# Nonstandard Generic Points

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**Abstract**

Starting from the Zariski topology, a natural notion of nonstandard generic point is introduced in complex algebraic geometry. The existence of this kind of point is a strong form of the Nullstellensatz. This notion is connected with the classical concept of generic point in the spectrum $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ of the corresponding algebra $\mathcal{A}_{n,\mathbb{C}}$. The nonstandard affine space $^*\mathbb{C}^n$ appears as an affine unfolding of the geometric space $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$. This affine space is the disjoint union of the sets whose elements are the nonstandard generic points of prime and proper ideals of $\mathcal{A}_{n,\mathbb{C}}$: this structure leads to the definition of algebraic points in $^*\mathbb{C}^n$. A natural extension to analytic points in $^*\mathbb{C}^n$ is given by Robinson's concept of generic point in local complex analytic geometry. The end of this paper is devoted to a generalization of this point of view to the real analytic case.

## 1 Introduction

Since the ancient Greeks, a *point* is a very simple geometrical figure without any internal structure. This mode of thinking is being questioned by the contemporary developments of algebraic geometry. Particularly with A.Grothendieck's works [7, 15], sophisticated concepts of points appeared thanks to an impressive ascent in abstraction : "*A R-valued point of a prescheme X is a morphism of* $\mathrm{Spec}(R)$ *into X*". One of the motivations of these constructions is to get infinitesimals well adapted to algebraic and geometrical structures.

This is the way David Mumford motivates the introduction of the concept of preschemes and schemes in *The Red Book of the Varieties and Schemes* :

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> *Another motivation for preschemes comes from the possibility of constructing via schemes an explicit and meaningful theory of infinitesimal objects.*

In a completely independent way, nonstandard analysis offers at once infinitesimals available in most elementary spaces [2, 3, 8, 9, 12, 16, 19]. Furthermore, this method provides every point of the euclidian space with a non trivial structure [4, 5].

It is natural to wonder on the connections which exist among both points of view. The purpose of this work is to begin to establish a link between them [1]. The motivation is obviously not to reconstruct algebraic geometry on a nonstandard basis. The idea is rather to throw light on some difficult concepts of algebraic geometry by getting them in touch with typical nonstandard entities.

Within the framework of this program, firstly we introduce a nonstandard notion of generic point of an algebraic sets in the affine space $\mathbb{C}^n$. Secondly, we establish a comparison between two kinds of generic point: the classical generic points which belong to the spectrum of the ring of polynomials and the nonstandard generic points which are points of the nonstandard affine space. Finally the concept of generic point is generalized to the analytic case.

This study is an extension of former works. The notion of nonstandard generic point in the case of local complex analytic geometry leading to the Rückert Nullstellensatz was introduced by A. Robinson, the creator of nonstandard analysis [20]. In the same spirit, the author of these lines also introduced a concept of nonstandard generic point adapted to the local real analytical geometry in a document not formally published [24]. A nonstandard point of view for generic points was also developped in [6].

We use the usual following notations: $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let $n$ be a positive integer and let $\mathcal{A}_{n,\mathbb{K}}$ be the ring $\mathbb{K}[X_1, \ldots, X_n]$ of polynomials with coefficients in $\mathbb{K}$ and $n$ indeterminates. Finally, let $\mathcal{O}_{n,\mathbb{K}}$ be the ring $\mathbb{K}\{X_1, \ldots, X_n\}$ of convergent power series with coefficients in $\mathbb{K}$ and $n$ indeterminates.

## 2 Nonstandard preliminaries

### 2.1 Choosing a nonstandard formalism

Although the nonstandard formalism generally used by the author is *Internal Set Theory* (I.S.T.) of E.Nelson [3, 2, 16], this work is stated in the more classical framework of *nonstandard extension of sets and structures* [8, 9, 12, 17]. This choice offers some conceptual advantages about the notion of external sets. Furthermore, the point of view according to which we look for solutions of polynomial or analytic equations with coefficients in $\mathbb{C}$ in an appropriate extension ${}^*\mathbb{C}$ of $\mathbb{C}$ is more in the spirit of field theory and algebraic geometry. In some way, we can consider that nonstandard generic points of an ideal $\mathcal{P}$ are *special imaginary solutions* of the equations belonging to $\mathcal{P}$. Lastly, this nonstandard formalism is developed in the

[1] This work is probably related to the approach of algebraic geometry by model theory [14]

usual mathematical language (i.e. set theory); we hope this choice will contribute to fill the too important gap between nonstandardists and the majority of working mathematicians.

## 2.2 A very short introduction to nonstandard extension

All the sets needed in this work ($\mathbb{N}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{A}_{n,\mathbb{K}}$ and $\mathcal{O}_{n,\mathbb{K}}$) are identified to suitable subsets of $\mathcal{O}_{n,\mathbb{C}}$. We choose a nonstandard extension ${}^*\mathcal{O}_{n,\mathbb{C}}$ of $\mathcal{O}_{n,\mathbb{C}}$ which is sufficiently rich[2]. All ordinary sets $X$ depending on $\mathcal{O}_{n,\mathbb{C}}$ are simultaneously extended to new objects ${}^*X$ such that $X \subset {}^*X$; every $x \in X$ is a *standard* element of ${}^*X$, every element of ${}^*X \setminus X$ is a *nonstandard* element of ${}^*X$. Hence, we get at once nonstandard extensions ${}^*\mathbb{N}$ of $\mathbb{N}$, ${}^*\mathbb{R}$ of $\mathbb{R}$, ${}^*\mathbb{C}$ of $\mathbb{C}$, ${}^*\mathcal{A}_{n,\mathbb{R}}$ of $\mathcal{A}_{n,\mathbb{R}}$, ${}^*\mathcal{A}_{n,\mathbb{C}}$ of $\mathcal{A}_{n,\mathbb{C}}$ and ${}^*\mathcal{O}_{n,\mathbb{R}}$ of $\mathcal{O}_{n,\mathbb{R}}$.

The main interest of this machinery lies in two important principles[3]. The first one asserts that $X$ and ${}^*X$ share the same properties. From the second one, we deduce the existence of many ideal objects in ${}^*X$.

### 2.2.1 The transfer principle

Given a formula $\varphi$ without free variable relating a mathematical property of objects like the preceding $X$, the $*$-transform ${}^*\varphi$ of $\varphi$ is obtained by replacing each object $Y$ in $\varphi$ by its extension ${}^*Y$. The first important property of a nonstandard extension is: *${}^*\varphi$ is true whenever $\varphi$ is true.* This is the *transfer principle*.

From this principle, we deduce that the $*$-transform behaves well with regard to set operations: ${}^*\emptyset = \emptyset$, ${}^*(A \setminus B) = {}^*A \setminus {}^*B$, ${}^*(A \times B) = {}^*A \times {}^*B$, if $f$ is a map from $A$ to $B$ then ${}^*f$ is a map from ${}^*A$ to ${}^*B$,... We also see that ${}^*\mathbb{R}$ is an ordered field extension of $\mathbb{R}$ and ${}^*\mathbb{C}$ is a field extension of $\mathbb{C}$, that $\mathcal{A}_{n,\mathbb{K}}$ is a $\mathbb{K}$-subalgebra of ${}^*\mathcal{A}_{n,\mathbb{K}}$ and that $\mathcal{O}_{n,\mathbb{K}}$ is a $\mathbb{K}$-subalgebra of ${}^*\mathcal{O}_{n,\mathbb{K}}$. From the transfert principle we also deduce that, for every $f \in {}^*(\mathbb{C}[X])$, there exists $x \in {}^*\mathbb{C}$ such that $f(x) = 0$; as ${}^*\mathbb{C}[X] \subset {}^*(\mathbb{C}[X])$, we get that the field ${}^*\mathbb{C}$ is algebraically closed.

### 2.2.2 The enlargement property or idealization principle

Let $U$ and $V$ be two arbitrary sets like the preceding $X$ and let $\varphi(u, v)$ a formula with two free variables representing a relation on $U \times V$. Suppose that for every finite set $F \subset U$ there exists $v \in V$ such that $\varphi(u, v)$ for every $u \in F$. Then, the second important property is: *There exists $w \in {}^*V$ such that ${}^*\varphi(u, w)$ for every $u \in U$.* This is the *enlargement property*.

Also called the *idealization principle*, this last property is a tool for getting many ideal objects in the nonstandard extension. For instance, there exists $\omega \in {}^*\mathbb{N}$ such that $k \leq \omega$ for all $k \in \mathbb{N}$: the element $\omega$ is a nonstandard integer which is *infinitely large*. In the same way, we get nonstandard real or complex numbers which are *infinitely small*, i.e. $\varepsilon \in^* \mathbb{R}$ or ${}^*\mathbb{C}$ such that $\forall n \in \mathbb{N} \setminus \{0\} \quad |\varepsilon| \leq 1/n$.

[2]Actually, we need a nonstandard extension of the superstructure $V({}^*\mathcal{O}_{n,\mathbb{C}})$ of ${}^*\mathcal{O}_{n,\mathbb{C}}$ and we ask this nonstandard extension to be an enlargement [8, 9, 12].

[3]A third property, the *Internal Definition Principle*, is also useful but we do not need it in this short introduction.

## 3 Algebraic sets and weak Nullstellensatz

An algebraic set of $\mathbb{C}^n$ is the set of roots of a family of polynomials $\mathcal{I} \subset \mathcal{A}_{n,\mathbb{C}}$. Without loss of generality (consider the ideal generated by $\mathcal{I}$), we may suppose that $\mathcal{I}$ is an ideal of the ring $\mathcal{A}_{n,\mathbb{C}}$. Let $Z(\mathcal{I})$ be the algebraic set defined by an ideal $\mathcal{I}$

$$Z(\mathcal{I}) = \{x \in \mathbb{C}^n \ ; \ \forall f \in \mathcal{I} \ f(x) = 0\}$$

There exists a finite family $(f_1, \dots, f_p)$ of polynomials which generates the ideal $\mathcal{I}$, so that

$$Z(\mathcal{I}) = \{x \in \mathbb{C}^n \ ; \ \forall i = 1, \dots, p \ f_i(x) = 0\}$$

The base of the study of the algebraic sets is the so called Nullstellensatz of Hilbert. It is possible to distinguish between a weak version and a strong version of this result.

**Weak Nullstellensatz.** *If $\mathcal{I}$ is a proper ideal of $\mathcal{A}_{n,\mathbb{C}}$, then the algebraic set $Z(\mathcal{I})$ is not empty.*

For each $x \in \mathbb{C}^n$, the set

$$\mathcal{M}_x = \{f \in \mathcal{A}_{n,\mathbb{C}} \ ; \ f(x) = 0\}$$

is a maximal ideal of the ring $\mathcal{A}_{n,\mathbb{C}}$. The maximal spectrum of $\mathcal{A}_{n,\mathbb{C}}$ is the set $\mathrm{Spec}_{\max}(\mathcal{A}_{n,\mathbb{C}})$ of the maximal ideals of $\mathcal{A}_{n,\mathbb{C}}$. The spectrum of the ring $\mathcal{A}_{n,\mathbb{C}}$ is the set $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ whose elements are the proper prime ideals of $\mathcal{A}_{n,\mathbb{C}}$. Thus, we have $\mathrm{Spec}_{\max}(\mathcal{A}_{n,\mathbb{C}}) \subset \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$.

**Equivalent version of the weak Nullstellensatz.** *The map $\theta : x \mapsto \mathcal{M}_x$ from $\mathbb{C}^n$ to $\mathrm{Spec}_{\max}(\mathcal{A}_{n,\mathbb{C}})$ is a bijection.*

With the help of $\theta$, it is possible to identify $\mathbb{C}^n$ with $\mathrm{Spec}_{\max}(\mathcal{A}_{n,\mathbb{C}})$. Thus, the spectrum $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ appears as an extension of the affine space $\mathbb{C}^n$. This is a way to add a new kind of 'points' to the affine space, namely the proper prime ideals which are not maximal.

## 4 A nonstandard glance at Zariski topology

The Zariski topology on $\mathbb{C}^n$ is a topology only defined with the help of algebraic sets. Hence, it is a well adapted tool for algebraic geometry. In this topology, a subset $U$ of $\mathbb{C}^n$ is open if and only if $\mathbb{C}^n \setminus U$ is an algebraic set. An equivalent condition is that there exists a finite family of polynomials $f_1, \dots, f_p \in \mathcal{A}_{n,\mathbb{C}}$ such that

$$U = \{x \in \mathbb{C}^n \ ; \ \exists i = 1, \dots, p \ f_i(x) \neq 0\} = \bigcup_{i=1}^{p} \mathbb{C}^n \setminus f_i^{-1}(\{0\})$$

Thus, the family $\left(\mathbb{C}^n \setminus f^{-1}(\{0\})\right)_{f \in \mathcal{A}_{n,\mathbb{C}}}$ is a basis for the Zariski topology. For each $f \in \mathcal{A}_{n,\mathbb{C}}$, the open set $\mathbb{C}^n_f = \mathbb{C}^n \setminus f^{-1}(\{0\})$ is called a distinguished open subset.

The Zariski topology is less fine that the usual one: every non empty Zariski open set is an usual open set which is dense. For this reason, a property $\mathcal{Q}(x)$ is called *generic* on $\mathbb{C}^n$ whenever $\mathcal{Q}(x)$ is true at every point $x$ belonging to a nonempty Zariski open set of $\mathbb{C}^n$. The space $\mathbb{C}^n$ is not separated in the Zariski topology ; nevertheless, the points of $\mathbb{C}^n$ are closed.

Let $X$ be a set on which a topology $\mathcal{T}$ is defined. Given a point $a$ of $X$, the nonstandard point of view introduces a kind of universal neighbourhood of $a$ for $\mathcal{T}$: the *halo* (also named the *monad*) of $a$. This is the subset of a nonstandard extension ${}^*X$ of $X$ defined by

$$\mathrm{hal}_{\mathcal{T}}(a) = \bigcap_{U \in \mathcal{V}(a)} {}^*U$$

which is the intersection of the nonstandard extension ${}^*U$ of the elements $U$ belonging to the set $\mathcal{V}(a)$ of open neighbourhoods of $a$ in $X$ for $\mathcal{T}$. We say that each point $x$ in $\mathrm{hal}(a)$ is infinitely close to $a$ and we write $x \simeq_{\mathcal{T}} a$ this relation. A general observation is that the local properties of the topological space $X$ at $a$ are condensed in $\mathrm{hal}_{\mathcal{T}}(a)$ [19, 22].

Now, our purpose is to apply this tool to the Zariski topology on the affine space. According to the general definition, the halo of an element $a$ of $\mathbb{C}^n$ in the Zariski topology is the set

$$\mathrm{hal}_Z(a) = \{x \in {}^*\mathbb{C}^n \ ; \ x \simeq_Z a\} = \bigcap_{U \in \mathcal{V}_Z(a)} {}^*U$$

where $\mathcal{V}_Z(a)$ is the set of open neighbourhoods of $a$ in $\mathbb{C}^n$ for the Zariski topology.

It is clear that $\mathcal{V}_Z(a)$ can be replaced by the set of all $U$ in the basis $\left(\mathbb{C}^n_f\right)_{f \in \mathcal{A}_{n,\mathbb{C}}}$ such that $a \in U$. Hence we get

$$\mathrm{hal}_Z(a) = \bigcap_{\substack{f \in \mathcal{A}_{n,\mathbb{C}} \\ f(a) \neq 0}} {}^*(\mathbb{C}^n \setminus f^{-1}(\{0\}))$$

In other words, we have the following characterization

$$x \simeq_Z a \iff \forall f \in \mathcal{A}_{n,\mathbb{C}} \ (f(a) \neq 0 \implies f(x) \neq 0)$$

or equivalently

$$x \simeq_Z a \iff \forall f \in \mathcal{A}_{n,\mathbb{C}} \ (f(x) = 0 \implies f(a) = 0)$$

This property brings to light a family of subsets of the ring $\mathcal{A}_{n,\mathbb{C}}$ : for each $x \in {}^*\mathbb{C}^n$ we consider the set $\{f \in \mathcal{A}_{n,\mathbb{C}} \ ; \ f(x) = 0\}$ which is similar to the maximal ideal $\mathcal{M}_y$ for $y \in \mathbb{C}^n$

$$\mathcal{M}_y = \{f \in \mathcal{A}_{n,\mathbb{C}} \ ; \ f(y) = 0\}$$

We see that this set is a prime and proper ideal of $\mathcal{A}_{n,\mathbb{C}}$. If $x$ is a standard element of ${}^*\mathbb{C}^n$ (i.e. if $x \in \mathbb{C}^n$) then, this set is equal to the maximal ideal $\mathcal{M}_x$.

**Definition 1.** *Given $x \in {}^*\mathbb{C}^n$, the* standard ideal null at $x$ *is the prime proper ideal $\mathcal{P}_x$ of $\mathcal{A}_{n,\mathbb{C}}$ defined by $\mathcal{P}_x = \{f \in \mathcal{A}_{n,\mathbb{C}} \ ; \ f(x) = 0\}$.*

We now return to the study of $\mathrm{hal}_Z(a) = \{x \in {}^*\mathbb{C}^n \ ; \ x \simeq_Z a\}$ for the Zariski topology.

**Proposition 1.** *Given $a \in \mathbb{C}^n$, for every $x \in {}^*\mathbb{C}^n$, we have*

$$x \simeq_Z a \iff \mathcal{P}_x \subset \mathcal{P}_a \iff \mathcal{P}_x \subset \mathcal{M}_a \iff a \in Z(\mathcal{P}_x)$$

*where $Z(\mathcal{P}_x)$ is the algebraic set of $\mathbb{C}^n$ defined by the standard ideal $\mathcal{P}_x$ null at $x$.*

# 5 Strong Nullstellensatz and nonstandard generic points

## 5.1 The classical strong Nullstellensatz

For each subset $F$ of $\mathbb{C}^n$, we define the ideal $I(F)$ of $\mathcal{A}_{n,\mathbb{C}}$ by

$$I(F) = \{f \in \mathcal{A}_{n,\mathbb{C}} \ ; \ \forall x \in F \ f(x) = 0\}$$

The algebraic set $Z(I(F))$ is obviously the closure of $F$ for the Zariski topology.

Now we can state the strong version of the Nullstellensatz in the case of a prime ideal.

**Strong Nullstellensatz.** *If $\mathcal{P}$ is a prime ideal of $\mathcal{A}_{n,\mathbb{C}}$, then $I(Z(\mathcal{P})) = \mathcal{P}$.*

An algebraic set $F$ of $\mathbb{C}^n$ is irreducible if it is impossible to have $F = F_1 \cup F_2$ where $F_1$ and $F_2$ are algebraic sets such that $F \neq F_1$ or $F \neq F_2$. It is well known that an algebraic set $F$ is irreducible if and only if $I(F)$ is a prime ideal, or equivalently, if there exists a prime ideal $\mathcal{P}$ such that $F = Z(\mathcal{P})$.

The Strong Nullstellensatz is linked to a nonstandard concept of generic point.

## 5.2 Introduction of nonstandard generic points

**Definition 2.** *A nonstandard generic point of a prime ideal $\mathcal{P}$ of $\mathcal{A}_{n,\mathbb{C}}$ is a point $x$ of ${}^*\mathbb{C}^n$ such that*

$$\forall f \in \mathcal{A}_{n,\mathbb{C}} \ (f(x) = 0 \iff f \in \mathcal{P})$$

In other words, $x \in {}^*\mathbb{C}^n$ is a generic point of a prime ideal $\mathcal{P}$ if and only if $\mathcal{P} = \mathcal{P}_x$ where $\mathcal{P}_x$ is the standard ideal null at $x$ introduced in the previous section.

We choose the name 'nonstandard generic point' to avoid confusion with another already existing notion of generic point. However, this appellation has the inconvenience to let believe that a nonstandard generic point $x$ is a nonstandard point, i.e. $x \in {}^*\mathbb{C}^n \setminus \mathbb{C}^n$. This is not always the case. We get at once the following characterization of 'standard nonstandard generic points'.

**Proposition 2.** *Let $\mathcal{P}$ be a prime proper ideal of $\mathcal{A}_{n,\mathbb{C}}$ and let $x \in {}^*\mathbb{C}^n$ be a nonstandard generic point of $\mathcal{P}$. Then, the following conditions are equivalent.*

1. *The point $x$ is standard.*
2. *The ideal $\mathcal{P}$ is a maximal ideal of $\mathcal{A}_{n,\mathbb{C}}$.*
3. *The algebraic set $Z(\mathcal{P})$ of $\mathbb{C}^n$ defined by $\mathcal{P}$ has only one element.*

From the definition, we see that a nonstandard generic point $x$ of a prime proper ideal $\mathcal{P}$ of $\mathcal{A}_{n,\mathbb{C}}$ is a point of the subset $Z_{^*\mathbb{C}}(\mathcal{P})$ of the nonstandard affine space ${}^*\mathbb{C}^n$ defined by the ideal $\mathcal{P}$

$$Z_{^*\mathbb{C}}(\mathcal{P}) = \{y \in {}^*\mathbb{C}^n \ ; \ \forall f \in \mathcal{P} \ f(y) = 0\}$$

Hence, an element $x \in {}^*\mathbb{C}^n$ is a non standard generic point of a prime ideal $\mathcal{P}$ of $\mathcal{A}_{n,\mathbb{C}}$ if and only if the following two conditions are satisfied

$$\begin{cases} x \in Z_{^*\mathbb{C}}(\mathcal{P}) \\ \forall f \in \mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P} \quad f(x) \neq 0 \end{cases}$$

Thus a nonstandard generic point of $\mathcal{P}$ is a solution in ${}^*\mathbb{C}^n$ of the equations belonging to $\mathcal{P}$ which is not solution of other equations in $\mathcal{A}_{n,\mathbb{C}}$: a nonstandard generic point of a prime proper ideal $\mathcal{P}$ characterizes $\mathcal{P}$.

Since the field ${}^*\mathbb{C}$ is an extension of the field $\mathbb{C}$ of the standard complexe numbers, we remark that *the framework of nonstandard generic points is in accordance to the usual situation in Galois theory and algebraic geometry in which solutions of polynomial equations are studied in an extension of the field which contains all coefficients of the equations.*

Furthermore, for every ideal $\mathcal{I}$ of $\mathcal{A}_{n,\mathbb{C}}$, we can defined three 'algebraic sets'. The first one is the usual algebraic set

$$Z(\mathcal{I}) = \{x \in \mathbb{C}^n \ ; \ \forall f \in \mathcal{I} \ f(x) = 0\}$$

The second one is the nonstandard extension ${}^*Z(\mathcal{I})$ of $Z(\mathcal{I})$. From the transfer principle, we deduce that

$${}^*Z(\mathcal{I}) = \{x \in {}^*\mathbb{C}^n \ ; \ \forall f \in {}^*\mathcal{I} \ f(x) = 0\}$$

The last one is

$$Z_{^*\mathbb{C}}(\mathcal{I}) = \{x \in {}^*\mathbb{C}^n \ ; \ \forall f \in \mathcal{I} \ f(x) = 0\}$$

But, there exists a finite set $\{f_1, \ldots, f_p\} \subset \mathcal{I}$ such that $\mathcal{I} = (f_1, \ldots, f_p)$. Hence,

$$\forall x \in \mathbb{C}^n (\forall f \in \mathcal{I} \ f(x) = 0 \Longleftrightarrow \forall k = 1, \ldots, p \ f_k(x) = 0)$$

Thus

$$\forall x \in {}^*\mathbb{C}^n (\forall f \in {}^*\mathcal{I} \ f(x) = 0 \Longleftrightarrow \forall k = 1, \ldots, p \ f_k(x) = 0 \Longleftrightarrow \forall f \in \mathcal{I} \ f(x) = 0)$$

which means that ${}^*Z(\mathcal{I}) = Z_{^*\mathbb{C}}(\mathcal{I})$.

Hence, for every nonstandard generic point $x$ of a prime proper ideal $\mathcal{P}$ of $\mathcal{A}_{n,\mathbb{C}}$, we have not only $f(x) = 0$ for every $f \in \mathcal{P}$ but also $f(x) = 0$ for every $f \in {}^*\mathcal{P}$.

Returning to the description of the halo of a given point $a \in {}^*\mathbb{C}^n$ for the Zariski topology, we can now assert that $\mathrm{hal}_Z(a)$ is the set of all generic points of all prime proper ideals $\mathcal{P}$ of $\mathcal{A}_{n,\mathbb{C}}$ such that $a \in Z(\mathcal{P})$.

**Remark** Every element of the localization $(\mathcal{A}_{n,\mathbb{C}})_{\mathcal{P}}$ of $\mathcal{A}_{n,\mathbb{C}}$ in $\mathcal{P}$ is written $\varphi = f/g$ where $(f, g) \in \mathcal{A}_{n,\mathbb{C}} \times ((\mathcal{A}_{n,\mathbb{C}}) \setminus \mathcal{P})$ ; thus, $\varphi$ is a function defined at some point $x \in {}^*\mathbb{C}^n$ if $g(x) \neq 0$. Then, all the elements of the localisation $(\mathcal{A}_{n,\mathbb{C}})_{\mathcal{P}}$ are defined at some $x \in {}^*\mathbb{C}^n$ if and only in $x$ is a nonstandard generic point of $\mathcal{P}$. That is to say, in ${}^*Z(\mathcal{P})$, the set of definition of all elements of $(\mathcal{A}_{n,\mathbb{C}})_{\mathcal{P}}$ is equal to the external set of nonstandard generic points of $\mathcal{P}$.

### 5.3 Existence of nonstandard generic points

The existence of nonstandard generic points is a consequence of the strong Nullstellensatz.

**Theoreme 1.** *Every prime and proper ideal of $\mathcal{A}_{n,\mathbb{C}}$ has a nonstandard generic point.*

*Proof.* Let $\mathcal{P}$ a prime and proper ideal of $\mathcal{A}_{n,\mathbb{C}}$. Given a finite subset $\mathcal{F} = \{f_1, \ldots, f_p\}$ of $\mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P}$ the product $f_1 \cdots f_p$ does not belong to $\mathcal{P}$. From the strong Nullstellensatz, we deduce that $f_1 \cdots f_p \notin I(Z(\mathcal{P}))$, i.e.

$$\exists y \in Z(\mathcal{P}) \; \forall k = 1, \ldots, p \quad f_k(y) \neq 0$$

From the idealization principle we get

$$\exists x \in {}^*Z(\mathcal{P}) \; \forall f \in \mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P} \quad f(x) \neq 0$$

that is to say, $x$ is a nonstandard generic point for $\mathcal{P}$. ∎

Conversely, this last result implies at once the strong Nullstellensatz: if $f \in I(Z(\mathcal{P}))$, then $f(y) = 0$ for every $y \in Z(\mathcal{P})$; the transfer principle implies that $f(x) = 0$ for every point $x$ of ${}^*Z(\mathcal{P})$ and thus also for a nonstandard generic point $x$ of $\mathcal{P}$, hence $f \in \mathcal{P}$.

### 5.4 A new version of the strong Nullstellensatz

From the existence of nonstandard generic point we deduce an equivalent version of the strong Nullstellensatz similar to the one of the weak Nullstellensatz.

**Equivalent version of the strong Nullstellensatz.** *The map* $\Theta : x \mapsto \mathcal{P}_x$ from ${}^*\mathbb{C}^n$ to $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ *is onto.*

This map $\Theta$ is an extension of

$$\begin{array}{ccccc} \theta & : & \mathbb{C}^n & \longrightarrow & \mathrm{Spec}_{\max}(\mathcal{A}_{n,\mathbb{C}}) \\ & & x & \longmapsto & \mathcal{M}_x \end{array}$$

Thus we get the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\theta} & \mathrm{Spec}_{\max}(\mathcal{A}_{n,\mathbb{C}}) \\ \big\downarrow & & \big\downarrow \\ {}^*\mathbb{C}^n & \xrightarrow{\Theta} & \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}}) \end{array}$$

in which the vertical arrows are canonical injections. The map $\theta$ is one to one and the map $\Theta$ is onto. With the help of the maps $\theta$ and $\Theta$, the set $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ may be interpreted as a kind of nonstandard extension of the space $\mathbb{C}^n$ similar to the extension ${}^*\mathbb{C}^n$. From this point of view, the prime proper ideals of $\mathcal{A}_{n,\mathbb{C}}$ which are not maximal are similar to nonstandard points of ${}^*\mathbb{C}^n$. Since the map $\Theta$ is onto, the nonstandard points in ${}^*\mathbb{C}^n$ are more numerous than the prime proper ideals in $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$.

### 5.5 Local existence of nonstandard generic points

Let $a$ be a point of $\mathbb{C}^n$ and $\mathcal{P}$ be a prime proper ideal of $\mathcal{A}_{n,\mathbb{C}}$ such that $a \in Z(\mathcal{P})$ (i.e. $\mathcal{P} \subset \mathcal{M}_a$). From the description of $\mathrm{hal}_Z(a)$ we deduce that every nonstandard generic point $x$ of $\mathcal{P}$ is infinitly close to $a$ for the Zariski topology. This does not means that $x$ is near $a$ for the usual topology. We want to rule on the local existence near the point $a$ of nonstandard generic points for the usual topology of $\mathbb{C}^n$ and its extension to ${}^*\mathbb{C}^n$. Hence, for $x \in {}^*\mathbb{C}^n$, we introduce the usual nonstandard relation $x \simeq a$ (we say that $x$ *is infinitely close to $a$ for the usual topology*) defined by

$$x \simeq a \iff \forall k \in \mathbb{N} \setminus \{0\} \ \|x - a\| < \frac{1}{k}$$

where $\| \ \|$ is a norm on $\mathbb{C}^n$ and also its extension to ${}^*\mathbb{C}^n$.

**Theoreme 2.** *Let $\mathcal{P}$ be a prime proper ideal of $\mathcal{A}_{n,\mathbb{C}}$ and $a \in \mathbb{C}^n$ a point of the algebraic set $Z(\mathcal{P})$. Then, there is a nonstandard generic point $x \in {}^*\mathbb{C}^n$ of $\mathcal{P}$ such that $x \simeq a$.*

*Proof.* Due to the idealization principle, it is sufficient to prove that, for every finite subset $\mathcal{F}$ of $\mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P}$ and every $k \in \mathbb{N}^*$, there exists $y \in Z(\mathcal{P})$ such that $\|y - a\| < 1/k$ and $f(y) \neq 0$ for every $f \in \mathcal{F}$.

Let $\mathcal{F}$ be a finite subset of $\mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P}$. For every $f \in \mathcal{F}$, the set $F = f^{-1}(\{0\}) \cap Z(\mathcal{P})$ is a Zariski-closed subset of $Z(\mathcal{P})$. Furthermore, the Nullstenllensatz implies that $F \neq Z(\mathcal{P})$. We know that a closed subset $F$ of an irreducible algebraic subset $V$ of $\mathbb{C}^n$ has an empty interior in $V$ or is equal to $V$. Thus, the set $\mathbb{C}^n_f \cap Z(\mathcal{P})$ is a dense subset of $Z(\mathcal{P})$ for the Zariski topology. Since the Zariski closure of a constructible set in an algebraic variety is equal to its closure for the usual (transcendant) topology, we get that $\mathbb{C}^n_f \cap Z(\mathcal{P})$ is an open dense subset of $Z(\mathcal{P})$ for the usual topology. Because a finite intersection of open dense subsets is a dense subset, we get the result. ∎

Let $\mathrm{hal}(a)$ be the halo of $a$ in ${}^*\mathbb{C}^n$ for the usual topology, that is to say the set whose elements are the points $x \in {}^*\mathbb{C}^n$ such that $x \simeq a$. For each $x \in \mathrm{hal}(a)$, the standard ideal $\mathcal{P}_x$ null at $x$ is such that $a \in Z(\mathcal{P}_x)$ since each $f \in \mathcal{P}_x$ is continuous at $a$, so $f(a) \simeq f(x) = 0$. Thus, $\mathcal{P}_x$ belongs to the set

$$\mathcal{H}_Z(\mathcal{M}_a) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}}) \ ; \ \mathcal{P} \subset \mathcal{M}_a\}$$

Consequently, the following map

$$\left\{ \begin{array}{cccc} \Theta_a & : \ \mathrm{hal}(a) & \longrightarrow & \mathcal{H}_Z(\mathcal{M}_a) \\ & x & \longmapsto & \mathcal{P}_x \end{array} \right.$$

is onto.

### 5.6 The status of the map $\Theta$

The author has the deep conviction that the map $\Theta$ is very natural. Nevertheless, in the framework of nonstandard analysis, this map is external. Actually, we can distinguish three kind of objects.

A *standard* object is an element of a set $X$ which appears initially in our work before we consider a nonstandard extension of $X$. For instance, any $x \in \mathbb{C}^n$, any $f \in \mathcal{A}_{n,\mathbb{C}}$ or any $\mathcal{P} \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ are standard. A standard map is a map between two standard sets.

An *internal* object is an element of one of the nonstandard extension ${}^*X$ which appears in our work. For instance, any $f \in {}^*\mathcal{A}_{n,\mathbb{C}}$ is an internal polynomial. An internal map is a map $g : {}^*X \to {}^*Y$ such that $g \in {}^*(\mathcal{P}(X \times Y))$ (the nice properties of nonstandard extensions imply that ${}^*(\mathcal{P}(X \times Y)) \subset \mathcal{P}({}^*X \times {}^*Y))$.

An *external* object is one which is neither standard nor internal. In some way, standard and internal objects are simpler and more natural than external one. However, the introduction of exernal objects is one of the interesting contributions of nonstandard analysis. We see that our map $\Theta$ is external because it is a map from the internal set ${}^*\mathbb{C}^n$ to the standard set $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$.

## 6 A link between two kinds of generic points

In the perspective of constructing the deep concepts of prescheme and scheme at the basis of algebraic topology, the set $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ is provided with a Zariski topology. As we shall see, all the points of this topological space are not closed. This last property is connected to the 'standard' notion of generic point.

For each ideal $\mathcal{I}$ of the ring $\mathcal{A}_{n,\mathbb{C}}$, let $V(\mathcal{I})$ be the subset of $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ defined by

$$V(\mathcal{I}) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}}) \ ; \ \mathcal{I} \subset \mathcal{P}\}$$

Since

$$x \in Z(\mathcal{I}) \iff \mathcal{I} \subset \mathcal{M}_x \iff \mathcal{M}_x \in V(\mathcal{I})$$

we can conceive the set $V(\mathcal{I})$ as a natural extension of the algebraic set $Z(\mathcal{I})$ in the set $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$.

The sets $Z(\mathcal{I})$ are the closed subsets for a topology: the Zariski topology on $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$. For each $f \in \mathcal{A}_{n,\mathbb{C}}$, let

$$\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})_f = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}}) \ ; \ f \notin \mathcal{P}\}$$

Since $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})_f = \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}}) \setminus V((f))$, this set is open and called a distinguished open set. The family $(\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})_f)_{f \in \mathcal{A}_{n,\mathbb{C}}}$ is a basis of the Zariski topology on $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$.

For each $\mathcal{P} \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$, the closure of $\{\mathcal{P}\}$ is equal to $V(\mathcal{P})$ ; thus, a point $\{\mathcal{P}\}$ is closed if and only if the ideal $\mathcal{P}$ is maximal.

A closed subset $F$ of $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ is irreducible if there does not exist closed subset $F_1$ and $F_2$ different from $F$ such that $F = F_1 \cup F_2$.

Then, the classical definition of a *generic point of a closed irreducible subset* $F$ of the space $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ is the following : it is an element $\mathcal{P}$ of $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ such that $F$ is the closure of $\{\mathcal{P}\}$. Thus, this is a point which is dense in $F$.

We verify than a closed subset $F$ is irreducible if and only if there exists $\mathcal{P} \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ such that $F = V(\mathcal{P})$, in which case $\mathcal{P}$ is the unique generic point of $F$.

The two kinds of generic points appear to be closely linked. The bridge between the two notions is the map $\Theta$.

**Proposition 3.** *Each $x \in {}^*\mathbb{C}^n$ is a nonstandard generic point of $\mathcal{P}_x = \Theta(x)$ and $\Theta(x)$ is the unique generic point of $V(\mathcal{P}_x)$ in Spec$(\mathcal{A}_{n,\mathbb{C}})$. Each $\mathcal{P}$ in Spec$(\mathcal{A}_{n,\mathbb{C}})$ is the unique generic point of $V(\mathcal{P})$ in Spec$(\mathcal{A}_{n,\mathbb{C}})$, and $\Theta^{-1}(\mathcal{P})$ is the set whose elements are the nonstandard generic points of $\mathcal{P}$.*

We provide every subset of $\mathbb{C}^n$ or $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ with the corresponding Zariski topology. For each $f \in \mathcal{A}_{n,\mathbb{C}}$, the map $\theta : x \mapsto \mathcal{M}_x$ sends the distinguished open set $\mathbb{C}^n_f$ on $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})_f \cap \mathrm{Spec}_{\max}(\mathcal{A}_{n,\mathbb{C}})$. Consequently, $\theta$ is a homeomorphism of $\mathbb{C}^n$ on $\mathrm{Spec}_{\max}(\mathcal{A}_{n,\mathbb{C}})$.

Despite the fact that it is an external object, it is natural to hope that the map $\Theta : x \mapsto \mathcal{P}_x$ is continuous in some natural meaning.

To this aim, we consider on the set ${}^*\mathbb{C}^n$ the $\mathbb{C}$-Zariski topology which is the Zariski topology defined by polynomials with coefficients in $\mathbb{C}$ (and not in ${}^*\mathbb{C}$). The closed sets for this topology are

$$Z_{^*\mathbb{C}}(\mathcal{I}) = \{x \in {}^*\mathbb{C}^n \ ; \ \forall f \in \mathcal{I} \ f(x) = 0\}$$

where $\mathcal{I}$ is an ideal of $\mathcal{A}_{n,\mathbb{C}}$. For each $f \in \mathcal{A}_{n,\mathbb{C}}$ the open set

$${}^*\mathbb{C}^n_f = \{x \in {}^*\mathbb{C}^n \ ; \ f(x) \neq 0\}$$

is called a distinguished open set; the family of the distinguished open sets is a basis of the $\mathbb{C}$-Zariski topology on ${}^*\mathbb{C}^n$.

It is a powerful point of view in Galois theory to consider this kind of topology when we are looking at a field extension. It is amusing to remark that this is also a particular case of a general construction for a nonstandard extension ${}^*X$ of a topological space $X$: if $\mathcal{T}$ is the family of open sets of $X$, we can consider on ${}^*X$ two interesting topology: firstly the *internal topology* whose open sets are the $U \in {}^*\mathcal{T}$ and secondly the *external topology* whose open sets are the ${}^*U$ for $U \in \mathcal{T}$. In the case of the space ${}^*\mathbb{C}^n$ interpreted as the nonstandard extension of the Zariski topological space $\mathbb{C}^n$, the $\mathbb{C}$-Zariski topology is exactly the external topology.

**Proposition 4.** *For the $\mathbb{C}$-Zariski topology on ${}^*\mathbb{C}^n$ and the Zariski-topology on the space Spec$(\mathcal{A}_{n,\mathbb{C}})$, the surjective map*

$$\begin{array}{cccc} \Theta & : & {}^*\mathbb{C}^n & \longrightarrow & \mathit{Spec}(\mathcal{A}_{n,\mathbb{C}}) \\ & & x & \longmapsto & \mathcal{P}_x \end{array}$$

*is continuous, open and closed.*

*Proof.* From the definition of the map $\Theta$ and of a generic point, we see that for every $f \in \mathcal{A}_{n,\mathbb{C}}$ and for every $x \in {}^*\mathbb{C}^n$

$$f \in \Theta(x) \Longleftrightarrow f(x) = 0$$

from which we deduce that

$$\Theta(x) \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})_f \Longleftrightarrow x \in {}^*\mathbb{C}^n_f$$

and, for every ideal $\mathcal{I}$ of $\mathcal{A}_{n,\mathbb{C}}$

$$\Theta(x) \in V(\mathcal{I}) \Longleftrightarrow x \in Z_{^*\mathbb{C}}(\mathcal{I})$$

Thus, we have $\Theta^{-1}(\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})_f) = {}^*\mathbb{C}^n_f$, $\Theta({}^*\mathbb{C}^n_f) = \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})_f$ and also $\Theta(Z_{^*\mathbb{C}}(\mathcal{I})) = V(\mathcal{I})$. ∎

**Remark** This situation suggests the following definition which is verified by the map $\Theta$. An *affine unfolding* of the geometric space $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ is a map $\Psi : k^n \to \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ such that

- $k$ is a field extension of $\mathbb{C}$;
- $\Psi$ is the natural map $k^n \to \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ defined by

  $$\forall x \in k^n \quad \Psi(x) = \{f \in \mathcal{A}_{n,\mathbb{C}} \ ; \ f(x) = 0\}$$

  which is an extension of the canonical bijection $\theta : \mathbb{C}^n \to \mathrm{Spec}_{\max}(\mathcal{A}_{n,\mathbb{C}})$;
- $\Psi$ is onto.

Then, the map $\Psi$ is continuous, open and closed for the $\mathbb{C}$-Zariski topology on $k^n$. Furthermore, for every $\mathcal{P}$ in $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$ and for each $s_{\mathcal{P}}$ in the stalk of $\mathcal{P}$ for the Grothendieck structure sheaf over $\mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})$, the function $s_{\mathcal{P}}$ is defined on $\Psi^{-1}(\mathcal{P})$.

## 7 Nonstandard generic points are nonsingular and generic

Let $\mathcal{P}$ be a prime proper ideal of $\mathcal{A}_{n,\mathbb{C}}$ and let $\{f_1, \ldots, f_p\}$ be a finite subset of $\mathcal{A}_{n,\mathbb{C}}$ such that $\mathcal{P} = (f_1, \ldots, f_p)$. We know that every nonstandard generic point $x$ of $\mathcal{P}$ belongs to the set $Z_{^*\mathbb{C}}(\mathcal{P}) = {^*Z}(\mathcal{P})$. From the transfert principle we deduce that this set is an internal algebraic set associated to the internal prime ideal ${^*\mathcal{P}}$ of ${^*\mathcal{A}_{n,\mathbb{C}}}$. Actually, we have more: the set $Z_{^*\mathbb{C}}(\mathcal{P})$ is an usual algebraic set of the affine space ${^*\mathbb{C}^n}$ defined by the ideal $(f_1, \ldots, f_p)$ of ${^*\mathbb{C}}[X_1, \ldots, X_n]$. Furthermore, this last algebraic set is irreducible. Thus the set ${^*Z}(\mathcal{P})$ may be interpreted in three ways: it is a closed set of the $\mathbb{C}$-Zariski topology of ${^*\mathbb{C}^n}$, it is an internal closed set of the internal Zariski topology of ${^*\mathbb{C}^n}$ and it is a closed irreducible set of the Zariski topology of ${^*\mathbb{C}^n}$.

We want to examine some properties of nonstandard generic points of $\mathcal{P}$ with regard to the set ${^*Z}(\mathcal{P})$.

The dimension of the affine variety $Z(\mathcal{P})$ is a number $d \in \mathbb{N}$ such that

1. $\forall \xi \in Z(\mathcal{P})$, the rank of the matrix $(\partial f_i/\partial X_j(\xi))$ is $\leq n - d$;
2. $\exists \xi \in Z(\mathcal{P})$ such that the rank of the matrix $(\partial f_i/\partial X_j(\xi))$ is $n - d$.

A point $\xi$ of $Z(\mathcal{P})$ is *nonsingular* if the second condition is satisfied.

By transfer, we see that ${^*Z}(\mathcal{P})$ is also of dimension $d$ in ${^*\mathbb{C}^n}$ and that a point $\xi \in {^*Z}(\mathcal{P})$ is *nonsingular* when the rank over ${^*\mathbb{C}}$ of the matrix $(\partial f_i/\partial X_j(\xi))$ is $n-d$.

**Proposition 5.** *A nonstandard generic point of $\mathcal{P}$ is a nonsingular point of ${^*Z}(\mathcal{P})$.*

*Proof.* From the definition of the dimension of $Z(\mathcal{P})$, we deduce the existence of a $(n-d) \times (n-d)$ submatrix $\Delta$ of $(\partial f_i/\partial X_j)$ such that the polynomial $\det(\Delta)$ takes non zero value at some point of $Z(\mathcal{P})$. Hence $\det(\Delta)$ does not belong to $\mathcal{P}$. Thus, a nonstandard generic point $x$ of $\mathcal{P}$ cannot be a root of the polynomial $\det(\Delta)$. ∎

Let $\mathcal{P}_{\text{gen}}$ be the subset of ${}^*\mathbb{C}^n$ whose elements are the nonstandard generic points of $\mathcal{P}$:

$$\mathcal{P}_{\text{gen}} = \Theta^{-1}(\mathcal{P}) = \{x \in {}^*\mathbb{C}^n \ ; \ \forall f \in \mathcal{A}_{n,\mathbb{C}} \ f(x) = 0 \Leftrightarrow f \in \mathcal{P}\}.$$

This is a subset of the algebraic set ${}^*Z(\mathcal{P})$ generally different from ${}^*Z(\mathcal{P})$.

**Proposition 6.** *The following conditions are equivalent:*

1. ${}^*Z(\mathcal{P}) = \mathcal{P}_{gen}$;
2. *the ideal* $\mathcal{P}$ *is maximal;*
3. $\exists x \in \mathbb{C}^n \quad Z(\mathcal{P}) = \{x\}$.

*Proof.* The last two conditions are equivalent and they obviously imply the first conditions. The standard set $Z(\mathcal{P})$ is not empty. Thus, there exist standard elements in ${}^*Z(\mathcal{P})$. If ${}^*Z(\mathcal{P}) \neq \{x\}$ for every standard $x \in {}^*Z(\mathcal{P})$, then any nonstandard generic point of $\mathcal{P}$ is not standard. ∎

The following result expresses a topological property of $\mathcal{P}_{\text{gen}}$ in ${}^*Z(\mathcal{P})$.

**Proposition 7.** *Let* $x \in {}^*\mathbb{C}^n$ *be a generic nonstandard point of a prime proper ideal* $\mathcal{P}$ *of* $\mathcal{A}_{n,\mathbb{C}}$. *Then, there exists an open neighbourhood* $U$ *of* $x$ *in* ${}^*Z(\mathcal{P})$ *for the internal Zariski topology such that every element of* $U$ *is a nonstandard generic point of* $\mathcal{P}$.

*Proof.* A set $\mathcal{F}$ is called *hyperfinite* if there exists $N \in {}^*\mathbb{N}$ and an internal bijection of $\mathcal{F}$ on $\{n \in {}^*\mathbb{N} \ ; \ n \leq N\}$. From the idealization principle we deduce the existence of an hyperfinite subset $\mathcal{F}$ of ${}^*(\mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P}) = {}^*\mathcal{A}_{n,\mathbb{C}} \setminus {}^*\mathcal{P}$ such that $f \in \mathcal{F}$ for every $f \in \mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P}$. Given a nonstandard generic point $x \in {}^*\mathbb{C}^n$ of $\mathcal{P}$, we have $f(x) \neq 0$ for every $f \in \mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P}$. Thus, the set $\mathcal{G} = \{g \in \mathcal{F} \ ; \ g(x) \neq 0\}$ is an hyperfinite subset of ${}^*(\mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P})$ which contains $\mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P}$.

Consequently, the point $x$ belongs to the internal open Zariski subset of ${}^*\mathbb{C}^n$:

$$V = \{z \in {}^*\mathbb{C}^n \ ; \ \forall g \in \mathcal{G} \quad g(z) \neq 0\}.$$

Every point $y \in U = V \cap {}^*Z(\mathcal{P})$ is such that

$$(y \in {}^*Z(\mathcal{P})) \wedge (\forall f \in (\mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P}) \quad f(y) \neq 0)$$

that is to say, $y$ is a nonstandard generic point of $\mathcal{P}$. ∎

From this proposition we cannot deduce that $\mathcal{P}_{\text{gen}}$ is an open set for the internal Zariski topology on ${}^*Z(\mathcal{P})$ because $\mathcal{P}_{\text{gen}}$ is generally an external set. Nevertheless, we say that $\mathcal{P}_{\text{gen}}$ is an *external open subset* of ${}^*Z(\mathcal{P})$ for the internal Zariski topology.

We know that a closed subset $W$ of an irreducible set $F$ has an empty interior in $F$ or is equal to $F$. Thus, the neighbourhood $U$ of $x$ given in the last proposition is dense in ${}^*Z(\mathcal{P})$ for the internal Zariski topology. Hence, the external open set $\mathcal{P}_{\text{gen}}$ is dense in ${}^*Z(\mathcal{P})$. In other words, the property *'x is a nonstandard generic point of* $\mathcal{P}$*'* is generic in ${}^*Z(\mathcal{P})$ for the internal Zariski topology.

**Remark** The set $\mathcal{P}_{\text{gen}}$ is clearly external. We have

$$\mathcal{P}_{\text{gen}} = {}^*Z(\mathcal{P}) \cap \bigcap_{f \in (\mathcal{A}_{n,\mathbb{C}} \setminus \mathcal{P})} {}^*\mathbb{C}^n_f$$

Therefore, among the external sets, $\mathcal{P}_{\text{gen}}$ has the structure of a *generalized prehalo* [23].

## 8 Algebraic and analytic points of ${}^*\mathbb{C}^n$

Let $(0)$ be the null ideal of $\mathcal{A}_{n,\mathbb{C}}$. Since $Z((0)) = \mathbb{C}^n$, we have ${}^*Z((0)) = {}^*\mathbb{C}^n$ and the set $(0)_{\text{gen}}$ of nonstandard generic points of $(0)$ is an external open dense in ${}^*\mathbb{C}^n$. Thus, we can say that *a generic point of ${}^*\mathbb{C}^n$ for the internal Zariski topology is a nonstandard generic point of* $(0)$. A point $x$ of ${}^*\mathbb{C}^n$ belongs to $(0)_{\text{gen}}$ if and only if $f(x) \neq 0$ for all polynomial $f \in \mathcal{A}_{n,\mathbb{C}}$ different from zero. Therefore, $(0)_{\text{gen}}$ is the set of $x \in {}^*\mathbb{C}^n$ such that $x \notin {}^*F$ for every algebraic set $F$ of $\mathbb{C}^n$ different from $\mathbb{C}^n$. This leads to the following definition.

**Definition 3.** *An* algebraic point *of ${}^*\mathbb{C}^n$ is an element of ${}^*\mathbb{C}^n$ which belongs to at least one nonstandard extension ${}^*F$ of an algebraic set $F$ of $\mathbb{C}^n$ different from $\mathbb{C}^n$.*

From the previous discussion, we get at once the next result.

**Proposition 8.** *Given $x \in {}^*\mathbb{C}^n$, the following conditions are equivalent.*

1. *$x$ is an algebraic point.*
2. *There exists $f \in \mathcal{A}_{n,\mathbb{C}} \setminus \{0\}$ such that $f(x) = 0$.*
3. *The prime proper ideal $\Theta(x)$ of $\mathcal{A}_{n,\mathbb{C}}$ is different from $(0)$.*

Given a standard $a \in \mathbb{C}^n$, we know that $\mathcal{P}_a = \mathcal{M}_a$ and $Z(\mathcal{M}_a) = \{a\}$. Thus, every standard element of $\mathbb{C}^n$ is an algebraic point of ${}^*\mathbb{C}^n$.

Finally, the nonstandard affine space ${}^*\mathbb{C}^n$ is decomposed in an external disjoint union

$$ {}^*\mathbb{C}^n = \bigcup_{\mathcal{P} \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}})} \mathcal{P}_{\text{gen}} $$

in which we find one 'big' external set $(0)_{\text{gen}}$. The complementary set ${}^*\mathbb{C}^n \setminus (0)_{\text{gen}}$ is the set ${}^*\mathbb{C}^n_{\text{alg}}$ of the algebraic points of $\mathbb{C}^n$. Then we have

$$ {}^*\mathbb{C}^n_{\text{alg}} = \bigcup_{\mathcal{P} \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}}) \setminus \{(0)\}} \mathcal{P}_{\text{gen}} $$

which is the disjoint union of the 'thin' external sets $\mathcal{P}_{\text{gen}}$ for $\mathcal{P} \neq (0)$. Every set $\mathcal{P}_{\text{gen}}$ is an external open dense subset of ${}^*Z(\mathcal{P})$ for the internal Zariski topology. Furthermore, for every prime proper ideal $\mathcal{P}$ of $\mathcal{A}_{n,\mathbb{C}}$, we have the following external union

$$ {}^*Z(\mathcal{P}) = \bigcup_{\mathcal{Q} \in \mathrm{Spec}(\mathcal{A}_{n,\mathbb{C}}) \text{ and } \mathcal{P} \subset \mathcal{Q}} \mathcal{Q}_{\text{gen}} $$

Few points of ${}^*\mathbb{C}^n$ are algebraic. One can widen the class of algebraic points by including points which are in a similar way analytics. A first difficulty is that an analytic function is not generally defined everywhere on the affine space.

**Definition 4.** *An element $x \in^* \mathbb{C}^n$ is an* analytic point *if there exists an analytic function $f : U \to \mathbb{C}$ defined on an open subset $U$ of $\mathbb{C}^n$ such that $x \in {}^*U$ and $f(x) = 0$.*

In the next sections, one suggests to establish a link between this notion and the usual frame of the Nullstensatz in the analytic case.

# 9 Convergent power series and the Rückert Nullstellensatz

This section is devoted to the description of the classical background of the Nullstellensatz in the space of convergent power series. A good reference for the reader not familiar with these topics is the book [21] of Jesús M. Ruiz.

## 9.1 The algebra $\mathcal{O}_{n,\mathbb{K}}$

We denote by $\mathbb{K}[[X]]$, where $X$ is a multi-indeterminate $X = (X_1, \dots, X_n)$, the set of formal power series with coefficients in $\mathbb{K}$ and $n$ indeterminates. A formal power series $f \in \mathbb{K}[[X]]$, written $f = \sum_\nu a_\nu X^\nu$ where $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ , $a_\nu \in \mathbb{K}$ and $X^\nu = X_1^\nu \dots X_n^\nu$, converges at $x \in \mathbb{K}^n$ to $s \in \mathbb{K}$ if, for all $\varepsilon > 0$ in $\mathbb{R}$, there exists a finite set $I_\varepsilon \subset \mathbb{N}^n$ such that $|\sum_{\nu \in J} a_\nu x^\nu - s| < \varepsilon$ for any finite set $J$ such that $I_\varepsilon \subset J \subset \mathbb{N}^n$. A formal power series $f$ always converges at $x = 0$ to its constant term $a_0$.

Let $D(f)$ be the interior of the set of points $x \in \mathbb{K}^n$ at which the formal power series $f$ is convergent. We say that $f$ is *convergent* if $D(f) \neq \emptyset$. Then $0 \in D(f)$. For $n = 1$, we know that $D(f)$ is an open disk. More generally, $D(f)$ is the union of the polycylinders

$$D_u = \{x = (x_1, \dots, x_n) \in \mathbb{K}^n \ ; \ x_i < |u_i| \ \ i = 1, \dots, n\}$$

for all $u \in \mathbb{K}^n$ such that every $u_i \neq 0$ and $f(u)$ is convergent.

Let $\mathcal{O}_{n,\mathbb{K}} = \mathbb{K}\{X_1, \dots, X_n\} = \mathbb{K}\{X\}$ be the set of convergent power series with $n$ indeterminates. For the usual operations of sum and product, the set $\mathbb{K}[[X]]$ is an integral domaine and a $\mathbb{K}$-algebra with $\mathcal{O}_{n,\mathbb{K}}$ as a subalgebra. Furthermore, $\mathcal{O}_{n,\mathbb{K}}$ is a local noetherian ring which is factorial.

In order to get most algebraic properties of $\mathcal{O}_{n,\mathbb{K}}$, two important tools are usually needed: the *Division Theorem* and the *Preparation Theorem*. Before giving these results, we have to introduce a new notion : a power series $f \in \mathbb{K}[[X]]$ with $X = (X_1, \dots, X_n)$ is *regular of order $p$ with respect to $X_n$* if $f(0, \dots, 0, X_n) = X_n^p g(X_n)$ with $g(0) \neq 0$.

**Rückert's Division Theorem.** *Let $\varphi \in \mathcal{O}_{n,\mathbb{K}}$ be a regular element of order $p$ with respect to $X_n$ and $f \in \mathcal{O}_{n,\mathbb{K}}$. Then, there exists a uniquely defined $(q, r) \in \mathcal{O}_{n,\mathbb{K}} \times \mathcal{O}_{n-1,\mathbb{K}}[X_n]$ such that $f = q\varphi + r$ with degree of $r$ less than $p$ (i.e. $d^o(r) < p$).*

We can obtain this last result with the help of the fixed point theorem for a suitable contractive map [21]. Taking $X_n^p$ for $f$ and $\varphi$ a regular element of order $p$ with respect to $X_n$, we get the following property.

**Weierstrass's Preparation Theorem.** *Let $\varphi \in \mathcal{O}_{n,\mathbb{K}}$ be a regular element of order $p$ with respect to $X_n$. Then there exists a polynomial $P = T^p + a_1 T^{p-1} + \dots + a_p \in \mathcal{O}_{n-1,\mathbb{K}}[T]$ with $a_1(0) = \dots = a_p(0) = 0$ and a unit $u$ de $\mathcal{O}_{n,\mathbb{K}}$ such that $\varphi = uP(X_n)$.*

## 9.2 Germ of analytic sets

The set $\mathcal{O}_{n,\mathbb{K}}$ is identified with the set of germs at 0 of $\mathbb{K}$-valued analytic functions defined on a neighbourhood of 0 in $\mathbb{K}^n$.

Given an ideal $\mathcal{I}$ of $\mathcal{O}_{n,\mathbb{K}}$, the definition of the zero set $Z(\mathcal{I})$ is not elementary because, interpreted as functions, the elements of $\mathcal{O}_{n,\mathbb{K}}$ have not a good common domain of definition. In fact $\bigcap_{f\in\mathcal{O}_{n,\mathbb{K}}} D(f) = \{0\}$.

We choose a generator system $\mathcal{F} = \{f_1, \cdots, f_q\}$ of $\mathcal{I}$ and introduce the set

$$Z(\mathcal{F}) = \{x \in \mathbb{K}^n \ ; \ f_1(x) = \ldots = f_q(x) = 0\} .$$

Then, the zero set $Z(\mathcal{I})$ of $\mathcal{I}$ is the germ at 0 of the set $Z(\mathcal{F})$. That is to say $Z(\mathcal{I})$ is the equivalence class of $Z(\mathcal{F})$ for the relation between subsets of $\mathbb{K}^n$

$$A \sim B \iff \exists U \in \mathcal{V}_0 \ A \cap U = B \cap U$$

where $\mathcal{V}_0$ is the set of neighbourhoods of 0 in $\mathbb{K}^n$. Obviously, this definition is independant of the choice of the generator system $\mathcal{F}$.

Now we can define the annulator ideal of $Z(\mathcal{I})$ in the following way

$$I(Z(\mathcal{I})) = \{f \in \mathcal{O}_{n,\mathbb{K}} \ ; \ \forall x \in Z(\mathcal{I}) \ f(x) = 0\}$$

where the assertion $\forall x \in Z(\mathcal{I}) \ f(x) = 0$ has to be interpreted in terms of germs of sets and functions.

It is clear that $\mathcal{I} \subset I(Z(\mathcal{I}))$. In the complex case, the relation between $\mathcal{I}$ and $I(Z(\mathcal{I}))$ is given by the following well known result.

**Rückert Complex Nullstellensatz.** *Let $\mathcal{P}$ be a prime ideal of the ring $\mathcal{O}_{n,\mathbb{C}}$. Then $\mathcal{P} = I(V(\mathcal{P}))$.*

# 10 Nonstandard generic points for complex analytic germs

## 10.1 Analytic germs from a nonstandard point of view

As we can expect, the nonstandard point of view allows us to consider that, in some sense, the elements of $\mathcal{O}_{n,\mathbb{K}}$ have a kind of infinitesimal common domain of definition. This conception was firstly introduced by A. Robinson [19, 20].

The point is that, for each standard element $f$ of $\mathcal{O}_{n,\mathbb{K}}$, every $x \in {}^*\mathbb{K}^n$ such that $x \simeq 0$ is contained in the nonstandard extension ${}^*D(f)$ of the domain $D(f)$ of $f$, so that the nonstandard extension of $f$ (also denoted $f$) is defined at $x$. Thus, we can consider that the halo of 0, $\mathrm{hal}(0) = \{x \in {}^*\mathbb{K}^n \ ; \ x \simeq 0\}$, is an external neighbourhood of 0 on which every element of $\mathcal{O}_{n,\mathbb{K}}$ is defined.

Given an ideal $\mathcal{I}$ of $\mathcal{O}_{n,\mathbb{K}}$, this property leads to a more direct definition of $Z(\mathcal{I})$

$$Z(\mathcal{I}) = \{x \in \mathrm{hal}(0) \ ; \ \forall f \in \mathcal{I} \ f(x) = 0\}$$

and of $I(Z(\mathcal{I}))$

$$I(Z(\mathcal{I})) = \{f \in \mathcal{O}_{n,\mathbb{K}} \ ; \ \forall x \in Z(\mathcal{I}) \ f(x) = 0\}.$$

It is a basic exercise of nonstandard analysis to prove that this last definition of $I(Z(\mathcal{I}))$ is equivalent to the classical one. This change of viewpoint lead A. Robinson to introduce a new concept of generic point.

## 10.2 Definition and examples of nonstandard generic points

**Definition 5.** *A* nonstandard generic point *of a prime ideal $\mathcal{P}$ of $\mathcal{O}_{n,\mathbb{K}}$ is an element $x \simeq 0$ of ${}^*\mathbb{K}^n$ such that*[4]

$$\forall f \in \mathcal{O}_{n,\mathbb{K}} \ \ (f \in \mathcal{P} \iff f(x) = 0)$$

For instance, zero is a nonstandard generic point of the maximal ideal of power series without constant term $\mathcal{M} = (X_1, \ldots, X_n)$.

The concept of nonstandard generic point is related to the uniqueness theorem for power series. In fact, a nonstandard version of this last result is the following: if $x \in \mathrm{hal}(0) \setminus \{0\}$ and $f \in \mathcal{O}_{1,\mathbb{C}}$ we have

$$f(x) = 0 \iff f = 0.$$

So, every $x \simeq 0$ in ${}^*\mathbb{C} \setminus \{0\}$ is a nonstandard generic point for the null ideal (0) of $\mathcal{O}_{1,\mathbb{C}}$.

Does the null ideal of $\mathcal{O}_{n,\mathbb{C}}$ have a nonstandard generic point? It is possible to answer this question using the idealization principle. For that purpose, we consider a hyperfinite subset $F$ of ${}^*\mathcal{O}_{n,\mathbb{C}} \setminus \{0\}$ which contains every element of $\mathcal{O}_{n,\mathbb{C}}$. As ${}^*\mathcal{O}_{n,\mathbb{C}}$ is an integral domain, the product of all the elements of $F$ is not equal to zero. Hence, we can find $x \simeq 0$ in ${}^*\mathbb{C}^n \setminus \{0\}$ such that $f(x)$ is different from 0 for every $f \in F$. So, $x$ is a nonstandard generic point for the null ideal (0) of $\mathcal{O}_{n,\mathbb{C}}$.

[4] A. Robinson's definition was only given in the complex case: $\mathbb{K} = \mathbb{C}$.

But it is more interesting to answer the same question in a quite constructive manner, more in the spirit of the uniqueness theorem. Given $\varepsilon_1 \simeq 0$ in $\mathbb{C}^*$, let $\varepsilon_2 \simeq 0$ in $\mathbb{C}^*$ be in the *micro-halo* of $\varepsilon_1$, that is to say

$$\forall k \in \mathbb{N} \setminus \{0\} \quad \frac{\varepsilon_2}{\varepsilon_1^k} \simeq 0$$

(for instance $\varepsilon_2 = \varepsilon_1^N$ with $N$ an infinitely large integer). Thus, for all $g \in \mathcal{O}_{1,\mathbb{C}} \setminus \{0\}$, we have $\dfrac{\varepsilon_2}{g(\varepsilon_1)} \simeq 0$.

If $f$ is a standard element of $\mathcal{O}_{2,\mathbb{C}} \setminus \{0\}$, we can write $f = \sum_{k=m}^{+\infty} a_k(X_1)X_2^k$ with $m$ an integer and $a_m$ an element of $\mathcal{O}_{1,\mathbb{C}} \setminus \{0\}$. Thus we have $f(\varepsilon_1, \varepsilon_2) = \varepsilon_2^m \left(a_m\left(\varepsilon_1\right) + \varepsilon_2 \mathcal{L}\right)$ where $\mathcal{L}$ is a limited complex number (i.e. non infinitely large); hence $f(\varepsilon_1, \varepsilon_2)$ cannot be equal to zero. Consequently, $(\varepsilon_1, \varepsilon_2)$ is a nonstandard generic point of the null ideal of $\mathcal{O}_{2,\mathbb{C}}$.

More generally, let $\varepsilon_1, \ldots, \varepsilon_n \in {}^*\mathbb{C} \setminus \{0\}$ be such that $\varepsilon_{k+1}$ is in the micro-halo of $\varepsilon_k$ for each $k = 1, \cdots, n-1$. Then, $(\varepsilon_1, \ldots, \varepsilon_n)$ is a nonstandard generic point for the null ideal of $\mathcal{O}_{n,\mathbb{C}}$.

From this we get a general formulation independent of the choice of the basis of $\mathbb{C}^n$.

*Let $x \simeq 0$ in ${}^*\mathbb{C}^n \setminus \{0\}$ with a Goze decomposition* [4, 5] $x = \varepsilon_1 v_1 + \varepsilon_1\varepsilon_2 v_2 + \cdots + \varepsilon_1 \ldots \varepsilon_n v_n$ *such that $\varepsilon_{k+1}$ is in the micro-halo of $\varepsilon_k$ for each $k = 1, \cdots, n-1$. Then $x$ is a nonstandard generic point for the null ideal $\{0\}$ of $\mathcal{O}_{n,\mathbb{C}}$.*

### 10.3 The Robinson theorem

The general result about existence of nonstandard generic point in the complex analytic case is the following [20].

**Robinson Theorem.** *Every prime and proper ideal $\mathcal{P}$ of $\mathcal{O}_{n,\mathbb{C}}$ has a nonstandard generic point.*

Given $a \in \mathbb{C}^n$, we introduce the ring $\mathcal{O}_{n,\mathbb{C}}(a)$ of complex analytic germs at $a$. For $x \simeq a$ let

$$\widetilde{\mathcal{P}}_x = \{f \in \mathcal{O}_{n,\mathbb{C}}(a) \ ; \ f(x) = 0\}.$$

Then $\widetilde{\mathcal{P}}_x$ is a prime and proper ideal of $\mathcal{O}_{n,\mathbb{C}}(a)$. Let $\mathrm{Spec}(\mathcal{O}_{n,\mathbb{C}}(a))$ be the spectrum of the ring $\mathcal{O}_{n,\mathbb{C}}(a)$. We can formulate Robinson's result in the following way.

**Equivalent version of Robinson Theorem.** *The map $\widetilde{\Theta}_a : x \mapsto \widetilde{\mathcal{P}}_x$ from hal$(a)$ to Spec$(\mathcal{O}_{n,\mathbb{C}}(a))$ is onto.*

For each prime and proper ideal $\mathcal{P}$ of $\mathcal{O}_{n,\mathbb{C}}(a)$, let $\mathcal{P}_{\mathrm{gen}}$ be the external set whose elements are the nonstandard generic points of $\mathcal{P}$. From the last result, we see that $\mathrm{hal}(a)$ is decomposed in an external disjoint union

$$\mathrm{hal}(a) = \bigcup_{\mathcal{P} \in \mathrm{Spec}(\mathcal{O}_{n,\mathbb{C}}(a))} \mathcal{P}_{\mathrm{gen}}$$

and the analytic points of $\mathbb{C}^n$ which are infinitly close to $a$ are the elements of

$$\mathrm{hal}(a)_{\mathrm{an}} = \mathrm{hal}(a) \setminus (0)_{\mathrm{gen}}.$$

## 11 The case of real analytic germs

### 11.1 A radically different situation

The situation is radically different in the real case because there exist prime and proper ideals of $\mathcal{O}_{n,\mathbb{R}}$ without any nonstandard generic point.

For instance, let $\mathcal{P}$ be the ideal $(X_1^2+X_2^2)$ of $\mathcal{O}_{2,\mathbb{R}}=\mathbb{R}\{X_1,X_2\}$. It is clear that $\mathcal{P}$ is a prime and proper ideal which cannot have a nonstandard generic point. The basic property which leads to the nonexistence of a generic point for $\mathcal{P}$ is

$$\xi_1^2+\xi_2^2=0 \implies \xi_1=\xi_2=0$$

which is true for $\xi_1,\xi_2\in\mathbb{R}$ and also for $\xi_1,\xi_2\in{}^*\mathbb{R}$. In fact, in the quotient ring $\mathcal{O}_{2,\mathbb{R}}/\mathcal{P}$, the elements $X_1$ and $X_2$ are not null but $X_1^2+X_2^2=0$. As a consequence, there is no order relation on the set $\mathcal{O}_{2,\mathbb{R}}/\mathcal{P}$ which is compatible with its ring structure.

### 11.2 Ordered ring structure and generic points

**Definition 6.** *An* order structure *on an integral domain $A$ is a partition $A = A_+\cup\{0\}\cup A_-$ such that*

- $A_- = -A_+ = \{-a\ ;\ a\in A_+\}$
- $(a,b)\in A_+ \implies (a+b\in A_+$ *and* $ab\in A_+)$

*We put $a\succ 0$ for $a\in A_+$, $a\prec 0$ for $a\in A_-$ and $a\prec b$ for $b-a\succ 0$. If such an order structure exists, we say that $A$ is an* ordered ring *(an* ordered field *if $A$ is a field).*

For instance, $\mathbb{R}$ is an ordered field for the usual order relation $<$ and there is no other order relation on it.

**Proposition 9.** *Let $\mathcal{P}$ be a prime and proper ideal of $\mathcal{O}_{n,\mathbb{R}}$ which has a nonstandard generic point. Then, the quotient ring $\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$ is an ordered ring.*

*Proof.* Let $\xi\simeq 0$ in ${}^*\mathbb{R}^n$ be a nonstandard generic point of $\mathcal{P}$ and let $A_+$ be the subset of $A=\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$ defined by

$$A_+=\{f\in A\ ;\ f(\xi)>0\}$$

and let $A_-=-A_+$. It is clear that

$$\forall f\in\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}\ \ (f\in A_+ \text{ or } f\in A_- \text{ or } f=0)$$

$$\forall (f,g)\in(\mathcal{O}_{n,\mathbb{R}}/\mathcal{P})^2\ \ \Big((f,g)\in A_+^2\implies f+g\in A_+ \text{ and } fg\in A_+\Big)$$

Hence $A=A_+\cup\{0\}\cup A_-$ is an order structure on the ring $A$. ∎

Given a standard prime and proper ideal $\mathcal{P}$ of $\mathcal{O}_{n,\mathbb{R}}$, the existence of an order relation on the quotient $\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$ is a necessary condition for the existence of a nonstandard generic point. Is this condition sufficient?

### 11.3 Another example: the cusp

Now we consider the prime ideal $\mathcal{P} = (X_2^2 - X_1^3)$ of $\mathcal{O}_{2,\mathbb{R}} = \mathbb{R}\{X_1, X_2\}$ generated by the polynomial $X_2^2 - X_1^3$. The zero set $Z(\mathcal{P})$ is the germ at 0 of a cusp.

As $X_2^2 - X_1^3$ is regular of order 2 with respect to $X_2$, for every $f \in \mathcal{O}_{2,\mathbb{R}}$, there exists in a unique way $(q, f_0, f_1) \in \mathcal{O}_{2,\mathbb{R}} \times \mathcal{O}_{1,\mathbb{R}} \times \mathcal{O}_{1,\mathbb{R}}$ such that $f = (X_2^2 - X_1^3)q + f_0 + X_2 f_1$. Thus the ring $\mathcal{O}_{2,\mathbb{R}}/\mathcal{P}$ is isomorphic to $\mathbb{R}\{X_1\}[X_2]/(X_2^2 - X_1^3)$. Every element $\alpha$ of this last ring can be written

$$\alpha = \sum_{n\geq 0} a_n X_1^n + X_2 \sum_{n\geq 0} b_n X_1^n$$

and thus also

$$\alpha = a_0 + \sum_{n\geq 0} a_{n+1} X_1^{n+1} + b_n X_2 X_1^n.$$

We agree that $\alpha \succ 0$ when the first nonzero element in the sequence

$$(a_0, a_1, b_0, \ldots, a_{n+1}, b_n, \ldots)$$

is $> 0$ in $\mathbb{R}$. In this way we get an order structure on the ring $\mathbb{R}\{X_1\}[X_2]/(X_2^2 - X_1^3)$ and also on $\mathcal{O}_{2,\mathbb{R}}/\mathcal{P}$.

We choose $\xi = (\xi_1, \xi_2) \simeq 0$ in ${}^*\mathbb{R}^2$ such that $\xi_2 > 0$ and $\xi_2^2 = \xi_1^3$. Every element $f$ of $\mathcal{P}$ is defined at $\xi$ and satisfies the condition $f(\xi) = 0$. Conversely, we consider a standard $f \in \mathcal{P}$ such that $f(\xi) = 0$. From the Rückert's Division Theorem, there exists $(q, f_0, f_1) \in \mathcal{O}_{2,\mathbb{R}} \times \mathcal{O}_{1,\mathbb{R}} \times \mathcal{O}_{1,\mathbb{R}}$ such that $f = (X_2^2 - X_1^3)q + f_0 + X_2 f_1$. Writing $f_0 = \sum_{n\geq 0} a_n X_1^n$ and $f_1 = \sum_{n\geq 0} b_n X_1^n$ we get

$$\sum_{n\geq 0} a_n \xi_1^n + \sum_{n\geq 0} b_n \xi_1^n \xi_2 = 0.$$

Let $\varepsilon$ be the infinitesimal $\sqrt{\xi_1}$ (also equal to $\sqrt[3]{\xi_2}$). We thus obtain the relation $a_0 + \sum_{n\geq 1} a_n \varepsilon^{2n} + b_{n-1}\varepsilon^{2n+1} = 0$. Then, the standard power serie with one indeterminate $Z$

$$\varphi = a_0 + \sum_{n\geq 1} a_n Z^{2n} + b_{n-1} Z^{2n+1}$$

is convergent and $\varepsilon$ is a nonstandard zero of its sum. By the Uniqueness Theorem for analytic functions, we deduce that $\varphi = 0$. Returning to $f$, we get $f = (X_2^2 - X_1^3)q$, that is to say $f \in \mathcal{P}$. Hence, we have proved that $\xi$ is a generic point for $\mathcal{P}$.

## 11.4 The main theorem in the real case

First we need a more accurate notion of nonstandard generic point.

**Definition 7.** *Let $\mathcal{P}$ be a prime and proper ideal of $\mathcal{O}_{n,\mathbb{R}}$ and an order structure $\prec$ on the ring $\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$. A* nonstandard generic point *of $(\mathcal{P}, \prec)$ is an element $\xi \simeq 0$ of ${}^*\mathbb{R}^n$ such that, for all $f \in \mathcal{O}_{n,\mathbb{R}}$, the sign of $f(\xi)$ in ${}^*\mathbb{R}$ is equal to the sign of the class of $f$ in $\mathcal{O}_n/\mathcal{P}$.*

It is clear that, if $\mathcal{P}$ is a prime and proper ideal of $\mathcal{O}_n$ and $\prec$ an order structure on the ring $\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$, a nonstandard generic point for $(\mathcal{P}, \prec)$ is also a nonstandard generic point for $\mathcal{P}$ in the sense of the preceding sections.

The following result was obtained by the author in 1978 [24] but never published. It is an application of the theory of Artin-Schreier on ordered fields [1, 10, 11] and of course, of nonstandard analysis.

**Theoreme 3.** *Let $\mathcal{P}$ be a prime and proper ideal of $\mathcal{O}_{n,\mathbb{R}}$ and $\prec$ an order structure on the ring $\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$. Then $(\mathcal{P}, \prec)$ have a nonstandard generic point.*

Then, we get a Real Nullstellensatz for a prime ideal.

**Corollary 1.** *Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_{n,\mathbb{R}}$ such that there exists an order structure on the ring $\mathcal{O}_n/\mathcal{P}$. Then $\mathcal{P} = I(Z(\mathcal{P}))$.*

Then, classical algebraic arguments lead us to a general Real Nullstellensatz, first obtained by J.J. Risler [18, 21].

**Risler Real Nullstellensatz.** *Let $\mathcal{I}$ an ideal of $\mathcal{O}_{n,\mathbb{R}}$. Then*

$$I(Z(\mathcal{I})) = \{f \in \mathcal{O}_{n,\mathbb{R}} \ ; \ \exists p \in \mathbb{N}^* \ \exists g_1, \dots, g_s \in \mathcal{O}_{n,\mathbb{R}} \quad f^{2p} + g_1^2 + \cdots + g_s^2 \in \mathcal{I}\}$$

## 11.5 Proof of theorem 3

### 11.5.1 The case of the null ideal $(0)$ of $\mathcal{O}_{n,\mathbb{R}}$

We consider a standard order structure on $\mathcal{O}_{n,\mathbb{R}}$ and a hyperfinite subset $F$ of ${}^*\mathcal{O}_{n,\mathbb{R}}$ which contains all its standard elements. From a generalization of a theorem of Artin ([10] page 290) proved by Risler [18], we know that there exists a point $\xi \in \cap_{f\in F} {}^*D(f)$ such that, for all $f \in F$, the sign of $f(\xi)$ in ${}^*\mathbb{R}$ is equal to the sign of $f$ in $\mathcal{O}_{n,\mathbb{R}}$. Such a $\xi$ is clearly a generic point for $(0)$.

### 11.5.2 The case of $\mathcal{O}_{1,\mathbb{R}}$

The only prime ideal of $\mathcal{O}_{1,\mathbb{R}}$ different from $(0)$ is the maximal ideal $(X)$. By the evaluation map $f \mapsto f(0)$, the quotient $\mathcal{O}_{1,\mathbb{R}}/(X)$ is isomorphic to $\mathbb{R}$ on which there exists a unique order structure. Hence, there exists a unique order structure $\prec$ on $\mathcal{O}_{1,\mathbb{R}}/(X)$ and $0 \in \mathbb{R}$ is a generic point for $((X), \prec)$.

#### 11.5.3 The general case

We will now argue by induction on the number $n$ of indeterminates. So we suppose the theorem is true for $\mathcal{O}_{n-1,\mathbb{R}}$. Let $\mathcal{P} \neq (0)$ be a prime and proper ideal of $\mathcal{O}_{n,\mathbb{R}}$ and $\prec$ an order structure on $\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$. After a possible change of variables, and using the Preparation Theorem, we may suppose that there is, in $\mathcal{P}$, an element $h = X_n^k + a_1 X_n^{k-1} + \cdots + a_k \in \mathcal{O}_{n-1,\mathbb{R}}[X_n]$ with $a_1(0) = \ldots = a_k(0) = 0$.

The relation $\prec$ defines in a unique way an order structure on the fraction field $\mathcal{K}_n(\mathcal{P})$ of $\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$. The set $\mathcal{P}' = \mathcal{P} \cap \mathcal{O}_{n-1,\mathbb{R}}$ is a prime and proper ideal of $\mathcal{O}_{n-1,\mathbb{R}}$ and $\prec$ defines an order structure on the fraction field $\mathcal{K}_{n-1}(\mathcal{P}')$ of $\mathcal{O}_{n-1,\mathbb{R}}/\mathcal{P}'$. Thus, $\mathcal{K}_n(\mathcal{P})$ is an ordered extension field of $\mathcal{K}_{n-1}(\mathcal{P}')$.

**Lemma 1.** *The field $\mathcal{K}_n(\mathcal{P})$ is a simple algebraic extension of $\mathcal{K}_{n-1}(\mathcal{P}')$.*

*Proof.* Denote by $\pi : \mathcal{O}_{n,\mathbb{R}} \to \mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$ and $\pi' : \mathcal{O}_{n-1,\mathbb{R}} \to \mathcal{O}_{n-1,\mathbb{R}}/\mathcal{P}$ the canonical projections. In $\mathcal{O}_{n,\mathbb{R}}/\mathcal{P}$ we get

$$\pi(X_n)^k + \pi'(a_1)\,\pi(X_n)^{k-1} + \cdots + \pi'(a_k) = 0.$$

Thus $\pi(X_n)$ is an element of $\mathcal{K}_n(\mathcal{P})$ which is algebraic on $\mathcal{K}_{n-1}(\mathcal{P}')$. Using the Division Theorem, we see that for every $f$ in $\mathcal{O}_{n,\mathbb{R}}$, there exists a polynomial $P$ with coefficients in $\mathcal{K}_{n-1}(\mathcal{P}')$ such that $\pi(f) = P(\pi(X_n))$. ∎

By induction, $(\mathcal{P}', \prec)$ has a generic point $\xi' = (\xi_1, \ldots, \xi_{n-1})$. The evaluation map $f \mapsto f(\xi)$ defines an external morphism $\tau'$ of ordered field from $\mathcal{K}_{n-1}(\mathcal{P}')$ to ${}^*\mathbb{R}$. Furthermore, the transfer principle implies that ${}^*\mathbb{R}$ is a real closed field. By the lemma and using the theory of real fields of Artin-Schreier[11], we can see that $\tau'$ has an extension $\tau$ from $\mathcal{K}_n(\mathcal{P})$ to ${}^*\mathbb{R}$ which is also a morphism of ordered field. Let $\xi_n = \tau(\pi(X_n))$. We are going to show that $\xi = (\xi_1, \ldots, \xi_{n-1}, \xi_n)$ is a generic point for $(\mathcal{P}, \prec)$.

From the definition of $\xi$ we obtain

$$\xi_n = -a_1(\xi') - a_2(\xi')\xi_n^{-1} - \cdots - a_k(\xi')\xi_n^{1-k}$$

As $a_i(0) = 0$ and $\xi' \simeq 0$, we see that $\xi_n$ must be infinitesimal.

For all $f \in \mathcal{O}_{n,\mathbb{R}}$, the sign of $\tau(\pi(f))$ is equal to the sign of $\pi(f)$. Furthermore, we can suppose that $f = hq + P$ where $h \in \mathcal{P} \cap \mathcal{O}_{n-1,\mathbb{R}}[X_n]$ and $P \in \mathcal{O}_{n-1,\mathbb{R}}[X_n]$. Thus, $h(\xi) = \tau(\pi(h)) = 0$, $\tau(\pi(f)) = \tau(\pi(P)) = P(\xi)$ and $f(\xi) = P(\xi)$. Hence, $\xi$ is a generic point for $(\mathcal{P}, \prec)$.

### 11.6 Equivalent version of theorem 3

Given $a \in \mathbb{R}^n$ we introduce the ring $\mathcal{O}_{n,\mathbb{R}}(a)$ of real analytic germs at $a$ and the halo $\mathrm{hal}(a)$ of $a$ in ${}^*\mathbb{R}^n$. For each $x \in \mathrm{hal}(a)$, we consider the set

$$\widehat{\mathcal{P}}_x = \{f \in \mathcal{O}_{n,\mathbb{R}}(a)\ ;\ f(x) = 0\}.$$

Then $\widehat{\mathcal{P}}_x$ is a prime and proper ideal of $\mathcal{O}_{n,\mathbb{R}}(a)$. Furthermore, there exists on the quotient ring $\mathcal{O}_{n,\mathbb{R}}(a)/\widehat{\mathcal{P}}_x$ an order relation $\prec_x$ defined by $0 \prec_x f$ if and only if $0 < f(x)$.

Let $\mathrm{Spec}_{\mathrm{Re}}(\mathcal{O}_{n,\mathbb{R}}(a))$ be the set whose elements are the pairs $(\mathcal{P}, \prec)$ where $\mathcal{P}$ is a prime proper ideal of $\mathcal{O}_{n,\mathbb{R}}(a)$ and $\prec$ an order relation on the quotient ring $\mathcal{O}_{n,\mathbb{R}}(a)/\mathcal{P}$. Then, we can formulate our last result in the following way.

**Equivalent version of theorem 3.** *The map $\widehat{\Theta}_a : x \mapsto (\widehat{\mathcal{P}}_x, \prec_x)$ from hal$(a)$ to Spec$_{Re}(\mathcal{O}_{n,\mathbb{R}}(a))$ is onto.*

For each standard $(\mathcal{P}, \prec)$ in $\mathrm{Spec}_{\mathrm{Re}}(\mathcal{O}_{n,\mathbb{R}}(a))$, let $\mathrm{Gen}(\mathcal{P}, \prec)$ be the external set whose elements are the nonstandard generic points of $(\mathcal{P}, \prec)$. Thus $\mathrm{hal}(a)$ is decomposed in an external disjoint union

$$\mathrm{hal}(a) = \bigcup_{(\mathcal{P},\prec)\in \mathrm{Spec}_{\mathrm{Re}}(\mathcal{O}_{n,\mathbb{R}}(a))} \mathrm{Gen}(\mathcal{P}, \prec)$$

Let $\mathcal{R}(\mathcal{O}_{n,\mathbb{R}}(a))$ be the set of order relation on the ring $\mathcal{O}_{n,\mathbb{R}}(a))$. It is natural to say that elements of

$$\mathrm{hal}(a)_{\text{Re-an}} = \mathrm{hal}(a) \setminus \bigcup_{\prec \in \mathcal{R}(\mathcal{O}_{n,\mathbb{R}}(a))} \mathrm{Gen}((0), \prec)$$

are *real analytic points* of ${}^*\mathbb{R}^n$ which are infinitely closed to $a$.

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