Gagliardo-Nirenberg inequalities in weighted Orlicz spaces equipped with a nonnecessarily doubling measure*

Agnieszka Kałamajska Katarzyna Pietruska-Pałuba

Abstract

We obtain Gagliardo-Nirenberg interpolation inequalities of the form $\|\nabla u\|_X \leq C_1\sqrt{\|u\|_Y\|\nabla^{(2)}u\|_Z} + C_2\|u\|_Y$, where X, Y, Z are Orlicz spaces related to a single measure which may not satisfy the doubling condition. Some examples among homogeneous, logarithmic and exponential spaces are given.

1 Introduction

Gagliardo-Nirenberg interpolation inequalities [18, 40]

$$\|\nabla^{(k)}u\|_q \leq \|u\|_p^{1-k/m}\|\nabla^{(m)}u\|_r^{k/m}, \tag{1.1}$$

where $u \in C_0^\infty(\mathbb{R}^n)$, $\frac{1}{q} = (1-\frac{k}{m})\frac{1}{p} + \frac{k}{m}\frac{1}{r}$, and $\nabla^{(l)}u = (D^\alpha u)_{|\alpha|=l}$, play an important role in the *apriori* estimates in linear and nonlinear PDE's and their applications to the regularity theory.

*This research was done while A.K. was visiting Institute of Mathematics of the Polish Academy of Sciences at Warsaw. This author wants to thank IM PAN for its hospitality. The work of both authors is supported by a KBN grant no. 1-PO3A-008-29, and also by the EC FP6 Marie Curie ToK programme SPADE2, MTKD-CT-2004-014508, and Polish MNiSW SPB-M.

Received by the editors February 2007.

Communicated by J. Mawhin.

2000 *Mathematics Subject Classification :* Primary 26D10; Secondary 46E35, 54C30, 35A25.

Key words and phrases : Gagliardo-Nirenberg inequalities, Orlicz-Sobolev spaces, nondoubling measures.

Physical motivations (see e.g. [1, 2, 11, 14, 42] and references therein) indicate a need to consider also PDE's with solutions not in the classical Sobolev spaces, but in Sobolev-like spaces related to Orlicz norms rather than L^p norms. Consequently it is natural to look for an extension of (1.1), where L^p–norms would be replaced with Orlicz norms.

On the other hand, it is motivated by numerous areas of mathematics (e.g. the theory of functions, imbedding theorems, spectral theory of differential operators, boundary value problems, regularity theory, degenerate P.D.E's, singular integral equations, theory of analytic functions) to investigate Sobolev spaces with respect to a general measure, not only the Lebesgue measure. We refer to [17, 26, 27, 30, 31, 38, 39, 44] and references therein for the theory and the motivations. Hence it is also natural to ask for extensions of (1.1) to L^s spaces equipped with a general Radon measure. For results in this direction we refer e.g. to [7, 10, 12, 13, 20, 21], Theorem 1 in Section 1.4.7 of [35] and their references.

In this paper we obtain the following variant of inequality (1.1) in the case $k = 1$, $m = 2$:

$$\|\nabla u\|_X \le C_1\sqrt{\|u\|_Y\|\nabla^{(2)}u\|_Z} + C_2\|u\|_Y, \tag{1.2}$$

where X, Y, Z are weighted Orlicz spaces, (if the measure is finite then Y is a proper subspace of X) and constants C_1, C_2 are independent of $u \in C_0^\infty(\mathbb{R}^n)$. The exact statement is given in Theorem 4.1, see also Remark 4.4.

There has been already a fair number of papers on Gagliardo-Nirenberg inequalities (see e.g. [8, 9, 16, 28, 32, 33, 36, 37]), but not much is known on inequalities similar to (1.2) within Orlicz spaces, even in the nonweighted case. Previous research in this area comes from two sources. First, Bang and coauthors [3, 4, 5, 6] examined the nonweighted case for one-variable function, within a single Orlicz space $L^M(dx)$. We have recently obtained variants of (1.2) (with possibly different Orlicz spaces) in the nonweighted case [22, 23, 24], and also in the weighted case [25], for measures which necessarily satisfy the doubling condition $\mu(2B) \le C\mu(B)$ (B is an arbitrary ball, $2B$ is the ball with the same center as B and twice the radius, and the constant C does not depend on B).

Here we extend some of techniques originating in [22] and generalized in [23]. Those techniques allowed previously to prove inequalities of the form (1.2), with $C_2 = 0$, in nonweighted Orlicz spaces.

The class of weights covered by our approach seems to be rather big. We require them to be absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^n$, $\mu(dx) = w(x)dx$, and further $w(x) = \exp(-\varphi(x))$, with $\varphi \in C^1$ and $|\nabla\varphi|$ in the given Orlicz space with respect to the measure μ. See Theorem 4.1 for the precise statement. In particular exponential-type measures $\mu(dx) = C_1\exp(-C_2|x|^\alpha)dx$ where $C_1, C_2 > 0, \alpha \ge 0$, so also the Gaussian measure $\mu(dx) = \frac{1}{\sqrt{2\pi n}}\exp(-|x|^2/2)dx$, are in many cases allowed in our inequalities (see Remarks 6.1 and 6.3). Moreover, typical weights considered in this paper do not satisfy the doubling condition and so the result obtained in this paper is complementary to the results obtained by independent techniques in the paper [25], where all the measures considered were doubling.

As nondoubling measures seem to be of separate interest (see e.g. [19],[34],[41] for some recent results), we hope to contribute in this direction too.

Some examples illustrating our approach within homogeneous Young functions, and also within functions of logarithmic and exponential type, are given in Section 6.

2 Preliminaries

2.1 Notation

Throughout the paper, the symbol $\nabla^{(k)}u$ stands for the k–th gradient of the mapping $u:\mathbb{R}^n \to \mathbb{R}$: the vector $(D^\alpha u)_{|\alpha|=k}$. If A is a vector or a matrix, by $|A|$ we denote its Euclidean norm induced by the standard scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$, while A^t stands for its transposition.

By c we denote a general constant which can change even within the same line. Upper case letters $C, D, \ldots$ are reserved for those constants whose value is relevant. By s^* we denote the Hölder conjugate to a number $s > 1$. We use the standard notation: $C_0^\infty(\mathbb{R}^n)$ stands for smooth compactly supported functions on $\mathbb{R}^n$, and $L^p(\mathbb{R}^n)$, $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $W^{k,p}(\mathbb{R}^n)$, $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for the L^p and Sobolev spaces respectively defined on $\mathbb{R}^n$. In general the L^p spaces (if not said otherwise then defined on $\mathbb{R}^n$) subordinated to the measure μ will be denoted by $L^p(\mu)$.

If f is defined on the set Ω, then by $f\chi_\Omega$ we denote this function extended by zero outside Ω.

Now we recall some preliminary facts about Orlicz spaces, referring e.g. to [29] or [43] for details.

2.2 $N-$functions

A function $\Phi:[0,\infty) \to [0,\infty)$ is called an $N-$function if it is differentiable, strictly convex, $\lim\limits_{\lambda\to 0^+} \Phi(\lambda)/\lambda = 0$ and $\lim\limits_{\lambda\to\infty} \Phi(\lambda)/\lambda = \infty$.

Note that in particular every $N-$function Φ satisfies $\Phi(0) = \Phi'_+(0) = 0$.

The *Legendre transform* of an $N-$function Φ, denoted by Φ^*, is defined as

$$\Phi^*(y) = \sup_{x\geq 0}[xy - \Phi(x)].$$

It is known that Φ^* is also an $N-$function. The Legendre transform is an involution, i.e. $(\Phi^*)^* = \Phi$. Moreover, for every $x, y \geq 0$ the Young inequality is satisfied:

$$xy \leq \Phi(x) + \Phi^*(y). \tag{2.1}$$

Functions Φ and Φ^* are called *mutually conjugate*.

Definition 2.1 (Δ_2-condition). *A differentiable function Φ: $[0,\infty) \to [0,\infty)$ such that $\Phi(0) = 0$ satisfies the Δ_2-condition if and only if*

$$\lambda\Phi'(\lambda) \leq c\Phi(\lambda), \tag{2.2}$$

with a positive constant c not depending on $\lambda > 0$.

For an $N-$function Φ, inequality (2.2) is equivalent to the doubling condition

$$\Phi(2\lambda) \leq c\Phi(\lambda), \tag{2.3}$$

with the constant $c > 0$ not depending on λ.

We will use the following auxiliary functions: for a given $N-$function Φ, we shall write

$$\Phi_1(\lambda) = \frac{\Phi(\lambda)}{\lambda}, \qquad \Phi_2(\lambda) = \frac{\Phi(\lambda)}{\lambda^2}, \qquad \widetilde{\Phi}(\lambda) = \frac{\Phi'(\lambda)}{\lambda}. \tag{2.4}$$

Observe that for any convex Φ one has $\Phi_2(\lambda) \leq \widetilde{\Phi}(\lambda)$, whereas for a function satisfying the Δ_2-condition one has $\widetilde{\Phi}(\lambda) \leq c\Phi_2(\lambda)$, with $c > 0$ independent of λ.

2.3 Weighted Orlicz spaces

Suppose that μ is a positive Radon measure on $\mathbb{R}^n$ and let $\Phi : [0, \infty) \to [0, \infty)$ be an $N-$function. The weighted space $L^\Phi(\mu)$ with respect to the measure μ is, by definition, the function space

$$L^\Phi(\mu) \stackrel{def}{=} \{f \text{ measurable}\colon \int_{\mathbb{R}^n} \Phi(\frac{|f(x)|}{K}) d\mu(x) \leq 1 \;\text{ for some }\; K > 0\},$$

equipped with the Luxemburg norm

$$\|f\|_{(\Phi,\mu)} = \inf\{K > 0\colon \int_{\mathbb{R}^n} \Phi(\frac{|f(x)|}{K}) d\mu(x) \leq 1\}.$$

This norm is complete and turns $L^\Phi(\mu)$ into a Banach space. Moreover, for $\Phi(\lambda) = \lambda^p$ with $p > 1$, the space $L^\Phi(\mu)$ coincides with the usual $L^p(\mu)$ space.

We recall the following two properties of Young functionals: for every $f \in L^\Phi(\mu)$ we have

$$\|f\|_{(\Phi,\mu)} \leq \int_{\mathbb{R}^n} \Phi(|f(x)|) d\mu(x) + 1, \tag{2.5}$$

and

$$\int_{\mathbb{R}^n} \Phi(\tfrac{f(x)}{\|f\|_{(\Phi,\mu)}})\, d\mu(x) \leq 1. \tag{2.6}$$

When Φ satisfies the Δ_2-condition, then (2.6) becomes an equality.

2.4 Domination and comparison of norms

We say that the function Φ dominates Ψ (symbolically: $\Psi \prec \Phi$) if there exist two positive constants K_1, K_2 such that

$$\Psi(\lambda) \leq K_1 \Phi(K_2 \lambda) \text{ for every } \lambda > 0. \tag{2.7}$$

We have:

$$\text{when } \Psi \prec \Phi, \text{ then } \|\cdot\|_{(\Psi,\mu)} \leq K \|\cdot\|_{(\Phi,\mu)}, \text{ with } K = K_2(K_1 + 1). \tag{2.8}$$

Functions Φ and Ψ are called equivalent (symbolically $\Phi_1 \asymp \Psi$) when $\Psi \prec \Phi$ and $\Phi \prec \Psi$. It is clear that equivalent $N-$functions give raise to equivalent Luxemburg norms.

If (2.7) holds for $\lambda > C$, with some positive C, then we say that Φ dominates Ψ at infinity. As the example of homogeneous spaces $L^p(dx)$ shows, it is not enough for (2.8) for the inclusion $L^\Phi \subset L^\Psi$ to hold. However, when $\mu(\mathbb{R}^n) < \infty$, then the inclusion $L^\Phi \subset L^\Psi$ is spared. Indeed, suppose $u \in L^\Phi$. Let s_0 be such a number that $\int \Phi(\frac{|u|}{s_0})\, d\mu < +\infty$. Then according to (2.7)

$$\begin{aligned}\int \Psi(\frac{|u|}{K_2 s_0})\, d\mu &= \int_{\{|u|>K_2 s_0 C\}} \Psi(\frac{|u|}{K_2 s_0})\, d\mu + \int_{\{|u|\le K_2 s_0 C\}} \Psi(\frac{|u|}{K_2 s_0})\, d\mu \\ &\le K_1 \int \Phi(\frac{|u|}{s_0})\, d\mu + \Psi(C)\mu(\mathbb{R}^n),\end{aligned}$$

which is finite, and therefore $u \in L^\Psi(\mu)$.

Also, if we restrict our attention to functions u with compact support, then the implication $\{u \in L^\Phi(\mu)\} \Rightarrow \{u \in L^\Psi(\mu)\}$ remains true for any Radon measure μ. This is so because any Radon measure is by definition locally finite.

2.5 Auxiliary estimates for $N-$functions

We will need the following two lemmas applied previously in [23]. For the reader's convenience we submit their proofs.

Lemma 2.1. *Suppose Φ is a nonnegative function such that $\Phi(\lambda)/\lambda^\alpha$ is nondecreasing, where $\alpha \ge 1$ is given. Then for any $\lambda, \mu > 0$*

$$\frac{\Phi(\lambda)}{\lambda^\alpha}\mu^\alpha \le \Phi(\lambda) + \Phi(\mu). \tag{2.9}$$

Proof. It can be readily seen: when $\mu \le \lambda$, then $\frac{\Phi(\lambda)}{\lambda^\alpha}\mu^\alpha \le \frac{\Phi(\lambda)}{\lambda^\alpha}\lambda^\alpha = \Phi(\lambda) \le \Phi(\lambda) + \Phi(\mu)$, and when $\lambda \le \mu$, then $\frac{\Phi(\lambda)}{\lambda^\alpha} \le \frac{\Phi(\mu)}{\mu^\alpha}$, so that $\frac{\Phi(\lambda)}{\lambda^\alpha}\mu^\alpha \le \Phi(\mu) \le \Phi(\lambda) + \Phi(\mu)$. ∎

This lemma applied for $\alpha = 2$ yields the following result.

Lemma 2.2. *Suppose that Φ is an $N-$function such that $\Phi_2(\lambda) = \Phi(\lambda)/\lambda^2$ is nondecreasing. Let $F : [0,\infty) \to [0,\infty)$ be a convex function with $F(0) = 0$. Then for any $\lambda, \mu, \rho > 0$*

$$\Phi_2(\lambda)\mu\rho \le \Phi(\lambda) + H(\mu) + J(\rho), \tag{2.10}$$

where $H(\lambda) = \frac{1}{2}\Phi(2F(\sqrt{\lambda}))$, $J(\lambda) = \frac{1}{2}\Phi(2F^*(\sqrt{\lambda}))$.

Proof. (2.10) follows from (2.9), the Young inequality $vw \leq F(v)+F^*(w)$, and from the convexity of Φ :

$$\begin{aligned}\Phi_2(\lambda)\mu\rho = \frac{\Phi(\lambda)}{\lambda^2}\mu\rho \leq & \Phi(\lambda) + \Phi(\sqrt{\mu\rho}) \\ \leq & \Phi(\lambda) + \Phi(F(\sqrt{\mu}) + F^*(\sqrt{\rho})) \\ \leq & \Phi(\lambda) + \frac{1}{2}\Phi(2F(\sqrt{\mu})) + \frac{1}{2}\Phi(2F^*(\sqrt{\rho})) \\ = & \Phi(\lambda) + H(\mu) + J(\rho).\end{aligned}$$ ■

Remark 2.1. The particular choice $F(\lambda) = F^*(\lambda) = \lambda^2/2$ in (2.10) results in the following inequality:

$$\Phi_2(\lambda)\mu\rho \leq \Phi(u) + \frac{\Phi(\mu) + \Phi(\rho)}{2}. \tag{2.11}$$

3 The basic lemma

We start with the following lemma. It extends Lemma 3.1 from [22] to the class of weighted Radon measures.

Lemma 3.1. *Suppose that $\Phi\colon [0,\infty) \to [0,\infty)$ is an $N-$function of class $C^1((0,\infty))$ such that $\Phi'(\lambda)/\lambda$ is bounded next to 0. Let $\mu(dx) = w(x)dx$ be a weighted Radon measure on $\mathbb{R}^n$, with weight function $w(x) = \exp(-\varphi(x))$, where $\varphi \in C^1(\mathbb{R}^n)$. Then for every $u \in C_0^\infty(\mathbb{R}^n)$ one has*

$$\int_{\mathbb{R}^n} \Phi(|\nabla u|)d\mu \leq \alpha_n \int_{\mathbb{R}^n} \widetilde{\Phi}(|\nabla u|)|\nabla^{(2)}u|\,|u|\,d\mu + \int_{\mathbb{R}^n} \Phi_1(|\nabla u|)|u|\,|\nabla\varphi|\,d\mu, \tag{3.1}$$

where the functions $\widetilde{\Phi}, \Phi_1$ are defined by (2.4), and $0 < \alpha_n < c\sqrt{n}$ with some constant $c > 0$ independent of n.

Proof. Set $\Omega = \mathrm{supp}\nabla u$, and let

$$I := \int_{\mathbb{R}^n} \Phi(|\nabla u|)d\mu = \int_\Omega \frac{\Phi(|\nabla u|)}{|\nabla u|^2}\langle\nabla u, \nabla u\rangle \exp(-\varphi)dx = \int_{\mathbb{R}^n} \langle S(x), \nabla u\rangle dx,$$

where

$$S(x) = \begin{cases} \exp(-\varphi(x))(\Phi(|\nabla u(x)|)|\nabla u(x)|^{-2})\nabla u(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \notin \Omega. \end{cases}$$

Let us show that $S \in W^{1,1}(\mathbf{R}^n, \mathbf{R}^n)$ (Sobolev space of vector-valued functions). By assumption we have $\exp(-\varphi(x)) \in W^{1,\infty}_{loc}(\mathbf{R}^n)$ and S is supported in Ω, a compact set. Therefore it suffices to show that

$$W(x) := \frac{\Phi(|\nabla u(x)|)}{|\nabla u(x)|^2}\nabla u(x) \cdot \chi_\Omega(x)$$

belongs to the space $W^{1,1}(\mathbf{R}^n, \mathbf{R}^n)$. One possibility to see this is to consider the sequence of functions

$$W_\varepsilon(x) := \frac{\Phi(|\nabla u(x)|)}{\varepsilon^2 + |\nabla u(x)|^2}\nabla u(x), \text{ where } \varepsilon > 0$$

and to show that W_ε converges to W in $W^{1,1}(\mathbf{R}^n, \mathbf{R}^n)$ as $\varepsilon \to 0$. We have for $x \in \Omega$ (in the sense of distributions)

$$\begin{aligned}\frac{\partial W_\varepsilon}{\partial x_i} &= \frac{\Phi'(|\nabla u|)}{\varepsilon^2 + |\nabla u|^2}\frac{\langle \nabla u, \frac{\partial}{\partial x_i}(\nabla u)\rangle}{|\nabla u|}\nabla u - \frac{\Phi(|\nabla u|)}{(\varepsilon^2 + |\nabla u|^2)^2}2\langle \nabla u, \frac{\partial}{\partial x_i}(\nabla u)\rangle \nabla u \\ &\quad + \frac{\Phi(|\nabla u|)}{\varepsilon^2 + |\nabla u|^2}\frac{\partial}{\partial x_i}(\nabla u).\end{aligned}$$

Using the fact that $|\nabla^{(2)}u| \leq C$, $\frac{\Phi(\lambda)}{\lambda^2} \leq \frac{\Phi'(\lambda)}{\lambda}$, and the boundedness of $\frac{\Phi'(\lambda)}{\lambda}$ next to 0, we get

$$\left|\frac{\partial W_\varepsilon}{\partial x_i}\right| \leq c\frac{\Phi'(|\nabla u|)}{|\nabla u|}\chi_\Omega = F(x) \in L^1(\mathbf{R}^n).$$

Moreover, for almost every $x \in \mathbf{R}^n$, one has $\frac{\partial W_\varepsilon}{\partial x_i}(x) \to K_i(x)$ as $\varepsilon \to 0$, where

$$K_i = \left\{ \left(\frac{\Phi'(|\nabla u|)}{|\nabla u|} - \frac{2\Phi(|\nabla u|)}{|\nabla u|^2} \right) \langle \frac{\nabla u}{|\nabla u|}, \frac{\partial}{\partial x_i}(\nabla u)\rangle \frac{\nabla u}{|\nabla u|} + \frac{\Phi(|\nabla u|)}{|\nabla u|^2}\frac{\partial}{\partial x_i}(\nabla u) \right\} \chi_\Omega.$$

Lebesgue's Dominated Convergence theorem implies $\frac{\partial W_\varepsilon}{\partial x_i} \to K_i$ in $L^1(\mathbf{R}^n)$. As simultaneously $W_\varepsilon \to W$ in $L^1(\mathbf{R}^n)$, we conclude that $W \in W^{1,1}(\mathbf{R}^n, \mathbf{R}^n)$ and so $\nabla W = (K_1, \ldots, K_n)$. This, together with the fact that $u \in C_0^\infty(\mathbf{R}^n)$, imply

$$I = -\int_\Omega \operatorname{div} S(x)u(x)dx.$$

By elementary differentiation we verify that for every $x \in \Omega$ one has

$$\operatorname{div} S = \left(\frac{\Phi_2'(|\nabla u|)}{|\nabla u|}[\nabla u]^t[\nabla^{(2)}u][\nabla u] + \Phi_2(|\nabla u|)(\Delta u - \langle \nabla\varphi, \nabla u\rangle) \right) \exp(-\varphi)$$

and

$$\Phi_2'(t) = \left(\frac{\Phi'(t)}{t^2} - 2\frac{\Phi(t)}{t^3} \right) = \frac{1}{t^2}\left(\Phi'(t) - \frac{\Phi(t)}{t} \right) - \frac{\Phi(t)}{t^3}. \tag{3.2}$$

Since Φ is convex, one has $\Phi(\lambda)/\lambda \leq \Phi'(\lambda)$ and so the first summand in (3.2) is nonnegative and does not exceed $\Phi'(t)/t^2$. Setting $v = \frac{\nabla u}{|\nabla u|}$ we check that $|\operatorname{div} S(x)| \leq (L_1 + L_2 + L_3)\exp(-\varphi)$, where

$$\begin{aligned} L_1 &= \widetilde{\Phi}(|\nabla u|)\,|\nabla^{(2)}u|, \\ L_2 &= \Phi_2(|\nabla u|)|\Delta u - v^t[\nabla^{(2)}u]v|, \\ L_3 &= \Phi_2(|\nabla u|)|\langle \nabla\varphi, \nabla u\rangle| \leq \Phi_1(|\nabla u|)|\nabla\varphi|. \end{aligned}$$

A direct computation (see [23], Lemma 6.1) shows that

$$|\Delta u - v^t[\nabla^{(2)}u]v| \leq \sqrt{n-1}|\nabla^{(2)}u|.$$

Since $\Phi_2(\lambda) \leq \widetilde{\Phi}(\lambda)$, the Lemma follows. ■

Remark 3.1. When Φ satisfies the Δ_2-condition, then the functions $\widetilde{\Phi}$ and Φ_2 are equivalent, so inequality (3.1) can be rewritten as

$$\int_{\mathbb{R}^n} \Phi(|\nabla u|)d\mu \leq C_n \left(\int_{\mathbb{R}^n} \Phi_2(|\nabla u|)|\nabla^{(2)} u|\,|u|\,d\mu + \int_{\mathbb{R}^n} \Phi_1(|\nabla u|)|u|\,|\nabla \varphi|\,d\mu \right). \quad (3.3)$$

Remark 3.2. Detailed analysis of the proof (see Lemma 3.1 in [23]) shows that an inequality slightly stronger than (3.1) holds, namely

$$\int_{\mathbb{R}^n} \Phi(|\nabla u|)d\mu \;\; \leq \;\; \int_{\mathbb{R}^n} \Phi_n(|\nabla u|)|\nabla^{(2)} u|\,|u|\,d\mu + \int_{\mathbb{R}^n} \Phi_1(|\nabla u|)|u|\,|\nabla \varphi|\,d\mu, \quad (3.4)$$

with $\Phi_n(\lambda) = \frac{\lambda \Phi'(\lambda) + (\sqrt{n-1}-1)\Phi(\lambda)}{\lambda^2}$.

4 The Gagliardo-Nirenberg inequalities

In this section we present and prove our main results.

Theorem 4.1. *Suppose that Φ is an $N-$function of class $C^1((0,\infty))$ such that $\frac{\Phi'(t)}{t}$ is bounded next to 0, and let H, J, J_1 be three other $N-$functions, for which the following conditions are satisfied for every $s, t, r > 0$:*

(C1)

$$\widetilde{\Phi}(s)tr \leq \Phi(s) + H(t) + J(r), \quad (4.1)$$

(C2)

$$\Phi_1(s)tr \leq \Phi(s) + H(t) + J_1(r), \quad (4.2)$$

where $\tilde{\Phi}$ and Φ_1 were defined by (2.4). Assume that $\mu(dx) = \exp(-\varphi(x))dx$, $\varphi \in C^1(\mathbb{R}^n)$, is a weighted measure on $\mathbb{R}^n$ for which $\|\nabla\varphi\|_{(J_1,\mu)} < \infty$.
Then for every $u \in C_0^\infty(\mathbf{R}^n)$ one has

$$\|\nabla u\|_{(\Phi,\mu)} \leq \beta_n \sqrt{\|u\|_{(H,\mu)} \|\nabla^{(2)} u\|_{(J,\mu)}} + C_{n,\varphi} \|u\|_{(H,\mu)}. \quad (4.3)$$

where $0 < \beta_n = c\sqrt[4]{n}$ and $C_{n,\varphi} = c\sqrt{n}\|\nabla\varphi\|_{(J_1,\mu)}$, with some constant $c > 0$ independent of n.

Applying an elementary inequality $2xy \leq x^2 + y^2$ we get the following.

Corollary 4.1. *Suppose that the assumptions of Theorem 4.1 are satisfied. Then*

$$\|\nabla u\|_{(\Phi,\mu)} \leq \beta_n \|u\|_{(H,\mu)} + \gamma_n \|\nabla^{(2)} u\|_{(J,\mu)}, \quad (4.4)$$

where $0 < \beta_n = c\sqrt[4]{n}$, $\gamma_n = c\sqrt{n}(1 + \|\nabla\varphi\|_{(J_1,\mu)})$, and $c > 0$ is independent of n and φ.

Proof of Theorem 4.1. First assume that $\|\nabla\varphi\|_{(J_1,\mu)} \neq 0$.
Fix four positive numbers s_1, s_2, s_3, s_4 and set $s = \sqrt{s_1 s_2} + s_3 s_4$. Starting with formula (3.1) applied to the function $\widetilde{u} = \frac{u}{s}$, we get

$$\mathcal{I} := \int_{\mathbb{R}^n} \Phi(|\nabla\widetilde{u}|)d\mu \leq \alpha_n \int_{\mathbb{R}^n} \widetilde{\Phi}(|\nabla\widetilde{u}|)\frac{1}{s^2}|u|\,|\nabla^{(2)}u|\,d\mu + \int_{\mathbb{R}^n} \Phi_1(|\nabla\widetilde{u}|)\frac{1}{s}|u|\,|\nabla\varphi|\,d\mu.$$

Take an arbitrary $\varepsilon > 0$, and estimate the integral in question by

$$\begin{aligned} \mathcal{I} &\leq \alpha_n\varepsilon \int_{\mathbb{R}^n} \widetilde{\Phi}(|\nabla u|)\frac{|u|}{s_1\varepsilon}\,\frac{|\nabla^{(2)}u|}{s_2}\,d\mu + \varepsilon\int_{\mathbb{R}^n} \Phi_1(|\nabla\widetilde{u}|)\frac{|u|}{s_3\varepsilon}\,\frac{|\nabla\varphi|}{s_4}\,d\mu \\ &=: \alpha_n\varepsilon\mathcal{I}_1 + \varepsilon\mathcal{I}_2 \end{aligned} \tag{4.5}$$

(we have used two obvious properties: $s^2 > s_1 s_2$ and $s > s_3 s_4$).
Now apply the assumption (4.1) to the first integral. This gives

$$\mathcal{I}_1 \leq \int_{\mathbb{R}^n} \Phi(|\nabla\widetilde{u}|)\,d\mu + \int_{\mathbb{R}^n} H(\frac{|u|}{s_1\varepsilon})\,d\mu + \int_{\mathbb{R}^n} J(\frac{|\nabla^{(2)}u|}{s_2})\,d\mu. \tag{4.6}$$

To estimate $\mathcal{I}_2$ we apply (4.2) instead of (4.1) and deduce that

$$\mathcal{I}_2 \leq \int_{\mathbb{R}^n} \Phi(|\nabla\widetilde{u}|)\,d\mu + \int_{\mathbb{R}^n} H(\frac{|u|}{s_3\varepsilon})\,d\mu + \int_{\mathbb{R}^n} J_1(\frac{|\nabla\varphi|}{s_4})\,d\mu. \tag{4.7}$$

Collecting (4.5), (4.6), (4.7) and rearranging, then choosing $\varepsilon = \varepsilon_n = \frac{1}{4(1+\alpha_n)}$ we get

$$\begin{aligned} \frac{3}{4}\mathcal{I} &\leq \alpha_n\varepsilon_n\left(\int_{\mathbb{R}^n} H(\frac{|u|}{s_1\varepsilon_n})\,d\mu + \int_{\mathbb{R}^n} J(\frac{|\nabla^{(2)}u|}{s_2})\,d\mu\right) \\ &\quad +\varepsilon_n\left(\int_{\mathbb{R}^n} H(\frac{|u|}{s_3\varepsilon_n})\,d\mu + \int_{\mathbb{R}^n} J_1(\frac{|\nabla\varphi|}{s_4})\,d\mu\right). \end{aligned} \tag{4.8}$$

Without loss of generality we can assume $\|u\|_{(H,\mu)} \neq 0$, and $\|\nabla^{(2)}u\|_{(J,\mu)} \neq 0$, as otherwise one has $u \equiv 0$ (since it has been assumed $u \in C_0^\infty(\mathbb{R}^n)$).
We set

$$s_1 = \frac{1}{\varepsilon_n}\|u\|_{(H,\mu)}, \quad s_2 = \|\nabla^{(2)}u\|_{(J,\mu)}, \quad s_3 = s_1, \quad s_4 = \|\nabla\varphi\|_{(J_1,\mu)}.$$

It follows from (4.8) and (2.6) that $\mathcal{I} < 1$, and from the very definition of the Luxemburg norm

$$\|\nabla u\|_{(\Phi,\mu)} \leq s.$$

so that

$$\|\nabla u\|_{(\Phi,\mu)} \leq \frac{1}{\sqrt{\varepsilon_n}}\sqrt{\|u\|_{(H,\mu)}\|\nabla^{(2)}u\|_{(J,\mu)}} + \frac{1}{\varepsilon_n}\|\nabla\varphi\|_{(J_1,\mu)}\|u\|_{(H,\mu)},$$

which is (4.3), because $\varepsilon_n^{-1} = 2 + 2\alpha_n$. Therefore we obtain the desired result.

Now we deal with the remaining case $\|\nabla\varphi\|_{(J,\mu)} = 0$. This condition yields $\varphi = const$ and $\mu(dx) = c\,dx$. For simplicity put $c = 1$.

In this case (3.1) has only one term. Take $\varepsilon = \varepsilon_n = \frac{1}{4\alpha_n}$, $s_1 = \frac{1}{\varepsilon_n}\|u\|_{(H,dx)}$, $s_2 = \|\nabla^{(2)}u\|_{(J,dx)}$ and apply (3.1) to the function $\tilde{u} = \frac{u}{\sqrt{s_1 s_2}}$, getting

$$\mathcal{I} \le \alpha_n\varepsilon_n \int_{\mathbb{R}^n} \widetilde{\Phi}(|\nabla u|)\,\frac{|u|}{s_1\varepsilon}\,\frac{|\nabla^{(2)}u|}{s_2}\,dx.$$

Applying (4.1) and rearranging we see that

$$\frac{1}{2}\mathcal{I} \le \frac{3}{4}\mathcal{I} \le \frac{1}{4}\left(\int_{\mathbb{R}^n} H(\frac{|u|}{s_1\varepsilon_n})\,dx + \int_{\mathbb{R}^n} J(\frac{|\nabla^{(2)}u|}{s_2})\,dx\right) \le \frac{1}{2}$$

and therefore $\mathcal{I} \le 1$, $\|\nabla u\|_{(\Phi,dx)} \le \sqrt{s_1 s_2}$. Hence

$$\|\nabla u\|_{(\Phi,dx)} \le \frac{1}{\sqrt{\varepsilon_n}}\sqrt{\|u\|_{(H,dx)}\|\nabla^{(2)}u\|_{(J,dx)}}$$

and $\frac{1}{\sqrt{\varepsilon_n}} = 2\sqrt{\alpha_n}$. The Theorem follows. ∎

Remark 4.1. Variants of inequality (4.3) in the particular case of the Lebesgue measure (with $C_{n,\varphi} = 0$) were examined in detail in [23].

Remark 4.2. In general, we cannot expect inequalities in multiplicative form

$$\|\nabla u\|_{(\Phi,\mu)} \le \beta_n\sqrt{\|u\|_{(H,\mu)}\|\nabla^{(2)}u\|_{(J,\mu)}}, \tag{4.9}$$

with β_n independent of $u \in C_0^\infty(\mathbb{R}^n)$, to hold. To see that, suppose that $\mu(\mathbb{R}^n) < \infty$, Φ, H, J satisfy the Δ_2-condition and $\int_{\mathbb{R}^n} H(|x|)\mu(dx) < \infty$. Let us consider an affine function $u(x) = \langle A, x\rangle$ where $A \neq 0$ and choose $f \in C_0^\infty(\mathbb{R}^n)$ with $f(x) \equiv 1$ on $B(0,1)$, $f(x) \equiv 0$ on $\mathbb{R}^n \setminus B(0,2)$, then let $u_R(x) = f(\frac{x}{R})u(x)$. Function $u_R(x)$ belongs to $C_0^\infty(\mathbb{R}^n)$, and for all $x \in \mathbb{R}^n$

$$u_R(x) \to u(x), \quad \nabla u_R(x) \to \nabla u(x), \quad \nabla^{(2)}u_R(x) \to \nabla^{(2)}u(x),$$

when $R \to \infty$. Using the Lebesgue Dominated Convergence Theorem we see that each of the quantities:

$$\int_{\mathbb{R}^n} \Phi(|\nabla u_R - \nabla u|)\mu(dx), \int_{\mathbb{R}^n} H(|u_R - u|)\mu(dx), \int_{\mathbb{R}^n} J(|\nabla^{(2)}u_R - \nabla^{(2)}u|)\mu(dx)$$

converges to 0 when $R \to \infty$. Using Theorem 9.4 in Chapter II.9 of [29] adapted to the case of general μ instead of the Lebesgue measure one gets:

$$\begin{aligned} u_R &\to u \text{ in } L^H(\mu), \\ \nabla u_R &\to \nabla u \text{ in } L^\Phi(\mu), \\ \nabla^{(2)}u_R &\to \nabla^{(2)}u = 0 \text{ in } L^J(\mu). \end{aligned}$$

Therefore if the inequality (4.9) has been true, it would apply to $u(x) = \langle A, x\rangle$ as well. But in such case the left hand side of (4.9) equals $\|\Phi(|A|)\|_{(\Phi,\mu)} \neq 0$, while the right hand side is zero, a contradiction. Inequality (4.9) cannot be satisfied.

Remark 4.3. Our constants in Theorem 4.1 do not depend on the measure chosen. For example the bounds on C_1 are the same for all weights considered (see Theorem 4.1).

Remark 4.4. Both $N-$functions: H and J_1 in (4.2) must essentially dominate Φ at infinity. Moreover, if u has compact support or $\mu(\mathbb{R}^n) < \infty$, then the condition $u \in L^H(\mu)$ implies $u \in L^\Phi(\mu)$ and the same holds with J_1 instead of H.

Indeed, by the symmetry argument (see (4.2)) it suffices to prove this observation for one of the functions, let us say for H. Applying (4.2) with $t = s$ we get

$$\Phi(s)r \le \Phi(s) + H(s) + J_1(r), \quad \text{for every } s, r > 0.$$

equivalent to

$$r \le 1 + \frac{H(s)}{\Phi(s)} + \frac{J_1(r)}{\Phi(s)}.$$

Therefore $r - 1 \le \liminf_{s\to\infty} \frac{H(s)}{\Phi(s)}$ for every $r \ge 0$ and consequently

$$\lim_{s\to\infty} \frac{H(s)}{\Phi(s)} = \infty.$$

This shows that H essentially dominates Φ next to infinity.

The second statement is a consequence of the discussion following the formula (2.7).

Remark 4.5. We do not impose any particular regularity assumptions on the measure μ other than: $\mu(dx) = \exp(-\varphi(x))dx$, $\varphi \in C^1(\mathbb{R}^n)$, $\|\nabla\varphi\|_{(J_1,\mu)} < \infty$ in Theorem 4.1. In particular our measure may not satisfy the doubling condition $\mu(2B) \le c\mu(B)$, where B is an arbitrary ball in $\mathbf{R}^n$, $2B$ denotes the ball with the same center as B and twice the radius. Inequalities within a restricted class of weights (which necessarily satisfy the doubling condition) were obtained in [25] by different methods.

5 The class of admissible $N-$functions

We are now going to discuss examples of triples (Φ, H, J) and (Φ, H, J_1) which can appear in inequalities (4.1) and (4.2). Triples (Φ, H, J) admissible in (4.1) were formerly analyzed in ([23]). The following result can be deduced from Theorem 7.1 and 7.2 in [23]. For readers' convenience we include its proof.

Proposition 5.1. *Suppose that the functions $\Phi, F, H, J\colon [0,\infty) \to [0,\infty)$ and $g : (0,\infty) \to (0,\infty)$ are such that:*
1. Φ and F are $N-$functions and $\Phi \in C^1((0,\infty))$;
2. g is strictly increasing, $\Phi(\lambda)/g(\lambda)$ is nondecreasing and the following inequality is satisfied with a constant C_g independent of $\lambda > 0$:

$$\frac{\Phi'(\lambda)}{\lambda} \le C_g \frac{\Phi(\lambda)}{g(\lambda)}; \tag{5.1}$$

3. $H(y) = ((\Phi \circ g^{-1})(2C_g F(y))$ and $J(z) = ((\Phi \circ g^{-1})(2C_g F^*(z))$.

Then the inequality

$$\frac{\Phi'(\lambda)}{\lambda}yz \leq \Phi(\lambda) + H(y) + J(z)$$

is satisfied for every $\lambda, y, z > 0$.

Remark 5.1. Note that the since for a convex Φ one has $\frac{\Phi(\lambda)}{\lambda} \leq \Phi'(\lambda)$ then (5.1) implies $\frac{\Phi(\lambda)}{\lambda^2} \leq C_g \frac{\Phi(\lambda)}{g(\lambda)}$. Therefore if g satisfies (5.1) then $g(\lambda) \leq C_g \lambda^2$ for every $\lambda > 0$. In particular within homogeneous functions only $g(\lambda) = \lambda^2$ is admitted. In such case we just get the Δ_2−condition.

Proof of Proposition 5.1. By (5.1) we have

$$L := \frac{\Phi'(\lambda)}{\lambda}yz \leq \frac{\Phi(\lambda)}{g(\lambda)}(C_g yz).$$

If $a := \frac{C_g yz}{g(\lambda)} \leq 1$ then $L \leq \Phi(\lambda)$ and the assertion is satisfied.
If $a > 1$, then we have $g(\lambda) < C_g yz$, so that $\lambda < g^{-1}(C_g yz)$. Therefore

$$\frac{\Phi(\lambda)}{g(\lambda)} \leq \frac{(\Phi \circ g^{-1})(C_g yz)}{(g \circ g^{-1})(C_g yz)} = \frac{(\Phi \circ g^{-1})(C_g yz)}{C_g yz}$$

and consequently

$$L \leq \Phi \circ g^{-1}(C_g yz) \leq (\Phi \circ g^{-1})(C_g(F(y) + F^*(z))).$$

If $F(y) \leq F^*(z)$, then $L \leq J(z)$, while if $F^*(z) \leq F(y)$, then $L \leq H(y)$. In either case the Proposition follows. ■

To approach inequality (4.2) we use a similar result.

Proposition 5.2. *Assume that* Φ *and* F *are* N−*functions,* $\frac{\Phi(\lambda)}{\lambda}$ *is nondecreasing and* $H_1(y) = \Phi(2F(y))$, $J_1(y) = \Phi(2F^*(y))$. *Then for every* $\lambda, y, z > 0$ *we have*

$$\frac{\Phi(\lambda)}{\lambda}yz \leq \Phi(\lambda) + H_1(y) + J_1(z).$$

Proof. By Lemma 2.1 and the Young inequality we have

$$\frac{\Phi(\lambda)}{\lambda}yz \leq \Phi(\lambda) + \Phi(yz) \leq \Phi(\lambda) + \Phi(F(y) + F^*(z)) \leq \Phi(\lambda) + H_1(y) + J_1(z). \quad ■$$

Remark 5.2. Note that if $H_1(y) = \Phi(2F(y))$ where Φ and F are as in the statement of Proposition 5.2, then again we have $\Phi \prec H_1$ next to infinity (see Remark 4.4) and Φ cannot be equivalent to H_1.

As a direct consequence of Proposition 5.1 and 5.2 we obtain the following result, which serves as a recipe for finding functions which can appear in (4.3) and (4.1).

Proposition 5.3. *Suppose that the functions $\Phi, F, H, H_1, J : [0,\infty) \to [0,\infty)$ and $g : (0,\infty) \to (0,\infty)$ are such that:*
1. Φ and F are $N-$functions and $\Phi \in C^1((0,\infty))$;
2. g is strictly increasing, $\Phi(\lambda)/g(\lambda)$ is nondecreasing and the following inequality is satisfied with the constant C_g independent of $\lambda > 0$

$$\frac{\Phi'(\lambda)}{\lambda} \le C_g \frac{\Phi(\lambda)}{g(\lambda)};$$

3. $H(y) = ((\Phi \circ g^{-1})(2C_g F(y))$ and $J(z) = ((\Phi \circ g^{-1})(2C_g F^*(z))$;
4. the function $R(z) := \frac{1}{2} g^{-1}(2C_g F(z))$ is an $N-$function and $J_1(z) = \Phi(2R^*(z))$.
Then for every $\lambda, y, z > 0$ we have

$$\frac{\Phi'(\lambda)}{\lambda} yz \le \Phi(\lambda) + H(y) + J(z) \quad \text{and} \quad \frac{\Phi(\lambda)}{\lambda} yz \le \Phi(\lambda) + H(y) + J_1(z).$$

6 Three examples

In this chapter we present three examples illustrating Theorem 4.1.

6.1 Inequalities within homogeneous functions

Our first example deals with homogeneous functions.

Proposition 6.1. *Let $p > q \ge 2$ and $r > 1$ be such numbers that $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$. Suppose that $\mu(dx) = \exp(-\varphi(x))dx$ is a Radon measure on $\mathbb{R}^n$, $\varphi \in C^1(\mathbb{R}^n)$ and $|\nabla\varphi| \in L^{\frac{pq}{p-q}}(\mu)$. Then for any $u \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\|\nabla u\|_{L^q(\mu)} \le \beta_n \sqrt{\|u\|_{L^p(\mu)} \|\nabla^{(2)} u\|_{L^r(\mu)}} + C_{n,\varphi} \|u\|_{L^p(\mu)}, \tag{6.1}$$

where $0 < \beta_n \le c\sqrt[4]{n}$, $C_{n,\varphi} \le c\sqrt{n}\|\nabla\varphi\|_{(J_1,\mu)}$, and the constant $c > 0$ is independent of n.

Proof. Let us take $g(\lambda) = \lambda^2$, $\Phi(\lambda) = \lambda^q$, $F(\lambda) = \lambda^s$ where $s = \frac{2p}{q}$ and apply Proposition 5.3. Then we have $H(y) \sim y^p$, $J(z) \sim z^r$, $R(z) \sim z^{\frac{p}{q}}$, $R^*(z) \sim z^{\frac{p}{p-q}}$ and $J_1(z) \sim z^{\frac{pq}{p-q}}$. Therefore Proposition follows. ■

Remark 6.1. In can easily be seen that the Gaussian measure $\gamma(dx) = \frac{1}{\sqrt{2\pi n}} \exp(-\frac{|x|^2}{2})dx$, exponential measure $\mu(dx) = \alpha\exp(-\alpha|x|)$ where $\alpha > 0$ is a given constant or an arbitrary measure of the form $\mu(dx) = C_1\exp(-C_2|x|^\beta)$ where $\beta \ge 1$ are allowed in every inequality of the form (6.1).

Remark 6.2. Proposition 6.1 deals with different class of weights than the weights in previous papers [7, 12, 13, 20, 21], [35, Theorem 1 in Section 1.4.7]. For example in the papers [12, 13, 20, 21] one assumes that the measure μ in $L^q(\mu)$ on the left hand side of the inequality (1.2) is doubling, while in Theorem 1, Section 1.4.7 of [35] it is assumed that such a measure satisfies the following s-regularity condition: there exists $s > 0$ such that $\sup\{r^{-s}\mu(B(x,r)) : x \in \mathbb{R}^n, r > 0\} < \infty$, where the symbol $B(x,r)$ denotes the ball with center x and radius r. The paper [7] is restricted to homogeneous weights. In present paper neither of these conditions is assumed.

The paper [10] deals with triples of measures Ndx, Wdx and Pdx. The authors obtain very general but additive (therefore weaker) inequalities of the form

$$\int_\Omega |\nabla^{(j)}u|^p N dx \le K\left\{\varepsilon^{-\varphi}\left(\int_\Omega |u|^q W dx\right)^{p/q} + \varepsilon^{\theta}\left(\int_\Omega |\nabla^{(m)}u|^r P dx\right)^{p/r}\right\},$$

where φ, θ are non-negative functions of m, j, p, q, r, while Ω is a bounded or unbounded domain in $\mathbb{R}^n$, $\varepsilon \in (0, \varepsilon_0)$, u is sufficiently smooth and N, W, P are weight functions satisfying certain additional conditions (we skip the detailed formulation which can be found in the paper). Therefore the approach presented there is different. Also, the techniques of [10] are independent from ours.

6.2 Logarithmic inequalities

Our next example applies to logarithmic $N-$functions

$$M_{s,\kappa}(t) = t^s(\ln(2+t))^\kappa.$$

Orlicz norms related to $M_{s,\kappa}$ and the measure μ are denoted by $\|\cdot\|_{(s,\kappa,\mu)}$. The result stated below generalizes Theorem 1.1 in [24].

Proposition 6.2. *Suppose that $\beta, \gamma \in \mathbb{R}$, $p, r > 1, p > q$ are given numbers such that additionally the following condition is satisfied*

$$(q > 2, \alpha \in \mathbb{R} \text{ or } q = 2, \alpha \ge 0) \text{ and } \left(\frac{2}{q} = \frac{1}{p} + \frac{1}{r}, \frac{2\alpha}{q} \le \frac{\beta}{p} + \frac{\gamma}{r}\right),$$

Let $\mu(dx) = \exp(-\varphi(x))dx$ is a Radon measure on $\mathbb{R}^n$, $\varphi \in C^1(\mathbb{R}^n)$ and $|\nabla\varphi| \in L^{M_{\eta,\kappa}}(\mu)$ where $\eta = q(\frac{p}{q})^*$ and $\kappa = -(\beta-\alpha)((\frac{p}{q})^* - 1) + \alpha$.

Then for any function $u \in C_0^\infty(\mathbb{R}^n)$ we have:

$$\|\nabla u\|_{(q,\alpha,\mu)} \le \beta_n \sqrt{\|u\|_{(p,\beta,\mu)}\| \, \|\nabla^{(2)}u\|_{(r,\gamma,\mu)}} + C_{n,\varphi}\|u\|_{(p,\beta,\mu)}, \tag{6.2}$$

where $0 < \beta_n \le c\sqrt[4]{n}$ and $C_{n,\varphi} \le c\sqrt{n}\|\nabla\varphi\|_{(J_1,\mu)}$ and the constant $c > 0$ is independent of n.

Proof. At first we note that each function $M_{s,\kappa}$, where $s > 1$, satisfies the Δ_2-condition, so that inequality (5.1) holds with $g(\lambda) = \lambda^2$. This gives $\frac{M'_{s,\kappa}(\lambda)}{\lambda} \sim M_{s-2,\kappa}(\lambda)$. An application of Proposition 5.3 with $\Phi(\lambda) = M_{q,\alpha}(\lambda)$, $g(\lambda) = \lambda^2$ and $F(\lambda) = \lambda^s(\ln(2+\lambda))^r$ where $s = \frac{2p}{q}$, $r = \frac{2(\beta-\alpha)}{q}$ and properties

$$M_{q,\alpha} \circ M_{\mu,\kappa} \sim M_{q\mu, q\kappa+\alpha}, \quad M^*_{\mu,\alpha} \sim M_{\mu^*, -\alpha(\mu^*-1)}$$

(see Theorem 7.1 in [29] for the latter property, the details are furnished in the proof of Theorem 1.2 in [24]) gives

$$H(y) \sim M_{p,\beta}(y) \text{ and } J(z) \sim M_{r,\gamma}(z). \tag{6.3}$$

Moreover, $R(z) \sim \sqrt{F(z)} \sim M_{\frac{p}{q},\frac{\beta-\alpha}{q}}(z)$ and $R^*(z) \sim M_{\rho,\delta}(z)$ where $\rho = (\frac{p}{q})^*$ and $\delta = -\frac{\beta-\alpha}{q}((\frac{p}{q})^* - 1)$. Therefore

$$J_1(z) \sim M_{q,\alpha} \circ M_{\rho,\delta}(z) \sim M_{\eta,\kappa}(z),$$

where η and κ are as in the statement of the Proposition. This ends the proof of the Proposition. ∎

Remark 6.3. Similarly as in the case of homogeneous functions every measure of the form $C_1\exp(-C_2|x|^\beta)$ where $C_1, C_2 > 0$ and $\beta \geq 1$ can appear in the inequality (6.2).

6.3 Exponential inequalities

Our concluding example deals with exponential functions. Such functions do not satisfy the Δ_2−condition but for the large λ's they satisfy the condition (5.1) with $g(\lambda) = \lambda^s$ for some positive $s < 2$. The detailed analysis of the example presented below for $\mu = \omega dx$ within certain class of weights ω introduced by Bloom and Kerman [15] was presented in [25]. Such measures satisfy the doubling condition. Some results within the Lebesgue measure were also obtained in [23]. Here we extend it to the Orlicz spaces $L^\Phi(\mu)$ for essentially larger class of measures.

Proposition 6.3. *Suppose that $p > 2, \alpha \in (0,2), s > 2$ are given numbers and*

$$\Phi(\lambda) \;=\; \lambda^p\exp(\lambda^\alpha), \quad H(y) = \lambda^{\frac{ps}{2}}\exp(\lambda^{\frac{s\alpha}{2-\alpha}}) \quad J(\lambda) = \lambda^{\frac{ps^*}{2}}\exp(\lambda^{\frac{s^*\alpha}{2-\alpha}}). \quad (6.4)$$

Let $\mu(dx) = \exp(-\varphi(x))dx$ be a Radon measure on $\mathbb{R}^n$, $\varphi \in C^1(\mathbb{R}^n)$ and $|\nabla\varphi| \in L^{J_1}(\mu)$ where $J_1(z) = z^\eta\exp(z^\kappa)$ where $\eta = p(\frac{s}{2})^*$ and $\kappa = \alpha(\frac{s}{2-\alpha})^*$.
Then for any function $u \in C_0^\infty(\mathbb{R}^n)$ we have:

$$\|\nabla u\|_{(\Phi,\mu)} \leq \beta_n\sqrt{\|u\|_{(H,\mu)}\| \,\|\nabla^{(2)}u\|_{(J,\mu)}} + C_{n,\varphi}\|u\|_{(H,\mu)}, \quad (6.5)$$

where $0 < \beta_n \leq c\sqrt[4]{n}$ and $C_{n,\varphi} \leq c\sqrt{n}\|\nabla\varphi\|_{(J_1,\mu)}$ and the constant $c > 0$ is independent on n.

Proof. Let us consider the function $g(\lambda) = \frac{\lambda^2}{\alpha\lambda^\alpha+p}$. Then Φ satisfies (5.1), g is increasing and so is Φ/g. Let us take $F(\lambda) = \lambda^s$ and apply Proposition 5.3. Indeed, one verifies that the functions H and J defined in Proposition 5.3 are equivalent to H and J defined in (6.4). The verification is based on the fact that they are equivalent for λ close to 0 and for λ close to infinity separately. In such a case we have $g(\lambda) \sim \lambda^2$ for small λ and $g(\lambda) \sim \lambda^{2-\alpha}$ for big λ, $\Phi(\lambda) \sim \lambda^p$ for small λ and $\Phi(\lambda) \sim \exp(\lambda^\alpha)$ for big λ. Now let us compute the function R from the statement of Proposition 5.3. We observe that $R(\lambda) \sim \lambda^{s/2}$ for small λ and $R(\lambda) \sim \lambda^{s/(2-\alpha)}$ for big λ. Therefore $R^*(\lambda) \sim \lambda^{(s/2)^*}$ for small λ and $R^*(\lambda) \sim \lambda^{(s/(2-\alpha))^*}$ for big λ. It implies that

$$\Phi(2R^*(z)) \sim z^{p(\frac{s}{2})^*} \text{ for small } z \text{ and } \Phi(2R^*(z)) \sim \exp(z^{\alpha(\frac{s}{2-\alpha})^*}) \text{ for big } z.$$

It follows that the function $J_1(z) = z^\eta \exp(z^\kappa)$, where η and κ are as in the statement of the Proposition, together with Φ, H, J, satisfy the assumptions of Proposition 4.1. This implies the statement of the Proposition. ∎

Acknowledgement. We would like to thank Piotr Hajłasz for helpful discussion.

References

[1] H. D. ALBER, *Materials with Memory - Initial–boundary Value Problems for Constitutive Equations with Internal Variables*, Lecture Notes in Mathematics, vol. 1682, Springer 1998.

[2] J. M. BALL, *Constitutive inequalities and existence theorems in nonlinear elastostatics*, Nonlinear analysis and mechanics: Heriot-Watt Symposium (Edinburgh, 1976), Vol. I, pp. 187–241. Res. Notes in Math., No. 17, Pitman, London, 1977.

[3] H. H. BANG, *A remark on the Kolmogorov–Stein inequality*, J. Math. Anal. Appl. **203** (1996), 861–867.

[4] H. H. BANG and H. M. GIAO, *On the Kolmogorov inequality for L_Φ–norm*, App. Anal. **81** (2002), no. 1, 1–11.

[5] H. H. BANG and H. M. LE, *On an inequality of Kolmogorov and Stein*, Bull. Austral. Math. Soc. **61** (2000), no. 1, 153–159.

[6] H. H. BANG and M. T. THU, *A Landau-Kolmogorov inequality for Orlicz spaces*, J. Inequal. Appl. **7** (2002), no. 5, 663–672.

[7] S. BARZA, V. I. BURENKOV, J. PEČARIČ, L. E. PERSSON, *Sharp multidimensional multiplicative inequalities for weighted L^p spaces with homogeneous weights*, Math. Ineq. Appl. **1** (1998), 53–67.

[8] O. W. BESOV, W. P. ILIN, S. and M. NIKOLSKI, *Integral Representations of Functions and Embeddings Theorems*, Nauka, Moscow 1975 (in Russian), english. trans.: *Integral representation of functions and embedding theorems*, J. Wiley, New York 1978.

[9] J. BOMAN, *Supremum norm estimates for partial derivatives of functions of several real variables*, Ill. J. Math. **16** (1972), no. 2, 203–216.

[10] R. C. BROWN and D. B. HINTON, *Weighted interpolation inequalities and embeddings in $\mathbb{R}^n$*, Can. J. Math., Vol. XLII, **6** (1990), 959-980.

[11] K. CHEŁMIŃSKI, *On monotone plastic constitutive equations with polynomial growth condition*, Math. Meth. Appl. Sci. **22** (1999), no. 7, 547–562.

[12] S.K. CHUA, *On weighted Sobolev interpolation inequalities*, Proc. Amer. Math. Soc., **121** (1994), no. 2, 441–449.

[13] S.K. CHUA, *Weighted Sobolev interpolation inequalities on certain domains*, J. London Math. Soc., **51** (1995), no. 3, 532–544.

[14] A. CIANCHI, *Some results in the theory of Orlicz spaces and applications to variational problems*. Nonlinear analysis, function spaces and applications, Vol. 6 (Prague, 1998), Acad. Sci. Czech Rep., Prague, 1999, pp. 50–92.

[15] S. BLOOM, R. KERMAN, *Weighted Orlicz space integral inequalities for the Hardy-Littlewood maximal operator*, Studia Math. **110**, no. 2 (1994), 149–167.

[16] Z. DITZIAN, *Remarks, questions and conjectures on Landau-Kolmogorov type inequalities*, Math. Inequal. Appl. **3** (2000), no. 1, 15–24.

[17] D. E. EDMUNDS and W. D. EVANS, *Spectral Theory and Differential Operators*, Oxford University Press, 1987.

[18] E. GAGLIARDO, *Ulteriori proprietà di alcune classi di funzioni in più variabili* (in Italian), Ricerche Mat. **8** (1959), 24–51.

[19] A. GOGATISHVILI, P. KOSKELA, *A non-doubling Trudinger inequality*, Studia Math. **170** (2005), no. 2, 113–119.

[20] C.E. GUTIERREZ and R.L. WHEEDEN, *Sobolev interpolation inequalities with weights*, Trans. Amer. Math. Soc., **323** (1991), no. 1, 263–281.

[21] A. KAŁAMAJSKA, *Pointwise multiplicative inequalities and Nirenberg type estimates in weighted Sobolev spaces*, Studia Math. **108** (1994), 275–290.

[22] A. KAŁAMAJSKA and K. PIETRUSKA-PAŁUBA, *Logarithmic version of interpolation inequalities for derivatives*, J. Lond. Math. Soc. **70**(3) (2004), 691–702.

[23] A. KAŁAMAJSKA and K. PIETRUSKA-PAŁUBA, *Interpolation inequalities for derivatives in Orlicz spaces*, Indiana Univ. Math. J. **55** (6) (2006), 1767–1789.

[24] A. KAŁAMAJSKA and K. PIETRUSKA-PAŁUBA, *Gagliardo-Nirenberg inequalities in logarithmic Orlicz spaces*, Colloq. Math. 106 (2006), 93-107.

[25] A. KAŁAMAJSKA and K. PIETRUSKA-PAŁUBA, *Gagliardo-Nirenberg inequalities in weighted Orlicz spaces*, Studia Math. **173** (1) (2006), 49–71.

[26] B. V. KHVEDELIDZE, *The method of the Cauchy type integrals in discontinuous boundary value problems of the theory of holomorphic functions of a complex variable* (in Russian), Itogi Nauki i Tekhniki, Seria Sovremennye problemi matematiki, Vol. 7, Moscow 1975, 5–162 (English. transl.: J. Soviet Math. 7,3 (1977), 309–415).

[27] V. KOKILASHVILI and M. KRBEC, *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific, Singapore, 1991.

[28] A. N. KOLMOGOROV, *On inequalities between upper bounds of consecutive derivatives of an arbitrary function defined on an infinite interval* (in Russian), Uchen. Zap. MGU, Mat. **30** , (1939), no. 3, 13–16.

[29] M. A. KRASNOSELSKII and Ya. B. RUTICKII, Convex Functions and Orlicz Spaces, P. Noordhoff Ltd. Groningen 1961.

[30] L. D. KUDRYAVCEV and S. M. NIKOLSKII, *Spaces of differentiable functions of several variables and imbedding theorems* (in Russian), Itogi Nauki i Tekhniki. Seriya "Sovremennye Problemi Matematyki. Fundamentalnye Napravleniya". Vol. 26 (1988), 5–148.

[31] A. KUFNER, *Weighted Sobolev Spaces*, John Wiley and Sons, Chichester, 1985.

[32] M. KWONG and A. ZETTL, *A norm inequalities for derivatives.* In: Ordinary and partial differential equations (Proc. Sixth Conf. Univ. Dundee, Dundee, 1980), Lect. Notes Math. 846, Springer, Berlin–New-York 1981, pp. 227–243.

[33] S. MACHIHARA and T. OZAWA, *Interpolation inequalities in Besov spaces*, Proc. Amer. Math. Soc. **131**, 5 (2003), no. 5, 1553–1556 (electronic).

[34] J. MATEAU, P. MATTILA, A. NICOLAU and J. OROBITG, *BMO for nondoubling measures*, Duke Math. J. **102** (2000), 533–565.

[35] V. G. MAZ'YA, Sobolev Spaces, Springer–Verlag 1985.

[36] V. G. MAZ'YA and T. SHAPOSHNIKOVA, *Pointwise interpolation inequalities for derivatives with best constants*, (Russian) Funktsjonal. Anal. i Prilozhen. **36** (2002), no. 1, 36–58; English transl.: Funct. Anal. Appl. **36** (2002), no. 1, 30–48.

[37] D. S. MITRINOVIĆ, J.E. PEČARIĆ, and A. M. FINK, *Inequalities Involving Functions and Their Derivatives,* Kluwer Acad. Publishers, Dordrecht–Boston–London, 1991.

[38] K.T. MYNBAEV and M. O. OTELBAEV, *Weighted function spaces and their applications to the study of boundary value problems for elliptic equations in divergent form* (in Russian), Izv. Vyssh. Uchebn. Zaved. Mat. 315,8 (1988), 4–55.

[39] S. M. NIKOLSKII, P. I. LIZORKIN and N. V. MIROSHIN, *Weighted function spaces and their applications to the study of boundary value problems in elliptic equations in divergent form* (in Russian), Izv. Vyssh. Uchebn. Zaved. Mat. 315,8 (1988), 4–55.

[40] L. NIRENBERG, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. di Pisa, **13** (1959), no. 3, 115-162.

[41] J. OROBITG, C. PÉREZ, *A_p weights for nondoubling measures in $\mathbb{R}^n$ and applications,* Trans. Amer. Math. Soc. **354** (2002), 2013–2033.

[42] W. POMPE, *Quasistatic and dynamic problems in viscoplasticity theory – nonlinearities with power growth*, Math. Methods Appl. Sci. **27** (2004), no. 11, 1347–1365.

[43] M. M. RAO and Z. D. REN, *Theory of Orlicz spaces*, M. Dekker, Inc. New York 1991.

[44] E. W. STREDULINSKY, *Weighted Norm Inequalities and Degenerate Elliptic Partial Differential Equations*, Lecture Notes in Math., Vol. 1074, Springer-Verlag, Berlin-Heidelberg-New-York-Tokyo 1984.

Institute of Mathematics,
Warsaw University,
ul. Banacha 2,
02-097 Warszawa, Polska (Poland),
emails: `kalamajs@mimuw.edu.pl` and `kpp@mimuw.edu.pl`