Constancy of some maps into f-manifolds

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Abstract

Constancy of some maps into certain f-manifolds is discussed

Introduction

In dealing with maps, such as harmonic and holomorphic ones, between manifolds existence questions form an essential part of their study. Mathematicians normally study such maps under certain conditions imposed on the manifolds and on the maps themselves. A vital question is that whether there exist such maps under the restrictions imposed. We highlight a few results in this line:

i) [**9**], If a harmonic map is constant on the boundary of a flat disk M of any dimension then it is constant on the whole disk M;

ii) [**5**], Let $\phi : M \to N$ be a holomorphic map between Kaehler manifolds with M compact and $rank(\phi) < \dim(M)$. If, for the respective Kaehler forms ω^M and ω^N, the cohomology classes satisfy that $[\phi^*\omega^N] = c[\omega^M]$ for some $c \in \mathbb{R}$ then ϕ is constant. In particular, one may take M to be a complex Grassmannian or a complex quadric.

Almost a decade ago mathematicians (see for example [**4**], [**6**], [**7**] and [**8**]) started to consider holomorphic maps and harmonic maps between metric f-manifolds. The main difference between an f-manifold and a complex one is that an almost complex structure is replaced with an f-*structure* by which we mean a 1-1 tensor field f satisfying $f^3 + f = 0$. In 2001, the following results were established:

iii) [**4**], Every holomorphic map from an almost Hermitian manifold into an almost $\mathcal{S}$-manifold is constant.

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iv) [**8**], Every holomorphic map from a semi-Kaehler manifold into a strongly pseudoconvex CR-manifold is constant.

In this work we give a non-existence result, *Theorem (2.1)*, for maps into certain f-manifolds under more general conditions. Our result improves the last two results above. One of the main features of our work is that it removes all restrictions on the domain manifold.

1 Preliminaries

For a Riemannian manifold $(N^{2n+\ell}, h)$ let φ denote a $(1,1)$- tensor field on N of rank $2n$ and nullity $\ell \geq 0$. Put $D = \varphi(TN)$ and $\mathcal{V} = Ker(\varphi)$. The distributions D and $\mathcal{V}$ over N are called *φ-horizontal* and *φ-vertical* respectively. The triple (N, h, φ) is then called *a metric f-manifold* (*M-f-manifold*) provided:

i) $\varphi^3 + \varphi = 0$;

ii) $h(X, Y) = 0, \ \forall X \in D, \ Y \in \mathcal{V}$;

iii) $h(X, Y) = h(\varphi X, \varphi Y) \ \forall X, \ Y \in D$.

We refer the conditions (ii) and (iii) as *h-compatibility of* φ. With no metric considered, the pair (N, φ) is called an *f-manifold.*

Suppose there is a global frame field $\{\xi_j\}_{j=1}^{\ell}$ for the φ-vertical bundle $\mathcal{V}$ with dual 1-forms $\{\eta^j\}_{j=1}^{\ell}$ satisfying

$$\varphi^2 = (-I + \sum_{j=1}^{\ell} \eta^j \otimes \xi_j), \quad \eta^j(\xi_a) = \delta_a^j$$

and

$$h(\varphi X, \ \varphi Y) = h(X, \ Y) - \sum_{j=1}^{\ell} \eta^j(X)\eta^j(Y).$$

Then $N = (N^{2n+\ell}; h, \varphi, \xi_j, \eta^j)$ is called *a globally framed metric f-manifold.* With no metric considered, $N = (N^{2n+\ell}; \varphi, \xi_j, \eta^j)$ will be called *a globally framed f-manifold.*

For the class of globally framed metric f-manifold $N = (N^{2n+\ell}; h, \varphi, \xi_j, \eta^j)$ we list some of its subclasses for later use ([**1**], [**2**], [**3**], [**4**]):

i) For $\ell = 0$, $N = (N^{2n}, h, \varphi)$ is called *an almost Hermitian manifold.*

ii) For $\ell \geq 1$ and set $\Omega(X, Y) = h(X, \varphi Y)$ then

$a°$) N is called *an almost $\mathcal{S}$-manifold* provided $d\eta^j = \Omega$, for each $j \in \{1, 2, \cdots, \ell\}$.

$b°$) N is called *an almost $\mathcal{C}$-manifold* provided $d\Omega = 0$ and $d\eta^j = 0$ for each $j = 1, 2, \cdots, \ell$.

iii) If $\ell = 1$, $N = (N^{2n+1}; h, \varphi, \xi, \eta)$ is called *an almost contact metric manifold* (almost CM-manifold). If further an almost CM-manifold is also an almost $\mathcal{S}$-manifold (so that $d\eta = \Omega$) then we drop the adjective "almost" and simply call *a contact metric manifold (* CM-manifold*)*. If an almost CM-manifold is also an almost $\mathcal{C}$-manifold, that is $d\Omega = 0$ and $d\eta = 0$, then it is called *almost cosymplectic.*

iv) An almost CM-manifold $(N^{2n+1}, h, \varphi, \xi, \eta)$ is called

$a°$) *nearly Sasakian* if $\forall X, Y \in \Gamma(TN)$

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = -2h(X, Y)\xi + \eta(X)Y + \eta(Y)X$$

where ∇ denotes the Levi-Civita connection;

b°) *almost trans–Sasakian of type* (α, β) if

$$d\eta = \alpha\Omega - \frac{1}{n}\eta \wedge \varphi^*(\delta\Omega) \text{ and } d\Omega = \Omega \wedge (\frac{1}{n}\varphi^*(\delta\Omega) - \beta\eta)$$

for some functions α, β on N, where δ is the codifferential operator and $\varphi^*(\delta\Omega)(X) = (\delta\Omega)(\varphi X)$. In particular, if $\alpha = \frac{1}{2n}(\delta\Omega)(\xi)$ and $\beta = \frac{1}{n}\delta\eta = div(\xi)$ then N is simply called *an almost trans-Sasakian manifold*;

c°) *nearly cosymplectic* if $(\nabla_X\varphi)X = 0$ for all $X \in \Gamma(TN)$;

d°) *quasi-K- cosymplectic* if

$$\mathcal{S}(X,Y) := (\nabla_X\varphi)Y + (\nabla_{(\varphi X)}\varphi)(\varphi Y) = \eta(Y)\nabla_{(\varphi X)}\xi \text{ for all } X, Y \in \Gamma(TN);$$

e°) *cosymplectic* if it is almost cosymplectic and normal (i.e. $N^1 = [\varphi, \varphi](X,Y) + 2d\eta(X,Y)\xi = 0$ for all $X, Y \in \Gamma(TN)$).

Remark:

i) [**10**], Every almost cosymplectic manifold is quasi K-cosymplectic.

ii) [**1**], Every almost contact metric manifold $(N^{2n+1}, h, \varphi, \xi, \eta)$ is cosymplectic if and only if $\nabla\varphi = 0$, that is φ is parallel.

Let $\phi : (M, g) \to (N, h, \varphi)$ be a smooth map of a Riemannian manifold into a M-f-manifold . Set $\mathcal{K} = \mathcal{K}_\phi = ker(d\phi), \mathcal{H} = \mathcal{H}_\phi = \mathcal{K}^\perp$ and $\overline{M} = \{p \in M : d\phi_p \neq 0\}$. If $\mathcal{K}$ (and therefore $\mathcal{H}$) forms a bundle then $\mathcal{K}$ and $\mathcal{H}$ are called the *vertical* and *horizontal distributions associated with* ϕ respectively.

Throughout our work the map ϕ will be smooth and $\overline{M}$ will be a dense open subset of M

Definition (1.1): A smooth map $\phi : M \to (N, \varphi)$ into an f-manifold is said to be

i) *φ-invariant* if $d\phi(TM)$ is invariant under φ, that is, $\varphi \circ d\phi(TM) = d\phi(TM)$.

ii) *properly φ-invariant* if it is φ-invariant of constant rank.

Remark: Note here that

a°) the φ-invariance of ϕ implies that $d\phi(TM) \subseteq D_\varphi$. Also every φ-invariant map into a surface is necessarily a submersion on $\overline{M}$;

b°) ϕ is properly φ-invariant if and only if the pull-back $\phi^{-1}(d\phi(TM)) \to \overline{M}$ together with $\phi^{-1}\varphi$ forms a complex bundle (and therefore $\mathcal{K} \to \overline{M}$ forms a bundle) over $\overline{M}$.

A smooth map $\phi : (M, \varphi^M) \to (N, \varphi^N)$ between f-manifolds is said to be (φ^M, φ^N)-*holomorphic* [resp: (φ^M, φ^N)-*antiholomorphic*] if

$$d\phi \circ \varphi^M = \varphi^N \circ d\phi, \quad [resp: \ d\phi \circ \varphi^M = -\varphi^N \circ d\phi].$$

We write $\pm(\varphi^M, \varphi^N)$-*holomorphic* to mean either (φ^M, φ^N)-holomorphic or (φ^M, φ^N)-antiholomorphic.

For a φ-invariant map $\phi : (M, g) \to (N, \varphi)$ of a Riemannian manifold into an f-manifold (not necessarily of constant rank), set $\forall p \in \overline{M}$,

$$\Psi_p(X_p) = \Psi_p^\phi(X_p) = \begin{cases} (d\phi)^{-1} \circ \varphi \circ d\phi(X_p) & , \ X_p \in \mathcal{H}_p \\ \\ 0 & , \ X_p \in \mathcal{K}_p \end{cases} \tag{1.1}$$

where $d\phi^{-1} = (d\phi|_{\mathcal{H}})^{-1}$.

Observe that for every φ-invariant map $\phi : (M, g) \to (N, \varphi)$ we have $d\phi \circ \Psi^{\phi} = \varphi \circ d\phi$. If ϕ is properly φ-invariant then Ψ becomes of constant rank and therefore it becomes an f-structure on M which we call *ϕ-associated f-structure.* In that case, we have that $\mathcal{H}_{\phi} = \mathcal{D}_{\Psi}$ and therefore we shall be using the letters $\mathcal{H}$ and D interchangeably for the same bundle. Setting $2m = rank(\mathcal{H})$ and $s = rank(\mathcal{K})$ we see that (M^{2m+s}, Ψ) becomes an f-manifold. Thus, every properly φ-invariant map is (Ψ,φ)-holomorphic as a map $\phi : (M, \Psi) \to (N, \varphi)$ between f-manifolds. However, this *ϕ-associated* f-structure Ψ need not be compatible with the prescribed metric g and therefore the triple (M^{2m+s}, g, Ψ) need not to be a M-f-manifold.

Definition (1.2): A map $\phi : (M, g) \to (N, h, \varphi)$ is said to be

a°) *horizontally weakly conformal* if $d\phi$ is surjective and satisfies that

$$h(d\phi(X), d\phi(Y)) = \lambda g(X, Y); \;\; \forall X, Y \in \mathcal{H}$$

for some smooth function $\lambda : M \to \mathbb{R}$

b°) (g, φ)-*pseudo horizontally weakly conformal* (or simply *pseudo horizontally weakly conformal* when no confusion arises) if ϕ is φ-invariant and Ψ^{ϕ} is compatible with the metric g.

Remark: Note here that

i) every horizontally weakly conformal map is also pseudo horizontally weakly conformal,

ii) the pseudo horizontal weak conformality of ϕ does not depend on the metric h.

2 Constancy of certain maps

For a smooth map $\phi : M \to (N, \varphi)$ of a manifold into an f-manifold with $d\phi(TM) \subseteq \mathcal{D}_{\varphi}$ we introduce the following two conditions: For every $p \in \overline{M}$, writing $q = \phi(p)$.

$\mathbf{C_N}$: there is a nonzero vector $\zeta_q \in \mathcal{V}_{\varphi}(q) = ker(\varphi_q)$ and a local 1-form ω defined on an open subset $\mathcal{U}$ of N containing q with $\omega(\zeta_q) = 1$ such that $\forall X \in T_qN$

$$\varphi^2(X) = -X + \omega(X)\zeta_q + v_x \quad \text{for some} \quad v_x \in \mathcal{V}(q) \cap ker(\omega_q);$$

$\mathbf{C}_{\phi}$: for $d\phi(T_pM) \neq 0$, the function $\gamma_q : d\phi(T_pM) \subset T_qN \to \mathbb{R}$ given by, (on an open subset $\mathcal{U} \subset N$ containing q) by

$$\gamma_q(X_q) = d\omega(X_q, \varphi X_q)$$

is not zero.

Observe that, one can easily deduce from the condition $\mathbf{C_N}$ that $\mathcal{D}_{\varphi}(q) \subseteq ker(\omega_q)$.

Note that, for a smooth map $\phi : M \to (N, \varphi)$, one has wide variety of choices of structures on N for which the conditions $\mathbf{C_N}$ and $\mathbf{C}_{\phi}$ are satisfied. We give the following examples. Let $\phi : M \to N$ be a smooth map into a M-$f.pk$- manifold. Consider the following cases:

1°) Let $N = (N^{2n+\ell}, h, \varphi, \xi_j, \eta^j)$ be an almost $\mathcal{S}$-manifold or, in particular, a CM-manifold then $\mathbf{C_N}$ and $\mathbf{C}_\phi$ are satisfied. Indeed, putting $\zeta = \xi_j$ and $\omega = \eta^j$ for some $j \in \{1, \cdots, \ell\}$, $\ell \geq 1$ we see that

$\mathbf{C_N} : \varphi^2(X) = -X + \omega(X)\xi_j + v$ with $v = \sum_{\substack{t=1\\t\neq j}}^{\ell} \eta^t(X)\xi_t \in \mathcal{V} \cap ker(\omega)$

$\mathbf{C}_\phi : \gamma(X) = d\eta^j(X, \varphi X) = \Omega(X, \varphi X) = -h(X, X) \neq 0, \ \forall X \in d\phi(TM) \subseteq D_\varphi$, where $\Omega(X, Y) = d\eta^j(X, Y) = h(X, \varphi Y)$.

2°) Let $N = (N^{2n+1}, h, \varphi, \xi, \eta)$ be a nearly Sasakian manifold then $\mathbf{C_N}$ and $\mathbf{C}_\phi$ are satisfied. For let $\zeta = \xi$ and $\omega = \eta$, then $\mathbf{C_N}$ is obvious. For $\mathbf{C}_\phi$, since N is nearly Sasakian, we have

$$(\nabla_X \varphi)X = -h(X, X)\xi + \eta(X)X$$

But then, for $X \in \Gamma(d\phi(T\overline{M})) \subseteq \Gamma(D^N)$, this gives $(\nabla_X \varphi)X = -h(X, X)\xi$ and therefore we get

$$\mathcal{S}(X, X) = -2h(X, X)\xi \tag{2.1}$$

where $\Gamma(V)$ denotes the set of all smooth local sections of a bundle $V \to N$. On the other hand, for $X \in \Gamma(d\phi(TM))$ we have

$$\begin{aligned} d\eta(X, \varphi X) &= -\eta([X, \varphi X]) = -\eta(\nabla_X(\varphi X) + \nabla_{(\varphi X)}(\varphi^2 X)) \\ &= -\eta((\nabla_X \varphi)X + (\nabla_{(\varphi X)}\varphi)(\varphi X)) = -\eta(\mathcal{S}(X, X)) \end{aligned}$$

So, by (2.1) we get

$$\gamma(X) = 2h(X, X) \neq 0.$$

3°) Let $N = (N^{2n+1}, h, \varphi, \xi, \eta)$ be an almost trans-Sasakian manifold of type (α, β) with $\alpha(q) \neq 0$, $\forall q \in \phi(M) \subseteq N$. Then $\mathbf{C_N}$ and $\mathbf{C}_\phi$ are satisfied. For let $\zeta = \xi$ and $\omega = \eta$, then $\mathbf{C_N}$ is obviously satisfied. For $\mathbf{C}_\phi$, since N is trans-Sasakian, we have

$$d\eta(X, \varphi X) = \alpha\Omega(X, \varphi X) - \frac{1}{n}(\eta \wedge \varphi^*(\delta\Omega))(X, \varphi X) = \alpha\Omega(X, \varphi X) = -\alpha h(X, X)$$

for all $X \in \Gamma(d\phi(T\overline{M}) \subseteq \Gamma(D^N)$. Thus $\gamma(X) = -\alpha h(X, X) \neq 0$ on $d\phi(T\overline{M})$.

However, for the following structure on N, the condition $\mathbf{C}_\phi$, for example, fails to hold:

Let $\phi : M \to (N^{2n+\ell}, h, \varphi, \xi_j, \eta^j)$ be a smooth map into a M-$f.pk$-manifold N such that $\mathcal{S}(X, X) \in D^N$ and $d\phi(TM) \subseteq D^N$, $\forall X \in D^N$ (or equivalently, $[X, \varphi X] \in D^N$, $\forall X \in D^N$). Then $\gamma(X) = 0$ on D^N and therefore on $d\phi(TM)$. Indeed, setting $\omega = \eta^j$, for any $j = 1, 2, ...\ell$ note that

$$\begin{aligned} \gamma(X) &= d\omega(X, \varphi X) = -\omega([X, \varphi X]) = -\omega(\nabla_X(\varphi X) + \nabla_{(\varphi X)}(\varphi^2 X)) \\ &= -\omega((\nabla_X \varphi)X + (\nabla_{(\varphi X)}\varphi)(\varphi X) + \varphi(\nabla_X X + \nabla_{(\varphi X)}(\varphi X))) \\ &= \omega((\nabla_X \varphi)X + (\nabla_{(\varphi X)}\varphi)(\varphi X)) = \omega(\mathcal{S}(X, X)) = 0. \end{aligned}$$

Remark: Observe that for almost $\mathcal{C}$-manifolds, nearly cosymplectic and quasi K-cosymplectic manifolds the condition that $\mathcal{S}(X,X) \in D, \quad \forall X \in D$ trivially holds.

Theorem (2.1): *Let $\phi : M \to (N^{2n+\ell}, \varphi)$ be a φ-invariant map from an arbitrary smooth manifold into an f-manifold, with $\ell = rank(\mathcal{V}_\varphi) \neq 0$ such that the conditions C_N and C_ϕ hold. Then ϕ is constant, that is $\overline{M} = \{p \in M : d\phi_p \neq 0\}$ is empty.*

Proof: Suppose $\overline{M}$ is not empty, then $d\phi_{p_0} \neq 0$ for some $p_0 \in M$. By condition C_ϕ, there is $Z = d\phi_{p_0}(W)$ with $\gamma_{q_o}(Z) \neq 0$ for some $W \in T_{p_0}M$. Set $H = \mathcal{K}^\perp$ with respect to any chosen Riemannian metric on M and let $X \in \Gamma(H)$ be a local section with $d\phi(X_{p_0}) = Z$. Recalling the endomorphism $\Psi_p : T_pM \to T_pM$ defined by (1.1), (note that then we have $\varphi \circ d\phi = d\phi \circ \Psi$), observe that

$$A := [d\phi(X), \varphi d\phi(X)] = d\phi([X, \Psi X])$$

which shows that $A_{q_o} \in d\phi(T_{p_0}M) \subseteq D_\varphi(q_o)$ and so $\omega(A_{q_o}) = 0$. On the other hand,

$$0 = \omega(A_{q_o}) = \omega([d\phi(X),\ \varphi d\phi(X)]_{q_o}) = -d\omega(d\phi(X_{p_0}),\ \varphi d\phi(X_{p_0})) = -\gamma_{q_o}(Z).$$

This contradicts the choice of $Z \in d\phi(T_{p_0}M)$ with $\gamma_{q_o}(Z) \neq 0$, so the result follows.

Remark: In the above theorem it is essential that $rank(\mathcal{V}_\varphi) = \ell > 0$ as the conditions $\mathbf{C_N}$ and $\mathbf{C}_\phi$ cannot possibly hold when $\ell = 0$. Also note that we do not impose any condition on M.

Corollary (2.2): *Let $\phi : (M, \varphi^M) \to (N, \varphi^N)$ be a $\pm(\varphi^M, \varphi^N)$- holomorphic map between f-manifolds with $d\phi(TM) \subseteq D^N$ and suppose that the conditions $\mathbf{C_N}$ and $\mathbf{C}_\phi$ hold. Then ϕ is constant.*

Proof: Observe that $\pm(\varphi^M, \varphi^N)$-holomorphicity, in this case, implies that ϕ is φ-invariant. So by Theorem (2.1), ϕ is constant.

Corollary (2.3): *Let $(M, J) \to (N, \varphi)$ be a $\pm(J, \varphi)$-holomorphic map from an almost complex manifold into an f-manifold such that the conditions $\mathbf{C}_N$ and $\mathbf{C}_\phi$ hold. Then ϕ is constant.*

Proof: Observe that since J is an almost complex structure, $\pm(J, \varphi)$-holomorphicity gives that $d\phi(TM) \subseteq D^N$. Then the result follows from Corollary (2.2).

We say that a *M-f.pk* manifold N is in $\star$*-category* if N is either nearly Sasakian or trans-Sasakian of type (α, β) with $\alpha(q) \neq 0, \quad \forall q \in N$ or an almost $\mathcal{S}$-manifold.

Corollary (2.4): *Let $\phi : (M, J) \to (N^{2n+\ell}, h, \varphi, \xi_j, \eta^j)$* be a *$\pm(J, \varphi)$-holomorphic map from an almost complex manifold into a manifold which is in* $\star$-category. *Then ϕ is constant.*

Proof: Since the target manifold is in $\star$-category, the conditions $\mathbf{C_N}$ and $\mathbf{C}_\phi$ are satisfied. Thus the result follows from Corollary (2.3).

In particular, when $(N^{2n+\ell}; h, \varphi, \xi_j, \eta^j)$ is an almost $\mathcal{S}$-manifold, Corollary (2.4) gives immediately the result in ([**4**], Theorem (5.2) and therefore Theorem (5.1)).

Corollary (2.5): *Let* $\phi : (M, g) \to (N^{2n+\ell}; h, \varphi, \xi_j, \eta^j)$ *be a* φ*-pseudo horizontally weakly conformal map from an arbitrary Riemannian manifold into a manifold which is in* $\star$*-category, then* ϕ *is constant.*

Proof: The φ-pseudo horizontal weak conformality of ϕ and N being in the $\star$-category imply that the hypothesis of Theorem(2.1) are satisfied. Thus the result follows immediately.

Following the terminology in [**8**], a strongly pseudoconvex CR-manifold $N = (N^{2n+1}; \varphi_{_D}, D^N, \eta)$ with its Levi distribution D^N of rank $2n$, 1-form η and positive definite Levi from $L(X, Y) = -d\eta\,(\varphi_{_D} X, Y)$ may be viewed as a contact metric manifold $(N^{2n+1}; h, \varphi, \xi, \eta)$ with $\varphi\mid_{_D} = \varphi_{_D}$, $h\mid_{_D} = L$ and η its contact form, ξ its characteristic vector field. Thus for such manifolds we have :

Corollary (2.6): *Any* $\pm(J, \varphi)$*-holomorphic map* $\phi : (M, J) \to N$ *from an almost complex manifold (or any* φ*-pseudo horizontally weakly conformal map* $\phi :$ $(M, g) \to N$ *from an arbitrary Riemannian manifold) into a strongly pseudoconvex almost CR-manifold is constant.*

This Corollary recovers the result given in ([**8**] , Proposition (2.5)). Note here that, in our work, the condition imposed therein that M is semi-Kaehler is removed. We also include the cases where ϕ is (J, φ)- antiholomorphic.

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