Blow-up for solution of a system of quasilinear hyperbolic equations involving the $p-$laplacian

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Abstract

We study the blow-up for the solution of a system of quasilinear hyperbolic equations involving the p-laplacian. We derive a differential inequality for a function involving some norms of the solution which yields the finite time blow-up.

1 Introduction

We are concerned with the blow-up of solutions of the initial boundary value problem for a class of quasilinear system of hyperbolic equations in a bounded domain $\Omega \subset \mathbf{R}^n$ $(n \geq 1)$ with a sufficiently smooth boundary $\partial\Omega$:

$$u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) - \Delta u_t + |v_t|^{\alpha_1} |u_t|^{\beta_1} \operatorname{sign}(u_t) = |u|^{m_1 - 1} u \text{ in } \Omega \times (0, T), \tag{1}$$

$$v_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x} \right|^{p-2} \frac{\partial v}{\partial x_i} \right) - \Delta v_t + |v_t|^{\alpha_2} |u_t|^{\beta_2} \operatorname{sign}(v_t) = |v|^{m_2 - 1} v \text{ in } \Omega \times (0, T), \tag{2}$$

$$u(x,t) = 0,\ v(x,t) = 0 \ \ on\ \partial\Omega \times (0,T), \tag{3}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \ \text{ in } \Omega, \tag{4}$$

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x) \ \text{ in } \Omega, \tag{5}$$

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where $0 < T \leq \infty$, p, m_i, α_i, β_i $(i = 1, 2)$ are positive numbers subjected to some appropriate restrictions.

The problem is related to a class of nonlinear evolutions equations of the type

$$\begin{cases} \varphi_{tt} + A(t)\varphi - B(t)\varphi_t + D(t)\varphi_t = F(t)\varphi \\ \varphi(0) = \varphi_0 \\ \varphi_t(0) = \varphi_1 \end{cases} \tag{6}$$

where A, B, D and F are some nonlinear operators. Issues of global existence under various conditions were considered in [1], [14]; see the references in these papers. Equations of this type arise in several areas of physics. The most common of them being the case when

$$A\varphi = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\vartheta_i\left(\frac{\partial \varphi}{\partial x}\right) \text{ and } B\varphi_t = \Delta\varphi_t$$

which describes the longitudinal motion of a viscoelastic bar obeying the nonlinear Voight model. Physically the strong damping term $-\Delta\varphi_t$ and the nonlinear dissipative damping term $D(t)\varphi_t$ play a dissipative role in the energy accumulation in the configurations of viscoelastic materials, while the nonlinear source term $F(t)\varphi$ leads to the gathering of energy in the configurations. The interaction between these terms may lead to a lack of synchronization in the energy accumulation and as a result the configuration may break or burn out in finite time, this mathematically is expressed through the finite time blow-up of the solution.

Here we consider an initial boundary value problem involving a system of nonlinear hyperbolic equations with slightly more general nonlinear damping terms.

The case without sources terms which lead to global existence was considered in [1]. The study of finite time blow-up involving one equation (thus one of the parameters $\alpha_i = 0$ and $m_1 = m_2$) was considered in [14], [15].

The approach in the present paper follows closely that of [2], [7], [9]. We refer also to the important papers devoted to related questions such as [4], [3], [5], [6], [8], [10], [11], [12](this paper treats hyperbolic systems with source terms without damping) and in the several references therein; the approach in some of these papers is mainly based on the potential well method which originated in the work of Sattinger [13]. We note that semilinear equations and systems (when $p = 2$) are the ones that have been widely studied. Nonlinear hyperbolic problems involving the p-Laplacian are becoming the object of increasing interest only in recent years.

The paper is organized as follows. In section 2, we state our main result. Section 3 is devoted to the proof of the main result through the derivation of a suitable differential inequality satisfied by a function involving some norms of the solution.

2 Preliminaries

We introduce some notations. By $L_p(\Omega)$ $(p \geq 1)$ we denote the set of integrable functions u on Ω, such that the norm

$$||u||_{L_p(\Omega)} = \left(\int_\Omega |u|^p \, dx\right)^{1/p} < \infty.$$

Let X be a Banach space. By the symbol $L_p(0,T,Y)$ we mean the functions $u(x,t)$ that are L_p-integrable from $[0,T]$ into X and with the norm

$$||u||_{L_p(0,T,X)} = \left(\int_0^T ||u(t)||_X^p \, dt\right)^{1/p}, \; 1 \leq p < \infty$$

and

$$||u||_{L_\infty(0,T,X)} = ess \sup_{t\in[0,T]} ||u(t)||_X \, .$$

For $p > 1$, we consider the function space

$$\overset{o}{W}{}_p^1(\Omega) = \left\{ u \in L_p(\Omega) : u|_{\partial\Omega} = 0, \; \frac{\partial u}{\partial x_i} \in L_p(\Omega), i = 1, ..., n \right\},$$

with the norm

$$||u||_{\overset{o}{W}{}_p^1(\Omega)} = \left(\sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx\right)^{1/p};$$

when $p = 2$, we denote $\overset{o}{W}{}_p^1(\Omega)$ by $H_0^1(\Omega)$. We denote by X^2 the Cartesian product of a set X with itself. The letter C will stand for all constants depending only on the data.

We introduce the functions

$$H_u(t) = -\frac{1}{2}\int_\Omega u_t^2 dx - \frac{1}{p}\int_\Omega \left|\frac{\partial u}{\partial x}\right|^p dx + \frac{1}{m_1+1}\int_\Omega |u|^{m_1+1} dx, \tag{7}$$

$$H_v(t) = -\frac{1}{2}\int_\Omega v_t^2 dx - \frac{1}{p}\int_\Omega \left|\frac{\partial v}{\partial x}\right|^p dx + \frac{1}{m_2+1}\int_\Omega |v|^{m_2+1} dx, \tag{8}$$

$$H(t) = H_u(t) + H_v(t) \tag{9}$$

$$F(t) = ||u(t)||_{L_2(\Omega)}^2 + ||v(t)||_{L_2(\Omega)}^2 + \int_0^t \int_\Omega \left[\left|\frac{\partial u(\tau)}{\partial x}\right|^2 + \left|\frac{\partial v(\tau)}{\partial x}\right|^2\right] dx d\tau; \tag{10}$$

for sake of simplicity we denote from now on the norm $||\cdot||_{L_p(\Omega)}$ by $||\cdot||_p$.

Our main result is

Theorem 1. *Let $U = (u, v)$ be a local weak solution of problem (1)-(5), in the sense that there exists a number $0 < T < \infty$ such that*

$$U \in \left[C\left(0, T, \overset{o}{W}{}_p^1(\Omega)\right)\right]^2 \cap C(0, T, L_{m_1+1}(\Omega)) \times C(0, T, L_{m_2+1}(\Omega)), \tag{11}$$

$$U_t = (u_t, v_t) \in [C(0, T, L_2(\Omega))]^2 \cap \left[L_2\left(0, T, H_0^1(\Omega)\right)\right]^2 \tag{12}$$

$$\int_0^T \int_\Omega \left[|v_t|^{\alpha_1} |u_t|^{\beta_1+1} + |v_t|^{\alpha_2+1} |u_t|^{\beta_2}\right] dxdt \textit{ is finite} \tag{13}$$

$$U_0 = (u_0, v_0) \in \left[\overset{o}{W}{}_p^1(\Omega)\right]^2, \; U_1 = (u_1, v_1) \in [L_2(\Omega)]^2 \tag{14}$$

and u satisfies (1)-(5) in the weak sense. Furthermore we assume that

$$H_u(0) \geq C_1 > 0,\ H_v(0) \geq C_2 > 0\ F'(0) = \int_\Omega [u_0u_1 + v_0v_1]\, dx > 0 \tag{15}$$

$$\alpha_i > 0,\ \beta_i > 0,\ m_i > 1\ (i = 1, 2),\ 0 < \alpha_1 - \alpha_2 < 1,\ 0 < \beta_2 - \beta_1 < 1 \tag{16}$$

$$\frac{\alpha_2 + 1}{\alpha_1} = \frac{\beta_2}{\beta_1} > \frac{m_1 + 1}{m_1},\ \frac{\alpha_1}{\alpha_2} = \frac{\beta_1 + 1}{\beta_2} > \frac{m_2 + 1}{m_2}, \tag{17}$$

$$\max_i \left\{2, \frac{n(m_i + 1)}{n + m_i + 1}\right\} \leq p < \min_i \{m_i + 1, n\}. \tag{18}$$

Then u blows up in finite time, i.e., there exists a $T_0 > 0$ such that

$$\lim_{t \to T_0^-} \left[||u(t)||_{m_1+1}^{m_1+1} + ||v(t)||_{m_2+1}^{m_2+1} + ||U_t(t)||_2^2\right] = \infty.$$

Remark 2. *The constants C_1 and C_2 will be chosen later. Some few words about questions related to the problem (1)-(5) are in order. The global existence without the source terms was considered in [1]. In particular it was shown that if $p \geq 2$, $0 < \beta_1 < 1 - \alpha_1$, $0 < \alpha_2 < 1 - \beta_2$, $0 < \alpha_1 < 1$, $0 < \beta_2 < 1$ and the above conditions (14) are imposed on the initial data, then a global weak solution exists and decay estimates under further conditions were derived.*

3 Proof of the theorem

The blow-up result will follow from a differential inequality satisfied by the function

$$W(t) = H^{1-\alpha}(t) + \varepsilon F'(t),$$

where α and ε are small parameters that will be chosen in the sequel. This idea goes back to Ball [2].

We start with the derivation of some useful informations on the function H which follow from a suitable identity.

Multiplying the equations (1) and (2) by u_t and v_t respectively and integrating over Ω, we get

$$\frac{d}{dt}\left[\frac{1}{2}||u_t||_2^2 + \frac{1}{p}\left|\left|\frac{\partial u}{\partial x}\right|\right|_p^p - \frac{1}{m_1 + 1}||u||_{m_1+1}^{m_1+1}\right] = -\left|\left|\frac{\partial u_t}{\partial x}\right|\right|_2^2 - \int_\Omega |v_t|^{\alpha_1}|u_t|^{\beta_1+1}\, dx$$

$$\frac{d}{dt}\left[\frac{1}{2}||v_t||_2^2 + \frac{1}{p}\left|\left|\frac{\partial v}{\partial x}\right|\right|_p^p - \frac{1}{m_2 + 1}||v||_{m_2+1}^{m_2+1}\right] = -\left|\left|\frac{\partial v_t}{\partial x}\right|\right|_2^2 - \int_\Omega |v_t|^{\alpha_2+1}|u_t|^{\beta_2}\, dx$$

This implies that

$$H_u' = \left|\left|\frac{\partial u_t}{\partial x}\right|\right|_2^2 + \int_\Omega |v_t|^{\alpha_1}|u_t|^{\beta_1+1}\, dx, \tag{19}$$

and

$$H_v' = \left\| \frac{\partial v_t}{\partial x} \right\|_2^2 + \int_\Omega |v_t|^{\alpha_2+1} |u_t|^{\beta_2} \, dx, \tag{20}$$

that is H_u and H_v are non decreasing. Thus from the assumption on $H_u(0)$ and $H_v(0)$, it follows that for all $t \geq 0$,

$$0 < H_u(0) \leq H_u(t) \leq \frac{1}{m_1+1} ||u||_{m_1+1}^{m_1+1} \leq C \left\| \frac{\partial u}{\partial x} \right\|_p^{m_1+1}, \tag{21}$$

$$0 < H_v(0) \leq H_v(t) \leq \frac{1}{m_2+1} ||u||_{m_2+1}^{m_2+1} \leq C \left\| \frac{\partial v}{\partial x} \right\|_p^{m_2+1}, \tag{22}$$

where we have used the fact that $\overset{o}{W}{}_p^1(\Omega)$ is embedded into $L_{m_i+1}(\Omega)$ since by the conditions (18) on p, $pn/(n-p) \geq \max_i \{m_i+1\}$.

By approximating u with sufficiently smooth functions with respect to t, we can see that F'' satisfies

$$\begin{aligned} F''(t) &= 2\int_\Omega \left(u_t^2 + v_t^2\right) dx + 2\int_\Omega (uu_{tt} + vv_{tt})\, dx \\ &\quad +2\int_\Omega \sum_{i=1}^n \left[\frac{\partial u}{\partial x_i}\frac{\partial u_t}{\partial x_i} + \frac{\partial v}{\partial x_i}\frac{\partial v_t}{\partial x_i}\right] dx \\ &= 2\left(||u_t||_2^2 - \left\| \frac{\partial u}{\partial x} \right\|_p^p - \int_\Omega |v_t|^{\alpha_1} |u_t|^{\beta_1} u \mathrm{sign} u_t dx + ||u||_{m_1+1}^{m_1+1} \right) \\ &\quad +2\left(||v_t||_2^2 - \left\| \frac{\partial v}{\partial x} \right\|_p^p - \int_\Omega |v_t|^{\alpha_2} |u_t|^{\beta_2} v \mathrm{sign} v_t dx + ||v||_{m_2+1}^{m_2+1} \right). \end{aligned} \tag{23}$$

Thus substituting F'' as expressed in (23) into the relation

$$\frac{d}{dt}\left(H^{1-\alpha}(t) + \varepsilon F'(t)\right) = (1-\alpha) H^{-\alpha} H + \varepsilon F''(t),$$

and using the definition of H, we get

$$\begin{aligned} &\frac{d}{dt}\left(H^{1-\alpha}(t) + \varepsilon F'(t)\right) \\ = &\ (1-\alpha) H^{-\alpha} H' + \varepsilon (2+p) ||U_t||_2^2 + 2p\varepsilon H \\ &+\varepsilon\left(2 - \frac{2p}{m_1+1}\right) ||u||_{m_1+1}^{m_1+1} + \varepsilon\left(2 - \frac{2p}{m_2+1}\right) ||v||_{m_2+1}^{m_2+1} \\ &-2\varepsilon \int_\Omega |v_t|^{\alpha_1} |u_t|^{\beta_1} u \mathrm{sign} u_t dx - 2\varepsilon \int_\Omega |v_t|^{\alpha 2} |u_t|^{\beta_2} v \mathrm{sign} v_t dx. \end{aligned} \tag{24}$$

Let us denote the two last integrals by I_1 and I_2, respectively and estimate them.

By Hölder's inequality, we have

$$\begin{aligned} I_1 &\leq \int_\Omega |v_t|^{\alpha_1} |u_t|^{\beta_1} |u|\, dx \\ &\leq \left(\int_\Omega \left(|v_t|^{\alpha_1} |u_t|^{\beta_1}\right)^{(m_1+1)/m_1} dx\right)^{m_1/(m_1+1)} \left(\int_\Omega |u|^{m_1+1} dx\right)^{1/(m_1+1)} \\ &\leq \left(\int_\Omega |v_t|^{\alpha_2+1} |u_t|^{\beta_2} dx\right)^{\beta_1/\beta_2} \left(\int_\Omega |u|^{m_1+1} dx\right)^{1/(m_1+1)} \end{aligned}$$

where in the last inequality we used the restrictions (16) and (17).

Next writing

$$\frac{1}{m_1+1} = \frac{\beta_2-\beta_1}{\beta_2} + \sigma_1;\ \sigma_1 = \frac{1}{m_1+1} - \frac{\beta_2-\beta_1}{\beta_2},$$

we have $\sigma_1 < 0$, and using Young's inequality we get

$$\begin{aligned} I_1 &\leq \left(\int_\Omega |v_t|^{\alpha_2+1} |u_t|^{\beta_2} dx\right)^{\beta_1/\beta_2} \left(\int_\Omega |u|^{m_1+1} dx\right)^{\sigma_1} \left(\int_\Omega |u|^{m_1+1} dx\right)^{(\beta_2-\beta_1)/\beta_2} \\ &\leq C\left[\int_\Omega |v_t|^{\alpha_2+1} |u_t|^{\beta_2} dx + \int_\Omega |u|^{m_1+1} dx\right] \left(\int_\Omega |u|^{m_1+1} dx\right)^{\sigma_1}. \end{aligned}$$

By (21) and (20), we get

$$2\varepsilon I_1 \leq C\varepsilon \left(H_u(t)\right)^{\sigma_1} \left[H_v'(t) + \int_\Omega |u|^{m_1+1} dx\right]. \tag{25}$$

Analogously we obtain

$$2\varepsilon I_2 \leq C\varepsilon \left(H_v(t)\right)^{\sigma_2} \left[H_u'(t) + \int_\Omega |v|^{m_2+1} dx\right];\ \sigma_2 = \frac{1}{m_2+1} - \frac{\alpha_1-\alpha_2}{\alpha_1}. \tag{26}$$

Combining these two last inequalities with (21)-(22), it follows that

$$2\varepsilon\left(I_1+I_2\right) \leq 2\varepsilon\left[H_u^{\sigma_1}(0) + H_v^{\sigma_2}(0)\right]\left[H'(t) + ||u||_{m_1+1}^{m_1+1} + ||v||_{m_2+1}^{m_2+1}\right].$$

Taking account of (24), the following relation holds:

$$\begin{aligned} &\frac{d}{dt}\left(H^{1-\alpha}(t) + \varepsilon F'(t)\right) \\ \geq\ &\left\{(1-\alpha) H^{-\alpha}(0) - 2\varepsilon C\left[H_u^{\sigma_1}(0) + H_v^{\sigma_2}(0)\right]\right\} H'(t) + \varepsilon(2+p)||U_t||_2^2 + 2p\varepsilon H \\ &+\varepsilon\left(2 - \frac{2p}{m_1+1} - C\left[H_u^{\sigma_1}(0) + H_v^{\sigma_2}(0)\right]\right)||u||_{m_1+1}^{m_1+1} \\ &+\varepsilon\left(2 - \frac{2p}{m_2+1} - C\left[H_u^{\sigma_1}(0) + H_v^{\sigma_2}(0)\right]\right)||v||_{m_2+1}^{m_2+1}. \end{aligned} \tag{27}$$

We take

$$\alpha \in \left(0, \min\left\{\frac{m_1-1}{2(m_1+1)}, \frac{m_2-1}{2(m_2+1)}, \frac{p-2}{p}\right\}\right), \tag{28}$$

and choose $\varepsilon > 0$ such that

$$(1-\alpha) H^{-\alpha}(0) - 2\varepsilon C\left[H_u^{\sigma_1}(0) + H_v^{\sigma_2}(0)\right] > 0.$$

Also we choose the constants C_1 and C_2 in (15) in such a way that

$$C_1^{\sigma_1} + C_2^{\sigma_2} \leq \min_i \left\{ \frac{1}{C} \left(2 - \frac{2p}{m_i + 1} \right) \right\}.$$

Then the inequalities hold:

$$2 - \frac{2p}{m_1 + 1} - C \left[H_u^{\sigma_1}(0) + H_v^{\sigma_2}(0) \right] > 0$$

$$2 - \frac{2p}{m_2 + 1} - C \left[H_u^{\sigma_1}(0) + H_v^{\sigma_2}(0) \right] > 0.$$

Thus from (19), (20) it follows that

$$\frac{d}{dt} \left(H^{1-\alpha}(t) + \varepsilon F'(t) \right) \geq C \left(||U_t||_2^2 + H(t) + ||u||_{m_1+1}^{m_1+1} + ||u||_{m_2+1}^{m_2+1} \right); \tag{29}$$

as a consequence we have that $W(t)$ is increasing since $H(t) > 0$ by (21)-(22). Therefore using the assumption that $F'(0) > 0$, we get

$$W(t) > 0, \ \forall t \geq 0.$$

We make a further restriction on α by requiring that $0 < \alpha < 1/2$. Then setting $\beta = 1/(1 - \alpha)$ (i.e., $2 > \beta > 1$) we claim the inequality

$$W'(t) \geq C W^{\beta}(t). \tag{30}$$

For the proof of (30), we consider two alternatives:

- If there exists a $t > 0$ such that $F'(t) < 0$, then

$$\left(H^{1-\alpha}(t) + \varepsilon F'(t) \right)^{\beta} \leq H(t). \tag{31}$$

  Thus (30) follows from (29).

- If there exists a $t > 0$ such that $F(t) \geq 0$, then using Holder's and Young's inequalities we get

$$\begin{aligned} [F'(t)]^{\beta} &= \left(2 \int_{\Omega} [u_t u + v_t v] \, dx + \left\| \frac{\partial u}{\partial x} \right\|_2^2 + \left\| \frac{\partial v}{\partial x} \right\|_2^2 \right)^{\beta} \\ &\leq C \left[||u||_2^{\lambda_1 \beta} + ||u_t||_2^{\mu_1 \beta} + ||v||_2^{\lambda_2 \beta} + ||v_t||_2^{\mu_2 \beta} + \left\| \frac{\partial u}{\partial x} \right\|_2^{2\beta} + \left\| \frac{\partial v}{\partial x} \right\|_2^{2\beta} \right] \end{aligned} \tag{32}$$

  where $\lambda_i^{-1} + \mu_i^{-1} = 1$. We take $\mu_i \beta = 2, \ i = 1, 2$. Thus $\mu_1 = \mu_2 = 2/\beta$ and

$$\lambda_1 = \lambda_2 = \lambda = \frac{2(1 - \alpha)}{1 - 2\alpha}.$$

  By the restrictions on α, we have

$$\lambda \beta = \frac{2}{1 - 2\alpha} \leq \min \{ m_i + 1 \}, \ 2\beta = \frac{2}{1 - \alpha} \leq p.$$

Thus from (32), using Hölder's inequality we have

$$\begin{aligned} [F'(t)]^{\beta} &\leq C\left[||u||_{m_1+1}^{\lambda\beta} + ||u_t||_2^2 + ||v||_{m_2+1}^{\lambda\beta} + ||v_t||_2^2 + \left|\left|\frac{\partial u}{\partial x}\right|\right|_p^{2\beta} + \left|\left|\frac{\partial v}{\partial x}\right|\right|_p^{2\beta} \right] \\ &= C\left\{ ||u||_{m_1+1}^{\lambda\beta-(m_1+1)} ||u||_{m_1+1}^{m_1+1} + ||U_t||_2^2 + ||v||_{m_2+1}^{\lambda\beta-(m_2+1)} ||v||_{m_2+1}^{m_2+1} \right. \\ &\quad \left. + \left|\left|\frac{\partial u}{\partial x}\right|\right|_p^{2\beta-p} \left|\left|\frac{\partial u}{\partial x}\right|\right|_p^{p} + \left|\left|\frac{\partial v}{\partial x}\right|\right|_p^{2\beta-p} \left|\left|\frac{\partial v}{\partial x}\right|\right|_p^{p} \right\}. \end{aligned}$$

¿From the estimates (21) and (22) we deduce that

$$\begin{aligned} [F'(t)]^{\beta} &\leq C\left\{ [H_u(0)]^{[\lambda\beta-(m_1+1)]/[m_1+1]} ||u||_{m_1+1}^{m_1+1} \right. \\ &\quad + ||U_t||_2^2 + [H_v(0)]^{[\lambda\beta-(m_2+1)]/[m_2+1]} ||v||_{m_2+1}^{m_2+1} \\ &\quad \left. + [H_u(0)]^{(2\beta-p)/(m_1+1)} \left|\left|\frac{\partial u}{\partial x}\right|\right|_p^{p} + [H_v(0)]^{(2\beta-p)/(m_2+1)} \left|\left|\frac{\partial v}{\partial x}\right|\right|_p^{p} \right\}. \end{aligned}$$

Thus

$$[F'(t)]^{\beta} \leq C\left[||u||_{m_1+1}^{m_1+1} + ||v||_{m_2+1}^{m_2+1} + ||U_t||_2^2 + \left|\left|\frac{\partial u}{\partial x}\right|\right|_p^{p} + \left|\left|\frac{\partial v}{\partial x}\right|\right|_p^{p} \right].$$

From the definition of H we have

$$H(t) + \frac{1}{p}\left[\left|\left|\frac{\partial u}{\partial x}\right|\right|_p^{p} + \left|\left|\frac{\partial v}{\partial x}\right|\right|_p^{p} \right] \leq \frac{1}{m_1+1} ||u||_{m_1+1}^{m_1+1} + \frac{1}{m_2+1} ||v||_{m_2+1}^{m_2+1}.$$

Thus

$$\begin{aligned} [F'(t)]^{\beta} &\leq C\left[||u||_{m_1+1}^{m_1+1} + ||v||_{m_2+1}^{m_2+1} + ||U_t||_2^2 + \left|\left|\frac{\partial u}{\partial x}\right|\right|_p^{p} + \left|\left|\frac{\partial v}{\partial x}\right|\right|_p^{p} + H(t) \right] \\ &\leq C\left[||u||_{m_1+1}^{m_1+1} + ||v||_{m_2+1}^{m_2+1} + ||U_t||_2^2 \right], \end{aligned}$$

and hence

$$\left(H^{1-\alpha}(t) + \varepsilon F'(t)\right)^{\beta} \leq C\left[||u||_{m_1+1}^{m_1+1} + ||v||_{m_2+1}^{m_2+1} + ||U_t||_2^2 \right]. \tag{33}$$

This inequality together with (29) imply (30).

Now integrating both sides of (30) over the interval $[0,t]$, it follows that there exists a $T_0 > 0$ such that

$$\lim_{t\to T_0^-} \left(H^{1-\alpha}(t) + \varepsilon F'(t)\right) = \infty.$$

This limit combined with (33), (31), (21) and (22) give

$$\lim_{t\to T_0^-} \left[||u(t)||_{m_1+1}^{m_1+1} + ||v(t)||_{m_2+1}^{m_2+1} + ||U_t(t)||_2^2 \right] = \infty.$$

The theorem is proved.

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