An estimate in Gottlieb ranks of fibration

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Dedicated to Professor Hiroo Shiga on his 60th birthday

Abstract

As an application of the Gottlieb sequence [11]([7]) of fibration, we give an upper bound of the rank of Gottlieb group $G(E) = \oplus_{i>0} G_i(E)$ of the total space E of a fibration $\xi : X \to E \to B$ and define the *Gottlieb type* $(a, b, c; s, t, u)$, which describes a rational homotopical condition of fibration with rank$G(E) = s + t + u$. We also note various examples showing the different situations that can occur. Finally we comment about an interaction with a Halperin's conjecture on fibration.

1 Introduction

The nth Gottlieb group $G_n(X)$ of a space X is the subgroup of the nth homotopy group $\pi_n(X)$ of X consisting of homotopy classes of maps $a : S^n \to X$ such that the wedge $(a|id_X) : S^n \vee X \to X$ extends to a map $F_a : S^n \times X \to X$ [3]. The nth evaluation subgroup $G_n(Y, X; f)$ of a map $f : X \to Y$ is the subgroup of $\pi_n(Y)$ represented by maps $a : S^n \to Y$ such that $(a|f) : S^n \vee X \to Y$ extends to a map $F_a : S^n \times X \to Y$. Note $G_n(Y) \subset G_n(Y, X; f)$ in general and $G_n(X) = G_n(X, X; id_X)$. Put $G(X) = \oplus_{i>0} G_i(X)$.

For a fibration $X \to E \to B$ of simply connected spaces, various inequations between their LS categories are known. For example, there is an upper bound of $\mathrm{cat}(E)$ by $\mathrm{cat}(X)$ and $\mathrm{cat}(B)$: $\mathrm{cat}(E) + 1 \leq (\mathrm{cat}(X) + 1)(\mathrm{cat}(B) + 1)$ [2, Prop.30.6]. It is well known that there is an inequation $\mathrm{rank}\pi_*(E) \leq \mathrm{rank}\pi_*(X) + \mathrm{rank}\pi_*(B)$ induced by the exact homotopy sequence of fibration. If both X and B have the rational homotopy types of homogeneous spaces, it is restricted as rank$G(E) \leq$

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rank$G(X)$ + rank$G(B)$ (see Proposition B). But, in general, we can't hope such a good inequation only between the ranks of Gottlieb groups of spaces X, E, B. In fact, rank$G(X)$, rank$G(E)$ and rank$G(B)$ can be arbitary natural numbers (see Example 1). So we must make a compromise.

In this paper, all spaces are simply connected with rational homology of finite type. Let $\xi : X \underset{j}{\to} E \underset{p}{\to} B$ be a Hurewicz fibration. Restricting the homomorphisms in the exact homotopy sequence of ξ yields a sequence

$$\cdots \to \pi_{n+1}(B) \underset{\partial}{\to} G_n(X) \underset{j_\sharp}{\to} G_n(E, X; j) \underset{p_\sharp}{\to} \pi_n(B) \to \cdots , \qquad (*)$$

which is called as the *Gottlieb sequence* of ξ [11]. The nth *Gottlieb homology group* of ξ, $GH_n(\xi)$, is defined by the subquotient Ker$p_\sharp$/Im$j_\sharp$ in $(*)$ [11](the n-th ω-homology group of j in [7]). We give an effective upper bound of rank$G_n(E)$ by adding the supplementary "rank$GH_n(\xi)$" and by expanding $G_n(B)$ somewhat.

Proposition A. *Let $\xi : X \to E \xrightarrow{p} B$ be a fibration. Then*

$$\mathrm{rank}G_n(E) \leq \mathrm{rank}G_n(X) + \mathrm{rank}GH_n(\xi) + \mathrm{rank}G_n(B, E; p)$$

for all $n > 1$.

If ξ is a fibre-homotopically trivial fibration, the left and right hand sides are equal. The gap between them may represent a distance from the triviality. Note that $GH(\xi) := \oplus_{n>0} GH_n(\xi) = 0$ and $G(B, E; p) = G(B)$ in this case. Although notice that there are rationally non-trivial examples as Example 2 (9),(10),(11) and (13). For the rational number field $\mathbb{Q}$, denote $G \otimes \mathbb{Q}$ as $G_{\mathbb{Q}}$ for an abelian group G and $f \otimes \mathbb{Q}$ as $f_{\mathbb{Q}}$ for a group homomorphism f. There is a monomorphism of $\mathbb{Q}$-spaces $G(Y)_{\mathbb{Q}} \to G(Y_{(0)})$ for the rationalization $Y_{(0)}$ of Y [2, p.378]. In the following without mention, suppose that the spaces $Y = X, E$ of ξ are finite complexes. Then $\dim G(Y_{(0)}) \leq \mathrm{cat}Y < \infty$ [2, Prop.28.8], $G(Y)_{\mathbb{Q}} \cong G(Y_{(0)})$ [9] and $G(Z, Y; f)_{\mathbb{Q}} \cong G(Z_{(0)}, Y_{(0)}; f_{(0)})$ for a map $f : Y \to Z$ [13]. Therefore we see $GH(\xi)_{\mathbb{Q}} \cong GH(\xi_{(0)})$ and it is possible to consider the sequence $(*)$ by the derivation argument of Sullivan model [1],[10],[11] (see Section 2). Proposition A is realized as an inclusion of positively graded $\mathbb{Q}$-spaces.

Theorem A. *Let $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration. Then there is a decomposition $G(E)_{\mathbb{Q}} = S \oplus T \oplus U$ with $S \subset G(X)_{\mathbb{Q}}$, $T \subset GH(\xi)_{\mathbb{Q}}$ and $U \subset G(B, E; p)_{\mathbb{Q}}$, whose dimensions are uniquely determined.*

Here, for $S = \oplus_{n>1} S_n, T = \oplus_{n>1} T_n$ and $U = \oplus_{n>1} U_n$, elements of S_n, T_n and U_n are respectively represented by the rationaliztions of elements α, β and γ of $G_n(E)$ satisfying the conditions: there are maps $\{F_\alpha, F'_\alpha\}$, F_β and F_γ which make respectively the homotopy commutative diagrams $(i), (ii)$ and (iii),

$$(i)\qquad \begin{tikzcd}
& X\vee S^n \arrow[rr, "j\vee id_{S^n}"] \arrow[dl, "inc_2"'] \arrow[dd, "(id_X|\ F'_\alpha|_{S^n})" near start] & & E\vee S^n \arrow[dl, "inc_1"'] \arrow[dd, "(id_E|\alpha)"] \\
X\times S^n \arrow[rr, "j\times id_{S^n}"' near end] \arrow[dr, dotted, "F'_\alpha"'] & & E\times S^n \arrow[dr, dotted, "F_\alpha"'] & \\
& X \arrow[rr, "j"] & & E,
\end{tikzcd}$$

$$(ii)\quad \begin{tikzcd}
E\vee S^n \arrow[r, "inc_1"] \arrow[d, "(p|*)"'] \arrow[dr, "(id_E|\beta)"] & E\times S^n \arrow[d, dotted, "F_\beta"] \\
B & E \arrow[l, "p"]
\end{tikzcd}
\qquad \text{and} \qquad
(iii)\quad \begin{tikzcd}
E\vee S^n \arrow[r, "inc_1"] \arrow[d, "(p|p\circ\gamma)"'] \arrow[dr, "(id_E|\gamma)"] & E\times S^n \arrow[d, dotted, "F_\gamma"] \\
B & E. \arrow[l, "p"]
\end{tikzcd}$$

Then our inclusion

$$i_\xi \ : \ G(E)_{\mathbb{Q}} = S \oplus T \oplus U \ \subset \ G(X)_{\mathbb{Q}} \oplus GH(\xi)_{\mathbb{Q}} \oplus G(B,E;p)_{\mathbb{Q}}$$

is given by

$$i_\xi \ ((\alpha_{(0)}, [\beta_{(0)}], \gamma_{(0)})) = (F'_\alpha|_{S^n(0)}, [\beta_{(0)}], p \circ \gamma_{(0)}),$$

where $[\beta_{(0)}] = [\beta'_{(0)}]$ if and only if $\phi := \beta_{(0)} - \beta'_{(0)} \in Imj_{\sharp\mathbb{Q}}$, i.e., ϕ can be embedded in the rationalized diagram of (i) with some F_ϕ and F'_ϕ.

Note that i_ξ depends on the choice of a map F'_α in (i) in general (see Remark 2). Anyway we can define a rational homotopy invariant of fibration.

Definition. We say that the fibration ξ is *Gottlieb type of* $(a,b,c;s,t,u)$ for $a = \mathrm{rank}G(X)$, $b = \mathrm{rank}GH(\xi)$, $c = \mathrm{rank}G(B,E;p)$, $s = \dim S$, $t = \dim T$ and $u = \dim U$.

We often say simply 'G-type of $(a,b,c;s,t,u)$'. Then $a \geq s \geq 0$, $b \geq t \geq 0$, $c \geq u \geq 0$ and $\mathrm{rank}G(E) = s+t+u$. It may be useful for estimating $\mathrm{rank}G(E)$. We see that i_ξ is 'equal' if and only if ξ is G-type of $(a,b,c;a,b,c)$ for some a,b,c. If a fibration is fibre-homotopically trivial, i_ξ is 'equal' with G-type of $(\mathrm{rank}G(X), 0, \mathrm{rank}G(B); \mathrm{rank}G(X), 0, \mathrm{rank}G(B))$, especially $G(E)_{\mathbb{Q}} = G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$. If a fibration satisfies $G(E)_{\mathbb{Q}} = G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$, it is rationally weak-homotopy trivial, that is, the rational connecting homomorphism $\partial_{\mathbb{Q}}$ is zero. But i_ξ may not be 'equal' as we can see in Example 2 (5) and (8). In §3, we give the proofs of Proposition A and Theorem A, and note various examples of small homotopy ranks, by the derivation argument of Sullivan model as in [11]. In §4, we comment about an interaction with G-type and a Halperin's conjecture on the rational cohomological splittings, i.e., $H^*(E;\mathbb{Q}) \cong H^*(X;\mathbb{Q}) \otimes H^*(B;\mathbb{Q})$ additively, of certain fibrations $X \to E \to B$.

2 Preliminary

We use the *Sullivan minimal model* $M(Y)$ of a simply connected space Y of finite type. It is a free $\mathbb{Q}$-commutative differential graded algebra (DGA) $(\Lambda V, d)$ with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i>1} V^i$ where $\dim V^i < \infty$ and a decomposable differential, i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda V =$ (the $\mathbb{Q}$-polynomial algebra over V^{even}) $\otimes$ (the $\mathbb{Q}$-exterior algebra over V^{odd}) and $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. Denote the degree of a homogeneous element x of a graded algebra as $|x|$ and the $\mathbb{Q}$-vector space of basis $\{v_i\}_i$ as $\mathbb{Q}\{v_i\}_i$. Then $xy = (-1)^{|x||y|} yx$ and $d(xy) = d(x)y + (-1)^{|x|} x d(y)$. A map $f : X \to Y$ has a minimal model which is a DGA-map $f^* : M(Y) \to M(X)$. Notice that $M(Y)$ determines the rational homotopy type of Y. Especially there is an isomorphism $Hom_i(V, \mathbb{Q}) \cong \pi_i(X)_{\mathbb{Q}}$. See [2] for a general introduction and the standard notations.

The detailed discussion of the followings are in [10],[11]. Let A be a DGA $A = (A^*, d_A)$ with $A^* = \oplus_{i \geq 0} A^i$, $A^0 = \mathbb{Q}$ and the augmentation $\epsilon : A \to \mathbb{Q}$. Define $Der_i A$ the vector space of derivations of A decreasing the degree by $i > 0$, where $\theta(xy) = \theta(x)y + (-1)^{i|x|} x\theta(y)$ for $\theta \in Der_i A$. We denote $\oplus_{i>0} Der_i A$ by $Der A$. The boundary operator $\delta : Der_* A \to Der_{*-1} A$ is defined by $\delta(\sigma) = d_A \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A$. For a DGA-map $\phi : A \to B$, define a ϕ-derivation of degree n to be a linear map $\theta : A^* \to B^{*-n}$ with $\theta(xy) = \theta(x)\phi(y) + (-1)^{n|x|} \phi(x)\theta(y)$ and $Der(A, B; \phi)$ the vector space of ϕ-derivations. The boundary operator $\delta_\phi : Der_*(A, B; \phi) \to Der_{*-1}(A, B; \phi)$ is defined by $\delta_\phi(\sigma) = d_B \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A$. Note $Der_*(A, A; id_A) = Der_*(A)$. For $\phi : A = (\Lambda Z, d_A) \to B$, the composition with the augmentation $\epsilon' : B \to \mathbb{Q}$ induces a chain map $\epsilon'_* : Der_n(A, B; \phi) \to Der_n(A, \mathbb{Q}; \epsilon)$. Define

$$G_n(A, B; \phi) := \mathrm{Im}(H(\epsilon'_*) : H_n(Der(A, B; \phi)) \to Hom_n(Z, \mathbb{Q})).$$

Especially

$$G_n(\Lambda Z, d_A) := \mathrm{Im}(H_n(\epsilon_*) : H_n(Der(\Lambda Z, d_A)) \to Hom_n(Z, \mathbb{Q})),$$

that is, $G_*(A, A; id_A) = G_*(A)$. Note that $z^* \in Hom_n(Z, \mathbb{Q})$ (z^* is the dual of the basis element z of Z^n) is in $G_n(A, B; \phi)$ if and only if z^* extends to a derivation θ of $Der_n(A, B; \phi)$ with $\delta_\phi(\theta) = 0$. For example, see [2, p.392-393]. Let $\xi : X \underset{j}{\to} E \underset{p}{\to} B$ be a fibration. Consider the rationalization of the Gottlieb sequence $(*)$ of ξ:

$$\cdots \to \pi_{n+1}(B)_{\mathbb{Q}} \underset{\partial_{\mathbb{Q}}}{\to} G_n(X)_{\mathbb{Q}} \underset{j_{\sharp\mathbb{Q}}}{\to} G_n(E, X; j)_{\mathbb{Q}} \underset{p_{\sharp\mathbb{Q}}}{\to} \pi_n(B)_{\mathbb{Q}} \to \cdots . \qquad (**)$$

Write $M(B) = (\Lambda W, d_B)$ and $M(X) = (\Lambda V, d)$. Then the model (not minimal in general) of E is given by $(\Lambda W \otimes \Lambda V, D)$ with $D \circ D = 0$, $D|_{\Lambda W} = d_B$ and $\overline{D} = d$. The DGA-maps $J : (\Lambda(W \oplus V), D) \to (\Lambda V, \overline{D}) = (\Lambda V, d)$ (projection) and $P : (\Lambda W, d_B) \to (\Lambda(W \oplus V), D)$ (injection) are the Sullivan models for j and p, respectively. They induce linearization maps $Q(J) : W \oplus V \to V$ and $Q(P) : W \to W \oplus V$. Then we obtain the model version of $(**)$ as

$$\cdots \to Hom_{n+1}(W, \mathbb{Q}) \underset{\partial_{\mathbb{Q}}}{\to} G_n(\Lambda V) \underset{Q(J)^*}{\to} G_n(\Lambda(W \oplus V), \Lambda V; J) \underset{Q(P)^*}{\to} Hom_n(W, \mathbb{Q}) \to \cdots$$

and $GH_n(\xi)_{\mathbb{Q}} \cong GH_n(\Lambda(W \oplus V), \Lambda V; J) := \mathrm{Ker}Q(P)^*/\mathrm{Im}Q(J)^*$ [11]. Note that there is a monomorphism $GH_n(\xi)_{\mathbb{Q}} \to Hom_n(V, \mathbb{Q}) \cong \pi_n(X)_{\mathbb{Q}}$ for $n > 1$.

Proposition B. *If X and B have the rational homotopy types of homogeneous spaces, then $G(E)_{\mathbb{Q}} \subset G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$ for any fibration $X \to E \to B$.*

Proof. We can put $M(X) \cong (\Lambda(x_1, \cdots, x_k, v_1, \cdots, v_l), d)$ with $|x_i|$ even for $1 \leq i \leq k$, $|v_i|$ odd for $1 \leq i \leq l$, $dx_* = 0$ and $dv_* \in \Lambda(x_1, \cdots, x_k)$ for some k and l [2, Proposition 15.16]. Also $M(B) \cong (\Lambda(y_1, \cdots, y_m, w_1, \cdots, w_n), d_B)$ with $|y_i|$ even for $1 \leq i \leq m$, $|w_i|$ odd for $1 \leq i \leq n$, $d_B y_* = 0$ and $d_B w_* \in \Lambda(y_1, \cdots, y_m)$ for some m and n. Since $G(X)_{\mathbb{Q}} \supset \mathbb{Q}\{v_1^*, \cdots, v_l^*\}$ and $G(B)_{\mathbb{Q}} \supset \mathbb{Q}\{w_1^*, \cdots, w_n^*\}$, we have $G_{odd}(E)_{\mathbb{Q}} \subset \pi_{odd}(E)_{\mathbb{Q}} \subset G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$. On the other hand, we know that $G_{2n}(Y)_{\mathbb{Q}} = 0$ $(n > 0)$ for any simply connected finite complex Y [2, Proposition 28.8]. ∎

Claim. A space X or a minimal model $M(X) = (\Lambda V, d)$ with $\dim H^*(X; \mathbb{Q}) < \infty$ ($\dim H^*(\Lambda V, d) < \infty$) and $\mathrm{rank}\pi_*(X) < \infty$ ($\dim V < \infty$) is said to be *elliptic*. If the fiber of a rationally weak-homotopy trivial fibration is elliptic, then $s > 0$ in its G-type since the dual of a top degree element in V is in $G(E)_{\mathbb{Q}}$. An elliptic minimal model $M(X) = (\Lambda V, d)$ with $dV^{even} = 0$ and $dV^{odd} \subset \Lambda V^{even}$ is said to be *pure*. For example, homogeneous spaces are pure. If the fiber X of a fibration has a pure model $M(X)$, then $t = 0$ in its G-type from $T \subset \pi_{odd}(X)_{\mathbb{Q}} = G(X)_{\mathbb{Q}}$.

Notation. Denote by $\sigma \otimes f$ for $\sigma \in Der_n(\Lambda Z)$ and $f \in \Lambda Z$ the derivation of degree $|\sigma \otimes f| = n - |f|$ on $(\Lambda Z, d)$ given by $(\sigma \otimes f)(z) := (-1)^{|z||f|}\sigma(z) \cdot f$, which satisfies for $x, y \in \Lambda Z$

$$(\sigma \otimes f)(xy) = (\sigma \otimes f)(x) \cdot y + (-1)^{|x||\sigma \otimes f|} x \cdot (\sigma \otimes f)(y)$$

and $(\sigma \otimes f) \circ D = -\sigma \circ D \otimes f$. Especially, note that $z^* \otimes f$ means the derivation sending z to f and extending by linearity.

3 Gottlieb type and examples

First we give two examples (1) and (2), which motivate our estimate. Note that these models are realized as certain fibrations of finite complexes $X \to E \to B$ since their cohomologies are finite. Especially the following spaces are elliptic.

Example 1. For any three natural numbers l, m and n, there is a fibration $\xi : X \to E \to B$ with $\mathrm{rank}G(X) = l$, $\mathrm{rank}G(E) = m$ and $\mathrm{rank}G(B) = n$. In the following models, the degrees of all elements $v_*, v'_*, w_*, w'_*, v, v', u$ are odd.

(1) Suppose $M(B) \cong (\Lambda(w_1, .., w_n), 0)$, i.e., $B \simeq_{(0)} S^{|w_1|} \times S^{|w_2|} \times \cdots \times S^{|w_n|}$ (B has the rational homotopy type of the product of n-odd spheres). Note that it induces $\mathrm{rank}G(B, E; p) = \mathrm{rank}G(B) = n$.

(a): $m > l+n-1$. If $m-l-n$ is even, for an even integer s (≥ 2), put $M(X) = (\Lambda(v_1,..,v_s,v,v'_1,..,v'_{l-1}),d)$ with $dv = v_1\cdots v_s$ and $dv_* = dv'_* = 0$. Put $Dv = v_1\cdots v_s + w_1v_1$. Then

$$G(E)_{\mathbb{Q}} = \mathbb{Q}\{v_2^*,\cdots,v_s^*,v^*,v'^*_1,\cdots,v'^*_{l-1},w_2^*,\cdots,w_n^*\}.$$

If $m-l-n$ is odd, for an odd integer $s(>1)$, put $M(X) = (\Lambda(v_1,..,v_{s+1},v,v'_1,..,v'_{l-1}), d)$ with $dv = v_1\cdots v_{s+1}$ and $dv_* = dv'_* = 0$. Put $Dv = dv + w_1v_1$, $Dv_3 = w_1v_2$. Then

$$G(E)_{\mathbb{Q}} = \mathbb{Q}\{v_3^*,\cdots,v_s^*,v_{s+1}^*,v^*,v'^*_1,\cdots,v'^*_{l-1},w_2^*,\cdots,w_n^*\}.$$

Thus $\mathrm{rank}G(E) = l+n+s-2$.

(b): $m = l+n-1$. Put $M(X) = (\Lambda(v_1,..,v_{l+2}),d)$ with $dv_3 = v_1v_2$ and $dv_i = 0$ for $i \neq 3$. Put $Dv_2 = w_1v_1$ and $Dv_i = dv_i$ for $i \neq 2$. Then

$$G(E)_{\mathbb{Q}} = \mathbb{Q}\{v_3^*,\cdots,v_{l+2}^*,w_2^*,\cdots,w_n^*\},$$

that is, $\mathrm{rank}G(E) = l+n-1$.

(c): $m < l+n-1$. If $l+n-m$ is even, put $M(X) = (\Lambda(v_1,..,v_l),0)$. Put $Dv_1 = 0,..,Dv_{l-1} = 0$ and $Dv_l = w_1\cdots w_iv_1\cdots v_k$ ($i+k$:even) for some $i \geq 1$ and $k \geq 0$. If $l+n-m$ is odd and $l > 1$, put $Dv_l = w_1v_1 + w_2\cdots w_i$ (i:odd) and $Dv_{l-1} = w_1v_2\cdots v_k$ (k:even) for $M(X) = (\Lambda(v_1,..,v_l),0)$. Then

$$G(E)_{\mathbb{Q}} = \mathbb{Q}\{v_{k+1}^*,\cdots,v_l^*,w_{i+1}^*,\cdots,w_n^*\},$$

that is, $\mathrm{rank}G(E) = l+n-(i+k)$.

If $l+n-m$ is odd and $l = 1$, put $M(X) = (\Lambda(v_1,v_2,v_3,v_4,v),d)$ with $dv_1 = \cdots = dv_4 = 0$ and $dv = v_1v_2v_3v_4$. Put $Dv = v_1v_2v_3v_4 + w_1\cdots w_kv_1$ (k:odd>1). Then

$$G(E)_{\mathbb{Q}} = \mathbb{Q}\{v^*,w_{k+1}^*,\cdots,w_n^*\},$$

that is, $\mathrm{rank}G(E) = 1+n-k$.

(2) Suppose $M(X) \cong (\Lambda(v_1,..,v_l),0)$, i.e, $X \simeq_{(0)} S^{|v_1|}\times S^{|v_2|}\times\cdots\times S^{|v_l|}$. Note that it induces $\mathrm{rank}G(X) = l$ and $\mathrm{rank}GH(\xi) = 0$.

(d): $m \geq l+n-1$ and $l+m+n$ is even. Put $M(B) = (\Lambda(w_1,..,w_{n+k-1},u),d_B)$ with $d_Bw_* = 0$ and $d_Bu = w_1\cdots w_k$ (k:even). Put $Dv_1 = w_1w_2$ and $Dv_i = 0$ for $i > 1$. Then

$$G(E)_{\mathbb{Q}} = \mathbb{Q}\{v_1^*,\cdots,v_l^*,w_3^*,\cdots,w_{n+k-1}^*,u^*\},$$

that is, $\mathrm{rank}G(E) = l+n+k-2$.

(e): $m \geq l+n-1$ and $l+m+n$ is odd. Put $M(B) = (\Lambda(w_1,..,w_{n+k-2},u),d_B)$ with $d_Bw_* = 0$ and $d_Bu = w_1\cdots w_{k-1}$ (k:odd). Put $Dv_1 = w_1w_2$, $Dv_2 = w_1v_1$ and $Dv_i = 0$ for $i > 2$. Then

$$G(E)_{\mathbb{Q}} = \mathbb{Q}\{v_2^*,\cdots,v_l^*,w_3^*,\cdots,w_{n+k-2}^*,u^*\},$$

that is, $\mathrm{rank}G(E) = l+n+k-4$.

(f): $m < l+n-1$. If $l > 1$, see the example in (c) . If $l = 1$, put $M(B) = (\Lambda(w_1,w_2,w_3,w_4,u,w'_1,\cdots,w'_{n-1}),d_B)$ with $d_Bw_* = d_Bw'_* = 0$ and $d_Bu =$

$w_1w_2w_3w_4$. Put $Dv = w_1w'_1\cdots w'_j$ (j:odd) or $Dv = w_1w_2w'_1\cdots w'_j$ (j:even>0). Then

$$G(E)_{\mathbb{Q}} = \mathbb{Q}\{v^*, w'^*_{j+1}, \cdots, w'^*_{n-1}\},$$

that is, $\mathrm{rank}G(E) = n - j$.

Remark 1. Even if $\mathrm{rank}G(E) = \mathrm{rank}G(X) + \mathrm{rank}G(B)$, notice that $G(E)_{\mathbb{Q}}$ may not be equal to $G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$. For example, put $M(X) = (\Lambda(v_1, v_2, v_3, v_4, v, v', v''), d)$ with $dv_* = dv' = dv'' = 0$, $dv = v_1v_2v_3v_4$ and $M(B) = (\Lambda(w), 0)$. Put $Dv = v_1v_2v_3v_4 + wv_4$, $Dv' = wv_1$ and $Dv'' = wv_2$. Then $\mathrm{rank}G(E) = 4 = 3 + 1 = \mathrm{rank}G(X) + \mathrm{rank}G(B)$ but $T = \mathbb{Q}\{v_3^*\} = GH(\xi)_{\mathbb{Q}}$ and ξ is G-type of $(3, 1, 1; 3, 1, 0)$, i.e., $G(E)_{\mathbb{Q}} = \mathbb{Q}\{v^*, v'^*, v''^*\} \oplus \mathbb{Q}\{v_3^*\} = G(X)_{\mathbb{Q}} \oplus GH(\xi)_{\mathbb{Q}} \neq G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$.

Proof of Proposition A. Consider the following commutative diagram:

$$\begin{tikzcd}
G_n(E) \arrow[r, "p^1_\sharp"] \arrow[d, "inc."'] & G_n(B, E; p) \arrow[d, "inc."] \\
G_n(E, X; j) \arrow[r, "p^2_\sharp"] & \pi_n(B)
\end{tikzcd}$$

where $p^i_\sharp$ is defined by $[p \circ a]$ for $a : S^n \to E$. From this, Ker $p^1_\sharp \subset$ Ker $p^2_\sharp$. From the definition of Gottlieb homology group, the sequence $G_n(X) \underset{j_\sharp}{\to}$ Ker $p^2_\sharp \to GH_n(\xi) \to 0$ is exact. Thus

$$\text{rank Ker}p^1_\sharp \leq \text{rank Ker}p^2_\sharp \leq \text{rank } G_n(X) + \text{rank } GH_n(\xi).$$

From rank $G_n(E) \leq$ rank Ker $p^1_\sharp$ + rank $G_n(B, E; p)$, we have done. ∎

Proof of Theorem A. Denote by U the image of $p_{\sharp\mathbb{Q}} : G(E)_{\mathbb{Q}} \to G(B, E)_{\mathbb{Q}}$ and put the kernel as K. Then $G(E)_{\mathbb{Q}} = K \oplus U$. Denote by S the kernel of the map

$$K \hookrightarrow G(E)_{\mathbb{Q}} \hookrightarrow G(E, X)_{\mathbb{Q}} \overset{proj.}{\to} G(E, X)_{\mathbb{Q}}/j_{\sharp\mathbb{Q}}(G(X)_{\mathbb{Q}})$$

and by T the image. Then $K = S \oplus T$ with the natural inclusion $S \overset{i}{\hookrightarrow} j_{\sharp\mathbb{Q}}(G(X)_{\mathbb{Q}})$ and $T \subset G(E, X)_{\mathbb{Q}}/j_{\sharp\mathbb{Q}}(G(X)_{\mathbb{Q}})$. By choosing a lift $\tilde{i}$ of i to $G(X)_{\mathbb{Q}}$, S injects into $G(X)_{\mathbb{Q}}$. Since $\overline{p}_{\sharp\mathbb{Q}}(T) = 0$ for $\overline{p}_{\sharp\mathbb{Q}} : G(E, X)_{\mathbb{Q}}/j_{\sharp\mathbb{Q}}(G(X)_{\mathbb{Q}}) \to \pi_*(B)_{\mathbb{Q}}$, we have

$$T \ \subset \ \mathrm{Ker}\overline{p}_{\sharp\mathbb{Q}} = \frac{\mathrm{Ker}(\ p_{\sharp\mathbb{Q}} : G(E, X)_{\mathbb{Q}} \to \pi_*(B)_{\mathbb{Q}}\ )}{j_{\sharp\mathbb{Q}}(G(X)_{\mathbb{Q}})} \ = \ GH(\xi)_{\mathbb{Q}}.$$

∎

Remark 2. An inclusion $i_\xi|_S : S \to G(X)_{\mathbb{Q}}$ in §1 corresponds to a lift $\tilde{i} : S \to G(X)_{\mathbb{Q}}$ in the proof of Theorem A. For example, consider the product fibration $S^3 \times S^3 \to S^7 \times S^3 \to S^4$ of the Hopf fibration $S^3 \to S^7 \to S^4$ (see Example 2 (1)) and the trivial fibration $S^3 \to S^3 \to *$. Put $M(S^3 \times S^3) = (\Lambda(v, v'), 0)$ with $|v| = |v'| = 3$ and $M(S^4) = (\Lambda(w_1, w_2), d_B)$ with $|w_1| = 4$ and $|w_2| = 7$. The model is given by $Dv = w_1$ and $Dv' = 0$. Then $S = \mathbb{Q}\{v'^*\}$ and a lift

$\tilde{i}: S \hookrightarrow G(X)_{\mathbb{Q}} = \mathbb{Q}\{v^*, v'^*\}$ is given by $\tilde{i}(v'^*) = av^* + v'^*$ for $a \in \mathbb{Q}$. Note that $\tilde{i}$ is not unique and depends on a. S is identified as $\mathbb{Q}\{av^* + v'^*\}$ in $G(X)_{\mathbb{Q}}$ and $j_{\sharp\mathbb{Q}}(av^* + v'^*) = v'^*$ in $\mathbb{Q}\{w_2^*, v'^*\} = G(E, X; j)_{\mathbb{Q}}$.

If a rationallized Gottlieb sequence $(**)$ deduces the short exact sequence $0 \to G_n(X)_{\mathbb{Q}} \to G_n(E, X; j)_{\mathbb{Q}} \to \pi_n(B)_{\mathbb{Q}} \to 0$ for all $n > 1$, the fibration ξ is said as *rationally Gottlieb-trivial* (r.G-trivial). Especially, if a fibration is r.G-trivial, $GH(\xi)_{\mathbb{Q}} = 0$. Recall that ξ is r.G-trivial if and only if $p_{\sharp\mathbb{Q}} : G_n(E, X; j)_{\mathbb{Q}} \to \pi_n(B)_{\mathbb{Q}}$ is surjective for $n > 1$ [11, Theorem 4.2 (2) $\Leftrightarrow$ (3)].

Example 2. The following examples are fibrations or the models of certain fibrations $\xi : X \xrightarrow{j} E \xrightarrow{p} B$. The degrees of elements of the models without mention are odd.

(1) The Hopf fibration : $S^3 \to S^7 \to S^4$, where $M(S^3) = (\Lambda v, 0)$ with $|v| = 3$, $M(S^4) = (\Lambda(w_1, w_2), d_B)$ with $|w_1| = 4$, $|w_2| = 7$, $d_B w_1 = 0$, $d_B w_2 = w_1^2$ and $Dv = w_1$. Note that there is a quasi-isomorphism $\rho : (\Lambda(w_2), 0) \to (\Lambda(w_1, w_2, v), D)$ with $\rho(w_2) = w_2 - w_1 v$. It is G-type of $(1, 0, 2; 0, 0, 1)$ from $G(B, E; p)_{\mathbb{Q}} = \mathbb{Q}\{w_1^*, w_2^*\}$ and $G(E)_{\mathbb{Q}} = G(B)_{\mathbb{Q}}$. Since δ_J-cycle v^* is exact by $\delta_J(w_1^*) = v^*$, $G(E, X; j)_{\mathbb{Q}} = \mathbb{Q}\{w_2^*\}$. Note that $GH(\xi)_{\mathbb{Q}} = 0$ but ξ is not r.G-trivial since $p_{\sharp\mathbb{Q}} : G(E, X; j)_{\mathbb{Q}} \to \pi_*(B)_{\mathbb{Q}} = \mathbb{Q}\{w_1^*, w_2^*\}$ in $(**)$ is not surjective.

(2) $S^5 \to E \to S^3 \times S^3$ with $M(S^5) = (\Lambda v, 0)$, $M(S^3 \times S^3) = (\Lambda(w_1, w_2), 0)$ and $Dv = w_1 w_2$. It is G-type of $(1, 0, 2; 1, 0, 0)$. Since $G(E, X)_{\mathbb{Q}} = \mathbb{Q}\{v^*, w_1^*, w_2^*\} \supset \mathbb{Q}\{w_1^*, w_2^*\} = \pi_*(B)_{\mathbb{Q}}$, ξ is r.G-trivial.

(3) $S^3 \times S^5 \to E \to S^3$ with $M(S^3 \times S^5) = (\Lambda(v_1, v_2), 0)$ $|v_1| = 3, |v_2| = 5$ and $M(S^3) = (\Lambda w, 0)$ and $Dv_2 = wv_1$. It is G-type of $(2, 0, 1; 1, 0, 0)$. Since $G(E, X)_{\mathbb{Q}} = \mathbb{Q}\{v_1^*, v_2^*\} \not\ni w^*$, ξ is not r.G-trivial.

(4) $M(X) = (\Lambda(v_1, v_2, v_3), d)$ and $M(B) = (\Lambda w, 0)$ with $dv_1 = dv_2 = 0$, $dv_3 = Dv_3 = v_1 v_2$, $Dv_1 = 0$ and $Dv_2 = wv_1$. Then ξ is G-type of $(1, 0, 1; 1, 0, 0)$. Since $G(E, X)_{\mathbb{Q}} \not\ni w^*$, ξ is not r.G-trivial.

(5) Recall the non-trivial fibration $\mathbb{C}P^2 \to E \to S^4$ of [11, Ex.4.4]. The model is given by $M(\mathbb{C}P^2) = (\Lambda(v_1, v_2), d)$ with $|v_1| = 2, |v_2| = 5$, $dv_1 = 0, dv_2 = v_1^3$ and $M(S^4) = (\Lambda(w_1, w_2), d_B)$ with $|w_1| = 4, |w_2| = 7$, $d_B w_1 = 0, d_B w_2 = w_1^2$, and $Dv_1 = 0$, $Dv_2 = v_1^3 + w_1 v_1$. Since $\delta_J(v_1^* - 3w_1^* \otimes v_1) = 0$, $GH(\xi)_{\mathbb{Q}} = \mathbb{Q}\{v_1^*\}$. In fact, $\delta_J(v_1^* - 3w_1^* \otimes v_1)(v_2) = v_1^*(dv_2) - 3(w_1^* \otimes v_1)(w_1 v_1) = v_1^*(v_1^3) - 3w_1^*(w_1 v_1) \cdot v_1 = 3v_1^2 - 3v_1^2 = 0$ and $\delta_J(v_1^* - 3w_1^* \otimes v_1)(z) = 0$ for $z = w_*, v_1$. It is G-type of $(1, 1, 1; 1, 0, 1)$ and $G(E)_{\mathbb{Q}} = G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$.

(6)[11, Ex.4.5] $M(X) = (\Lambda(v_1, v_2, \cdots, v_{n+1}, v), d)$ (n:odd) with $dv_* = 0$, $dv = v_1 v_2 \cdots v_{n+1}$ and $M(B) = (\Lambda w, 0)$. Put $Dv = v_1 v_2 \cdots v_{n+1} + wv_{n+1}$ and $Dv_* = 0$. Then $GH(\xi)_{\mathbb{Q}} = T = \mathbb{Q}\{v_1^*, \cdots, v_n^*\}$ if $n > 1$. In fact, $\delta_J(v_i^* + (-1)^i w^* \otimes v_1 \cdots \check{v}_i \cdots v_n) = 0$ for $i \leq n$. Thus ξ is G-type of $(1, n, 1; 1, n, 0)$ if $n > 1$. If $n = 1$, ξ is G-type of $(1, 0, 1; 1, 0, 1)$. Note that it is rationally trivial. In fact, there is a DGA-isomorphism $\rho : (\Lambda(w, v_1, v_2, v), D) \to (\Lambda w, 0) \otimes (\Lambda(v_1, v_2, v), d)$ given $\rho(v_1) = v_1 - w$ and $\rho(z) = z$ for the other elements z.

(7) $M(X) = (\Lambda(v_1, .., v_n, v), d)$ with $dv_* = 0$ and $dv = v_1 \cdots v_n$ (n:even). If $Dv = v_1 \cdots v_n + w_1 w_2 w_3 v_n$ for $M(B) = (\Lambda(w_1, w_2, w_3), 0)$, then ξ is G-type of $(1, 0, 3; 1, 0, 0)$ if $n > 2$. Since $G(E, X)_{\mathbb{Q}} \ni w_1^*, w_2^*, w_3^*$, ξ is r.G-trivial. If $n = 2$, ξ is G-type of $(1, 0, 3; 1, 0, 3)$. Note that it is rationally trivial by $\rho(v_1) = v_1 - w_1 w_2 w_3$ as in (6).

(8) $M(X) = (\Lambda(v_1, v_2, v_3, v_4, v), d)$ with $dv_* = 0$ and $dv = v_1 v_2 v_3 v_4$. $M(B) = (\Lambda(w, w', u), d_B)$ with $d_B w = d_B w' = 0$ and $d_B u = ww'$. Put $Dv_* = 0$ and $Dv = v_1 v_2 v_3 v_4 + wv_4$. Then ξ is G-type of $(1, 3, 1; 1, 0, 1)$. Especially $GH(\xi)_{\mathbb{Q}} = \mathbb{Q}\{v_1^*, v_2^*, v_3^*\}$ and $G(E)_{\mathbb{Q}} = G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$.

(9) $M(X) = (\Lambda(v_1, v_2, v_3, v_4, v_5, v), d)$ with $dv_1 = dv_2 = dv_3 = dv_4 = 0$, $dv_5 = v_1 v_4$, $dv = v_1 v_2 v_3 v_5$ and $M(B) = (\Lambda(w, w', u), d_B)$ with $d_B w = d_B w' = 0$, $d_B u = ww'$. Put $Dv = v_1 v_2 v_3 v_5 + wv_4$ and $Dv_* = dv_*$. Then ξ is G-type of $(1, 0, 1; 1, 0, 1)$ and $G(E)_{\mathbb{Q}} = G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$. Since $G(E, X)_{\mathbb{Q}} \not\ni w^*$, ξ is not r.G-trivial.

(10) $M(X) = (\Lambda(v_1, v_2, v_3, v_4, v, v', v''), d)$ with $dv_1 = dv_2 = dv_3 = dv_4 = dv' = dv'' = 0$, $dv = v_1 v_2 v_3 v_4$ and $M(B) = (\Lambda(w, w', u), d_B)$ with $d_B w = d_B w' = 0$, $d_B u = ww'$. Put $Dv = v_1 v_2 v_3 v_4 + wv_4$, $Dv' = wv_3$, $Dv'' = w' v_3$ and $Dv_* = 0$. Then ξ is G-type of $(3, 2, 1; 3, 2, 1)$ with $GH(\xi)_{\mathbb{Q}} = T = \mathbb{Q}\{v_1^*, v_2^*\}$.

(11) $M(X) = (\Lambda v, 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u), d_B)$ with $d_B w_* = 0$ and $d_B u = w_1 w_2 w_3 w_4$. If $Dv = w_1 w_2$, then ξ is G-type of $(1, 0, 3; 1, 0, 3)$. Since $G(E, X)_{\mathbb{Q}} \ni w_1^*, w_2^*, w_3^*, w_4^*, u^*$, ξ is r.G-trivial. Note that $G(B)_{\mathbb{Q}} = \mathbb{Q}\{u^*\}$ but $G(B, E)_{\mathbb{Q}} = \mathbb{Q}\{w_3^*, w_4^*, u^*\}$. Thus $G(E)_{\mathbb{Q}} \neq G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$ but $G(E)_{\mathbb{Q}} = G(X)_{\mathbb{Q}} \oplus G(B, E; p)_{\mathbb{Q}}$.

(12) $M(X) = (\Lambda(v, v'), 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u), d_B)$ with $d_B w_* = 0$ and $d_B u = w_1 w_2 w_3 w_4$. Put $Dv = w_1 w_2$ and $Dv' = w_3 w_4 + w_1 v$. Then ξ is G-type of $(2, 0, 4; 1, 0, 1)$ from $G(B, E)_{\mathbb{Q}} = \mathbb{Q}\{w_2^*, w_3^*, w_4^*, u^*\}$. Here $w_1^* \notin G(B, E)_{\mathbb{Q}}$ from the reason that the D-cocycle $w_1 w_2 v$ is not exact. Since $G(E, X)_{\mathbb{Q}} \not\ni w_1^*$, ξ is not r.G-trivial. Note that $G(E)_{\mathbb{Q}} \subsetneqq G(X)_{\mathbb{Q}} \oplus G(B)_{\mathbb{Q}}$.

(13) There is a rationally non-trivial fibration $\eta : S^2 \vee S^2 \to E' \to S^3$, which is rationally constructed as [5, (6.5)]. Then η is G-type of $(0, 0, 1; 0, 0, 0)$ since the Gottlieb rank of the one point union of spheres is zero [13, Theorem 5.4] and $GH(\eta)_{\mathbb{Q}} = 0$ from degree argument. The pull back fibration $\xi : S^2 \vee S^2 \to E \to S^3 \vee S^3$ of η by the map $(id_{S^3}|*) : S^3 \vee S^3 \to S^3$. Then ξ is G-type of $(0, 0, 0; 0, 0, 0)$ and $G(E, S^2 \vee S^2; j)_{\mathbb{Q}} = 0$ from degree argument. Since $\mathrm{rank}\pi_*(S^3 \vee S^3) = \infty$, ξ is not r.G-trivial.

Remark 3. In general (especially without finite condition), if a fibration $X \to E \to B$ has a section, then $G_n(B, E; p) = G_n(B)$ for all n. In fact, we can see that $G_n(B, E; p) \subset G_n(B)$ as follows. Put $s : B \to E$ a section of ξ, i.e.,

$p \circ s \simeq id_B$. For an element $a \in G(B, E; p)$, there is a map $F' : E \times S^n \to B$ satisying $F' \circ inc_1 \simeq (p|a)$. Put $F := F' \circ (s \times id_{S^n})$. Then we have $F \circ inc_2 = F' \circ (s \times id_{S^n}) \circ inc_2 = F' \circ inc_1 \circ (s \vee id_{S^n}) \simeq (p|a) \circ (s \vee id_{S^n}) \simeq (id_B|a)$ in the diagram:

$$\begin{tikzcd}
& B \vee S^n \arrow[rr, "s\vee id_{S^n}"] \arrow[dl, "inc_2"'] \arrow[dd, "(id_B|a)"] & & E \vee S^n \arrow[dl, "inc_1"'] \arrow[dd, "(p|a)"] \\
B \times S^n \arrow[rr, "s\times id_{S^n}"'] \arrow[dr, dotted, "F"'] & & E \times S^n \arrow[dr, "F'"'] & \\
& B \arrow[rr, equal] & & B,
\end{tikzcd}$$

that is, the left triangle homotopically commutes. Thus we have $a \in G_n(B)$. For example, since the free loop fibration of X $\xi_X : \Omega X \to LX \to X$ has the section s: $s(q)$ =the constant loop map to q for a point q of X, $G(X, LX; p) = G(X)$. Finally, the rationalized fibration $\xi_{(0)} : X_{(0)} \to E_{(0)} \to B_{(0)}$ of ξ has a section if and only if a model of it has the property : $(D - d)V \subset \Lambda W \otimes \Lambda^+ V$, where $\Lambda^+ V$ is the ideal of ΛV generated by positive degree elements [2]. The rationalized fibrations of (3) $\sim$ (10) and (13) of Example 2 satisfy it. For them we see $G(B, E; p)_{\mathbb{Q}} = G(B_{(0)}, E_{(0)}; p_{(0)}) = G(B_{(0)}) = G(B)_{\mathbb{Q}}$.

Remark 4. From a fixed fiber X, base B and G-type, we can not determine the rational homotopy equivalent class of fibration $X \to E \to B$ uniquely. We give such two examples (i) and (ii), in which X is the product of spheres, E is finite and $B = K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. Note that there is a free S^1-action on X for each fibration, which is rationally realized as a Borel fibration $X \to ES^1 \times_{S^1} X \to BS^1$ [4, Proposition 4.2].

(i) A fibration $S^3 \times S^5 \times S^9 \to E \to K(\mathbb{Z}, 2)$ is rationally given as

$$(\mathbb{Q}[w], 0) \to (\Lambda(w, x, y, z), D) \to (\Lambda(x, y, z), 0),$$

where $|w| = 2$, $|x| = 3$, $|y| = 5$, $|z| = 9$. From degree argument, it is given by one of (1) $Dx = w^2$ and $Dy = Dz = 0$, (2) $Dy = w^3$ and $Dx = Dz = 0$, (3) $Dz = w^5$ and $Dx = Dy = 0$ or (4) $Dz = wxy + w^5$ and $Dx = Dy = 0$. Then E has the rational homotopy type of (1) $S^2 \times S^5 \times S^9$, (2) $S^3 \times \mathbb{C}P^2 \times S^9$, (3) $S^3 \times S^5 \times \mathbb{C}P^4$ or (4) a 16-dimensional c-symplectic space, i.e., E satisfies that $[w^8] \neq 0 \in H^{16}(E; \mathbb{Q})$, respectively. The G-types of (1), (2) and (3) are $(3, 0, 1; 3, 0, 0)$ and the G-type of (4) is $(3, 0, 1; 1, 0, 0)$.

(ii) A fibration $\xi_\alpha : S^2 \times S^3 \to E \to K(\mathbb{Z}, 2)$ is rationally given as

$$(\mathbb{Q}[w], 0) \to (\Lambda(w, x, y, z), D) \to (\Lambda(x, y, z), d),$$

where $|w| = |x| = 2$, $|y| = |z| = 3$, $dy = x^2$, $dz = 0$, $Dz = wx$ and $Dy = x^2 + \alpha w^2$ for $\alpha \in \mathbb{Q} - \{0\}$. There are infinitely many rationally different classes of fibrations $\{\xi_\alpha\}$ and they are G-type of $(2, 0, 1; 2, 0, 0)$.

4 Appendix

In this section, we put the G-types of fibrations $\xi : X \to E \underset{p}{\to} B$ as $(a, b, c; s, t, u)$.

Lemma A. *When $B \simeq_{(0)} S^{2n+1}$, ξ is rationally trivial if and only if $u = 1$.*

Proof. The 'only if ' part is trivial from

$$u = \mathrm{rank}G(B, E; p) = \mathrm{rank}G(B) = \mathrm{rank}G(S^{2n+1}) = 1.$$

Show the 'if' part. Put $(\Lambda w, 0) \to (\Lambda w \otimes \Lambda V, D) \to (\Lambda V, d)$ be the model of ξ. When $u = 1$, $\delta_E(w^*) = \delta_E(\sigma)$ for some $\sigma \in Der_{2n+1}\Lambda V$. Then $\sigma \otimes w \in Der_0(\Lambda w \otimes \Lambda V, D)$ (see Notation in §2). Put the algebra isomorphism $\psi : \Lambda w \otimes \Lambda V \to \Lambda w \otimes \Lambda V$ as $\psi := id - \sigma \otimes w$. Then we can define a differential D' on $\Lambda w \otimes \Lambda V$ by $D' := \psi^{-1} \circ D \circ \psi$, which is said as *change of KS-basis* in [9, page 119]. In fact, we can easily check $D'(xy) = D'(x)y + (-1)^{|x|}xD'(y)$ and $D' \circ D' = 0$. From $D \circ \psi = \psi \circ D'$, we have an isomorphism of models

$$\begin{tikzcd}
(\Lambda w, 0) \arrow[r, "inc."] \arrow[d, equal] & (\Lambda w \otimes \Lambda V, D') \arrow[r, "proj."] \arrow[d, "\psi", "\cong"'] & (\Lambda V, d) \arrow[d, equal] \\
(\Lambda w, 0) \arrow[r, "inc."] & (\Lambda w \otimes \Lambda V, D) \arrow[r, "proj."] & (\Lambda V, d).
\end{tikzcd}$$

Notice that $(\Lambda w \otimes \Lambda V, D') = (\Lambda w, 0) \otimes (\Lambda V, d)$. In fact,

$$D' = (id + \sigma \otimes w) \circ D \circ (id - \sigma \otimes w) = D - (D \circ \sigma + \sigma \circ D) \otimes w$$

$$= D - \delta_E(\sigma) \otimes w = D - \delta_E(w^*) \otimes w = D - (w^* \circ D) \otimes w \underset{(*)}{=} d,$$

where $d(w) := 0$. For $Dv = dv + w \cdot \tau(v)$ with $v \in V$ and $\tau \in Der_{2n}\Lambda V$, $(*)$ is given by $((w^* \circ D) \otimes w)(v) = (-1)^{|v|}(w^* \circ D)(v) \cdot w = (-1)^{|v|}\tau(v) \cdot w = (-1)^{|v|+|\tau(v)|}w \cdot \tau(v) = w \cdot \tau(v)$. Thus ξ is rationally trivial. ■

Suppose that X is an F_0-space, i.e., $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \cdots, x_l]/(f_1, \cdots, f_l)$ with a regular sequence $(f_1, \cdots, f_l)$ in $\mathbb{Q}[x_1, \cdots, x_l]$, where $|x_*|$ are even. Then $M(X)$ is given by $(\Lambda(x_1, \cdots, x_l, v_1, \cdots, v_l), d)$ with $dx_i = 0, dv_i = f_i \in \Lambda(x_1, \cdots, x_l)$ for $i = 1, \cdots, l$ and $\mathrm{rank}G(X) = \mathrm{rank}\pi_{odd}(X) = l$. Halperin conjectures that any fibration $X \xrightarrow{j} E \to B$ c-splits (is T.N.C.Z.), i.e., $H^*(E; \mathbb{Q}) \cong H^*(X; \mathbb{Q}) \otimes H^*(B; \mathbb{Q})$ additively ($j^* : H^*(E; \mathbb{Q}) \to H^*(X; \mathbb{Q})$ is surjective) [2, p.516]. It is proved in many cases, for example, when $l \leq 3$ [8] and when X is a homogeneous space [12]. It is equivalent to that any fibration $X \to E \to S^{2n+1}$ $(n > 0)$ is rationally trivial [9, Theorem 2.2]. It is known that there are interactions with certain numerical invariants as in [9, §4] and [14].

Corollary A. *Let X be an F_0-space. The followings are equivalent.*
(1) Any fibration $X \to E \to B$ c-splits.
(2) For any fibration $X \to E \to S^{2n+1}$, $u = 1$.
(3) For any fibration $X \to E \to S^{2n+1}$, $\mathrm{rank}G(E) = \mathrm{rank}G(X) + 1$.

(4) Any fibration $X \to E \to S^{2n+1}$ *is G-type of* $(\mathrm{rank}G(X), 0, 1; \mathrm{rank}G(X), 0, 1)$.

Proof. "$(1) \Rightarrow (2), (3), (4)$" follows from [9, Theorem 2.2]. "$(2) \Rightarrow (1)$" follows from Lemma A. "$(3) \Rightarrow (2)$" is given as follows. Since $M(X)$ is pure, $t = 0$ (see Claim in §2). Then we have $u \geq 1$ from the inequations $s + u = s + t + u = \mathrm{rank}G(E) = \mathrm{rank}G(X) + 1$ and $s \leq \mathrm{rank}G(X)$. "$(4) \Rightarrow (2)$" is trivial. ■

Note that the condition: " *(5) Any fibration* $X \to E \to B$ *is G-type of* $(\mathrm{rank}G(X), 0, \mathrm{rank}G(B); \mathrm{rank}G(X), 0, \mathrm{rank}G(B))$ " is sufficient but not necessary for the conditions in Corollary A. In fact, there is a c-split fibration $S^4 \to E \to S^3 \times S^5$ with the model

$$(\Lambda(w_1, w_2), 0) \to (\Lambda(w_1, w_2, x_1, v_1), D) \to (\Lambda(x_1, v_1), d)$$

where $|w_1| = 3, |w_2| = 5, |x_1| = 4, |v_1| = 7$, $dx_1 = Dx_1 = 0$, $dv_1 = x_1^2$ and $Dv_1 = x_1^2 + w_1 w_2$. It is G-type of $(1, 0, 2; 1, 0, 0)$.

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