \sum_k (-1)^k I_k(tD^*D_{|\mathcal{E}^+}, \bar{\nabla}_+^2)\,,$$

where

$$I_k(tD^*D_{|\mathcal{E}^+}, \bar{\nabla}_+^2) := \int_{\triangle_k} e^{-s_0 tD^*D}\bar{\nabla}_+^2 e^{-s_1 tD^*D}\bar{\nabla}_+^2 \ldots e^{-s_{k-1}tD^*D}\bar{\nabla}_+^2 e^{-s_k tD^*D}.$$

On the other hand, from the first part of the proposition 2 we get

$$(e^{-tDD^*})_{|\mathcal{E}^-} = \mathcal{U}(e^{-tD^*D})_{\mathcal{E}^+}\mathcal{U}^{-1}.$$

Using this relation and the equality $\bar{\nabla}_- = \mathcal{U}\bar{\nabla}_+\mathcal{U}^{-1}$ we obtain

$$I_k(tD^*D_{|\mathcal{E}^-}, \bar{\nabla}_-^2) = \mathcal{U} I_k(tD^*D_{|\mathcal{E}^+}, \bar{\nabla}_+^2)\mathcal{U}^{-1},$$

therefore

$$(e^{-tDD^*+\bar{\nabla}^2})_{|\mathcal{E}^-} = \mathcal{U}(e^{-tD^*D+\bar{\nabla}^2})_{|\mathcal{E}^+}\mathcal{U}^{-1}. \tag{3.15}$$

A similar discussion, using again the proposition 2, implies the following relation in $\Omega^*(B)$ with C^ℓ-topology

$$(e^{-tDD^*+\bar{\nabla}^2})_{|\mathcal{E}^+} = \mathcal{U}(e^{-tD^*D+\bar{\nabla}^2})_{|\mathcal{E}^-}\mathcal{U}^{-1} + o(1) \quad \text{at } t = 0. \tag{3.16}$$

The relations (3.15) and (3.16) imply the desired assertion:

$$\mathrm{STr}(e^{-\tilde{\mathbb{F}}_t})_{|\mathcal{E}^+\oplus\mathcal{E}^-} = o(1)\ ; \text{ at } t = 0.$$

Therefore

$$\mathrm{STr}\ e^{-\bar{\mathbb{F}}_t} = \mathrm{Tr}\,(e^{-t\partial_u^2+\bar{\nabla}_0^2})_{|\mathcal{E}_0} - \mathrm{Tr}\,(e^{-t\partial_u^2+\bar{\nabla}_0^2})_{|\mathcal{E}_0} + o(1).$$

Since ∂_u commutes with $\bar{\nabla}_0^2$, the relation (2.21) can be used to get

$$\mathrm{STr}_N\ e^{-\mathsf{F}_t}(t,u) = -\frac{e^{\frac{-u^2}{t}}}{\sqrt{\pi t}}\mathrm{str}\,(\mathrm{e}^{-\nabla_0^2})_{|\,\ker \mathrm{A}}.$$

Summarizing these discussions, we get the following relation in $\Omega^*(B)$ with C^ℓ-topology

$$\int_0^{\frac{1}{2}} \mathrm{STr}_N\ e^{-\tilde{\mathbb{F}}_t}(t,u)\,du = -\frac{1}{2}\,\mathrm{Ch}(\ker A, \nabla_0) + o(1)\ ;\ \text{at } t = 0. \tag{3.17}$$

3.3 Index theorem for a family of Dirac operators

We recall that all the differential operators and the geometric structures that we have on F are of the product form in the collar neighborhood U. So they can be smoothly extended to the double fibration $F \sqcup_{F'} F$. In the other direction, let $\tilde{\nabla}$ be a connection on $\mathcal{E} \to F \sqcup_{F'} F$ and let $\bar{\nabla}_i$'s denote the connections on the half-cylinder fibrations discussed in the previous subsections. Then

$$\nabla := \sum_i f_1 \bar{\nabla}_i g_1 + f_2 \tilde{\nabla} g_2, \tag{3.18}$$

defines a connection on $\mathcal{E} \to B$. Here functions f_1, f_2, g_1 and g_2 are defined in subsection 2.3. Let $\tilde{\mathbb{F}}_t$ be the rescaled supercurvature on the double fibration $F \sqcup_{F'} F$ and let $\tilde{K}_t$ be the kernel of the associated heat operator. We denote by $\bar{K}_{t\epsilon_i}$ and $K_{t\epsilon_i}$ respectively the fundamental solutions of $e^{-\bar{\mathbb{F}}_t}$ and $e^{-\mathbb{F}_t}$ with respect to the boundary condition P^{ϵ_i}. At first we prove the following asymptotic formula at $t = 0$

$$K_{t\epsilon_i} = \sum_i f_1 \bar{K}_{t\epsilon_i} g_1 + f_2 \tilde{K}_t g_2 + o(1). \tag{3.19}$$

For this purpose we use the relation (3.7) and the Voltera formula to deduce

$$e^{-\mathbb{F}_t} = e^{-t\mathsf{D}^2} + \sum_{k=1}^{\dim B} (-1)^k I_k(t\mathsf{D}, \nabla \oplus \nabla) + o(1),$$

$$I_k(t\mathsf{D}, \nabla \oplus \nabla) := \int_{\triangle_k} e^{-s_0 t\mathsf{D}^2}(\nabla \oplus \nabla)e^{-s_1 t\mathsf{D}^2}(\nabla \oplus \nabla)\ldots e^{-s_{k-1}t\mathsf{D}^2}(\nabla \oplus \nabla)e^{-s_k t\mathsf{D}^2}.$$

In this formula the boundary conditions are implicit in the family of heat operators $e^{-t\mathsf{D}^2}$. The following relation is the family version of (2.24)

$$e^{-t\mathsf{D}^2} = f_1 e^{-t\bar{\mathsf{D}}^2} g_1 + f_2 e^{-t\tilde{\mathsf{D}}^2} g_2 + o(1).$$

On the other hand, since $\nabla = \sum_i f_1 \bar{\nabla}^i g_1 + f_2 \tilde{\nabla} g_2$, and $f_1 g_1 = g_1$ and $f_2 g_2 = g_2$ we get

$$\begin{aligned} I_k(t\mathsf{D}, \nabla \oplus \nabla) = &\sum_{\epsilon_i = \pm} f_1 I_k(\bar{\mathsf{D}}_{\epsilon_i}, f_1(\bar{\nabla}^i \oplus \bar{\nabla}^i)g_1)g_1 + f_2 I_k(\tilde{\mathsf{D}}, f_2(\tilde{\nabla} \oplus \tilde{\nabla})g_2)g_2 \\ &+ \text{a finite sum of operators of the form } M(t)h\,e^{-st\bar{\mathsf{D}}^2}\,kN(t) + o(1) \end{aligned}$$

In the last line of the above expression $M(t)$ and $N(t)$ are vertical smoothing operators with differential form coefficients whose kernels goes exponentially toward zero outside of the diagonal when t goes to 0 while h and k are smooth functions on F such that $supp(hk) \subseteq [\frac{1}{4}, 1] \times F' \subset U$. So, using (2.15) and (2.23), we deduce that $M(t)h\, e^{-st\bar{\mathsf{D}}^2}\, kN(t) = o(1)$ when $t \to 0$. Using again the relations (2.15) and (2.23) we have

$$g_1 e^{-t\bar{\mathsf{D}}^2} g_1 = e^{-t\bar{\mathsf{D}}^2} g_1 + o(1) \text{ and } f_1 e^{-t\bar{\mathsf{D}}^2} f_1 = f_1 e^{-t\bar{\mathsf{D}}^2} + o(1).$$

Therefore

$$f_1 I_k(\bar{\mathsf{D}}_{\epsilon_i}, f_1(\bar{\nabla}^i \oplus \bar{\nabla}^i) g_1) g_1 = f_1 I_k(\bar{\mathsf{D}}_{\epsilon_i}, \bar{\nabla}^i \oplus \bar{\nabla}^i) g_1 + o(1)$$

which completes the proof of the relation (3.19) by considering the relation (3.7). Now take the supertrace of the both sides of the relation (3.19). Clearly the contribution of the interior term $f_2 \tilde{K}_t g_2$ is zero since it appears twice with opposite signs. The contribution of the boundary terms $\bar{K}_{t\epsilon_i} g_1$ on $[1/2, 1]$ vanish too when t goes toward 0 (see relation (2.15) and (2.23)). Since $f_1 = g_1 = 1$ on $[0, 1/2]$, using relations (3.13) and (3.17), we obtain the following asymptotic formula at $t = 0$

$$\mathrm{STr}\,(e^{-\mathbb{F}^\epsilon_t}) = -\frac{1}{2} \sum_{\epsilon_i = \pm} \epsilon_i \, \mathrm{Ch}(\mathbb{A}_{it}) - \frac{1}{2} \sum_{\epsilon_i = 0} \mathrm{Ch}(\ker A_i, \nabla_0) + o(1) \in \Omega^*(B) \quad (3.20)$$

The involved differential forms in the above expressions are closed, so we can regard this relation in $H^*_{dr}(B)$. According to the proposition 6, and its analogous for the superconnection $\mathbb{A}$, the class of the Chern forms do not depend on the parameter t. Therefore

$$\mathrm{Ch}[\mathrm{ind}(D, P^\epsilon)] = \\ -\frac{1}{2} \sum_{\epsilon_i = +} \mathrm{Ch}[\mathrm{ind}\, A_i] + \frac{1}{2} \sum_{\epsilon = -} \mathrm{Ch}[\mathrm{ind}\, A_i] - \frac{1}{2} \sum_{\epsilon_i = 0} \mathrm{Ch}(\ker A_i) \in H^*_{dr}(B).$$

which completes the proof of the theorem 4.

Remark 4. *The above proof is based on the assumption that* $\dim \ker A_b$ *is independent of* $b \in B$ *for the APS boundary condition. This assumption is satisfied for some interesting cases, e.g. for the family of signature operators or for a family of Dirac operator twisted by flat vector bundles provided that the fibers carry metrics with positive scalar curvature. In general this assumption may be relaxed by considering smooth perturbations of the boundary family or by putting more general spectral boundary condition using, e.g. the spectral projections introduced in [8]. In the later case the standard tool for analyzing the associated heat kernel will be the family version of the Melrose b-calculus.*

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