Bound on Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$, Part II

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Abstract

In the note we give an uniform bound for the multiple point Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$, improving the bound from [3].

1 Introduction

In this paper we improve the bounds on multiple points Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$, obtained in [3].
For X, a smooth projective variety (over $\mathbb{C}$) with an ample line bundle L and for $P_1, ..., P_r$, different points on X, we define the Seshadri constant of L in $P_1, ..., P_r$ as follows (cf [2]).

Definition 1. *The Seshadri constant of L in $P_1, ..., P_r$ is defined as the number*

$$\varepsilon(L, P_1, ..., P_r) := inf\,\{\frac{LC}{mult_{P_1}C + ... + mult_{P_r}C} \mid C \text{ is a curve on } X\},$$

or, equivalently

$$\varepsilon(L, P_1, ..., P_r) := \ sup\ \{\varepsilon \mid \pi^*L - \varepsilon(E_1 + ... + E_r) \text{ is numerically effective}\},$$

where $\pi : \tilde{X} \longrightarrow X$ is the blow-up of X in $P_1, ..., P_r$.

As we are interested only in case of surfaces, from now on we assume that $\dim X = 2$.

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Remark 2. *1. It follows from the definition that for an ample line bundle L on X,*

$$0 < \varepsilon(L, P_1, ..., P_r) \leq \sqrt{\frac{L^2}{r}}.$$

2. As $\varepsilon(L, P_1, ..., P_r)$ is lower semi-continuous, for $P_1, ..., P_r$ generic on X we write $\varepsilon(L, r)$ instead of $\varepsilon(L, P_1, ..., P_r)$.

It is still an open problem whether $\varepsilon(L, P_1, ..., P_r)$ may attain the maximal possible value $\sqrt{\frac{L^2}{r}}$, in case this value is irrational.

Suppose, we can find a curve C on X, such that $\frac{LC}{\mathrm{mult}_{P_1} C + ... + \mathrm{mult}_{P_r} C} < \sqrt{\frac{L^2}{r}}$. Such a curve is called *Seshadri submaximal curve.* Then, the Seshadri constant $\varepsilon(L, P_1, ..., P_r)$ is rational, what follows from the fact that there is a finite number of Seshadri submaximal curves on a surface, see for example [8]. The existence of such a curve may follow from the Riemann-Roch theorem, and then the Seshadri constant is rational.

Definition 3. *A curve C in a linear system $|L|$ on a surface, passing through r points with multiplicities $m_1, ..., m_r$ is Riemann-Roch expected if*

$$h^0(L) - \sum_{i=1}^{r} \binom{m_i + 1}{2} \geq 1.$$

In [9] Syzdek studied Riemann-Roch expected Seshadri submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with different polarizations L. She gave a list of the Riemann-Roch expected submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$. She also proved that there exists a number R_0 (depending on the type of the polarization), such that for $r \geq R_0$, there are no Riemann-Roch expected submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$.

In [3], as well as in this note, we considered the situation, when L and r are such, that there are no Riemann-Roch expected submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$, so $\varepsilon(L, r)$ "should" attain the maximal value $\sqrt{\frac{L^2}{r}}$. In [3] we then gave a uniform lower bound for the Seshadri constant on $\mathbb{P}^1 \times \mathbb{P}^1$, proving that $\varepsilon(L, r) \geq \sqrt{\frac{L^2}{r}}\sqrt{1 - \frac{1}{2r+1}}$. In this note we improved this bound, namely we proved the following theorem.

Theorem 4. *Let L be a line bundle in $\mathbb{P}^1 \times \mathbb{P}^1$, of type (a, b). Let r be such, that there exist no Riemann-Roch expected submaximal curves on $(\mathbb{P}^1 \times \mathbb{P}^1, L)$. Then*

$$\varepsilon(L, r) \geq \sqrt{\frac{2ab}{r}}\sqrt{1 - \frac{1}{4.5r}}.$$

In some cases we were able to improve the bound further.

Theorem 5. *Under the assumptions of the previous theorem, if*
i) r is odd
or
ii) r is even and L is of type (a, a),
then

$$\varepsilon(L, r) \geq \sqrt{\frac{L^2}{r}}\sqrt{1 - \frac{1}{5r}}.$$

The proof of this theorem uses the result of Harbourne and Roé from [5], and we are grateful to them for their suggestions on the subject.

2 Important facts

The main result used in the proof of theorem 4 will be the following theorem of Harbourne and Roé (cf [5], Theorem I.2.1, here we quote and use only the second part of the theorem).

Theorem 6. *Let X be a smooth projective surface with an ample line bundle L. Let $\alpha_0(X, L, m_1, ..., m_r)$ denote the least degree (with respect to L) of an irreducible curve passing through r general points with multiplicities $m_1, ..., m_r$ in these points. If all the multiplicities are equal, we write $\alpha_0(X, L, m^{\times r})$. Let $\mu \geq 1$ be a real number. Then, if*

$$\alpha_0(X, L, m^{\times r}) \geq m\sqrt{L^2\left(r - \frac{1}{\mu}\right)} \tag{1}$$

for every integer $1 \leq m < \mu$ and if

$$\alpha_0(X, L, m^{\times r-1}, m+k) \geq \frac{mr+k}{r}\sqrt{L^2\left(r - \frac{1}{\mu}\right)} \tag{2}$$

for every integer $1 \leq m < \frac{\mu}{r-1}$ and every integer k with

$$k^2 < \frac{r}{r-1}\min\{m, m+k\},$$

then

$$\varepsilon(L, r) \geq \sqrt{\frac{L^2}{r}}\sqrt{1 - \frac{1}{r\mu}}.$$

Observe, that as in our case $r \geq 9$ (for $r < 9$ there are always Riemann-Roch expected submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$, see [9]) and $\mu \leq 5$, the condition (2) is empty. Thus, to get the bound from theorem 6, we need only to check (1). For this, we use the lemma proved by Xu.

Lemma 7. (See [11], Lemma 1). *Let C be a reduced and irreducible curve on a surface X, passing through a general point $P \in X$ with multiplicity $m \geq 2$. Then*

$$C^2 \geq m^2 - m + 1.$$

3 Proofs

3.1 Proof of theorem 4

As mentioned above, to prove theorem 4 we need to check that the condition (1) is satisfied for all positive $m < \mu = 4.5$. Thus, we have to check (1) for $m = 1, 2, 3, 4$ and then our result will follow from theorem 6.

Let us take an irreducible curve C of type (α,β) on $\mathbb{P}^1\times\mathbb{P}^1$. The degree of C with respect to L (L is of type (a,b)) is $LC=a\beta+b\alpha$. Assume that C passes through r general points with multiplicity m. The condition (1) says then that:

$$a\beta+b\alpha\geq m\sqrt{2ab\left(r-\frac{1}{4.5}\right)}. \tag{3}$$

As $2\sqrt{\alpha\beta ab}\leq a\beta+b\alpha$, the condition (3) will be implied by

$$2\alpha\beta\geq m^2\left(r-\frac{1}{4.5}\right). \tag{4}$$

Thus, we need to check that (3) or (4) is satisfied for $m=1,2,3,4$. Let us consider cases.
Case I. $m=1$. We have to prove that $a\beta+b\alpha\geq\sqrt{2ab(r-\frac{1}{4.5})}$, for any irreducible curve C passing through r general points with multiplicity one. As the points of multiplicity one give independent conditions on the dimension of linear system, the curve C must be Riemann-Roch expected. According to our assumptions, C is then not submaximal, what means

$$\frac{LC}{r}\geq\sqrt{\frac{L^2}{r}},$$

so (3) follows immediately.
Case II. $m=2$. Here we have to prove that $2\alpha\beta\geq 4\left(r-\frac{1}{4.5}\right)$. As C passes through r general points with multiplicity 2 at each point, from lemma 7, we have

$$2\alpha\beta-4(r-1)\geq 4-2+1,$$

so

$$2\alpha\beta\geq 4r-1.$$

If $2\alpha\beta\geq 4r$ then inequality (4) follows, and $2\alpha\beta=4r-1$ means that $0=1\mathrm{mod}2$, what is impossible.
Case III. $m=3$. We have to check that $2\alpha\beta\geq 9\left(r-\frac{1}{4.5}\right)=9r-2$, and this is exactly guaranteed by lemma 7 in this case.
Case IV. $m=4$. We have to check that $2\alpha\beta\geq 16\left(r-\frac{1}{4.5}\right)=16r-\frac{32}{9}$. From lemma 7, we have now $2\alpha\beta\geq 16r-4+1=16r-3$, and we are done. ■

3.2 Proof of theorem 5

If we prove that

$$a\beta+b\alpha\geq m\sqrt{2ab\left(r-\frac{1}{5}\right)}, \tag{5}$$

for $m=1,2,3,4$, then again the result will follow from theorem 6. For $m=1,2,4$ the proof goes analogously as the proof of theorem 4, so we skip the calculations.

The only case needing considerations is $m = 3$. We have to show that

$$a\beta + b\alpha \geq 3\sqrt{2ab\left(r - \frac{1}{5}\right)}, \tag{6}$$

what will follow from

$$2\alpha\beta \geq 9\left(r - \frac{1}{5}\right). \tag{7}$$

From lemma 7 we have that

$$2\alpha\beta \geq 9r - 2.$$

Thus, if $2\alpha\beta \geq 9r - 1$, we are done. We have to consider the case $2\alpha\beta = 9r - 2$. This is clearly impossible if r is odd, so the proof of part (i) is finished.
For (ii), assume then that r is even, $r = 2k$, and $2\alpha\beta = 9r - 2$. If the curve C is not submaximal, then $\frac{LC}{3r} \geq \sqrt{\frac{L^2}{r}}$ and inequality (6) follows.
So, assume that C is submaximal. We have then

$$2\sqrt{\alpha\beta ab} \leq b\alpha + a\beta < 3\sqrt{2abr}. \tag{8}$$

From $2\alpha\beta = 9r - 2$ and $r = 2k$ we obtain $\alpha\beta = 9k - 1$, so (8) becomes

$$2\sqrt{9k-1} \leq \frac{b\alpha + a\beta}{\sqrt{ab}} < 6\sqrt{k}, \tag{9}$$

or equivalently

$$4(9k-1) \leq \frac{(b\alpha + a\beta)^2}{ab} < 36k. \tag{10}$$

Taking $a = b$, we get

$$4(9k-1) \leq (\alpha + \beta)^2 < 36k. \tag{11}$$

Thus $(\alpha+\beta)^2 = 36k - j$, for $j = 1, 2, 3, 4$. We have to exclude these possibilities. We will use the fact that a square of a natural number modulo a prime number must again be a square.
If $j = 1$, then $(\alpha+\beta)^2 = 36k - 1$. This means that $(\alpha+\beta)^2 = -1 \mathrm{mod} 3$, what is impossible.
If $j = 2$, then $(\alpha+\beta)^2 = 36k - 2 = 2(18k-1)$ and this is impossible as $(18k-1)$ is odd.
If $j = 3$, then $(\alpha+\beta)^2 = 36k - 3 = 3(12k-1)$ and this is impossible as $(12k-1)$ is not divisible by 3.
If $j = 4$ then $(\alpha+\beta)^2 = 36k - 4$. This again means that $(\alpha+\beta)^2 = -1 \mathrm{mod} 3$, what is impossible. ■

Remark 8. Observe, that not every curve on $(\mathbb{P}^1 \times \mathbb{P}^1, L)$ satisfies the bound given by theorem 6. Take for example L of type $(7, 1)$ and $C \equiv (5, 1)$, $r = 15$. Take $\mu = 2$ in theorem 6. Then it is easy to check that $\frac{LC}{r} < \sqrt{\frac{L^2}{r}}\sqrt{1 - \frac{1}{2r}}$. Of course, the assumption (1) is also not satisfied. The reader may look into [9] for more examples.

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References

[1] Bauer, Th.: Seshadri constants on algebraic surfaces. Math. Ann. **313**, (1999) 547-583

[2] Demailly, J.-P.: *Singular Hermitian metrics on positive line bundles.* Lect. Notes Math. 1507, Springer 1992, 87-104

[3] De Volder, C., Tutaj-Gasińska, H. A bound on Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$. to appear in Bull. Belgian Math. Soc.

[4] Harbourne, B.: Seshadri constants and very ample divisors on algebraic surfaces. J. Reine Angew. Math. **559**, (2003) 115-122

[5] Harbourne, B., Roé, J.: Discrete behaviour of Seshadri constants on surfaces. http://mat.uab.es/~jroe/papers/SeshSurfaces.pdf

[6] Küchle, O.: Multiple point Seshadri constants and the dimension of adjoint linear series. Ann. Inst. Fourier **46**, (1996) 63-71

[7] Nagata, M., On the 14-th problem of Hilbert. Am. J. Math. **81** (1959) 766-772

[8] Syzdek, W.: Nagata submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$. Ann. Polon. Math. **80** (2003) 223-230

[9] Syzdek, W.: Seshadri constants and geometry of surfaces. preprint, http://www.uni-essen.de/ mat903/alggeo.html

[10] Szemberg, T.: Global and local positivity of line bundles. Habilitationsshrift Essen 2001

[11] Xu, G.: Ample line bundles on smooth surfaces, J. Reine Angew. Math. **469**, (1995) 199-209

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