A characteristic property of nest algebras

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Abstract

In this short note, we prove a characteristic property of nest algebras. This property is closely related to the extreme point structure of nest algebras.

The terminology and notation of this note concerning reflexive algebras and extreme points may be found in [2] and [6]. For simplicity, we give the definition of Property EP which is closely related to the extreme point structure.

Property EP: Let $\mathcal{L}$ be a subspace lattice. For any $B \in \mathcal{B}(\mathcal{H})$, if $BXB = 0$ for any rank-one operator X in Alg$\mathcal{L}$, then $BYB \in$ Alg$\mathcal{L}$ for any rank one operator Y in $\mathcal{B}(\mathcal{H})$. We say that Alg$\mathcal{L}$ has Property EP.

During the last decade there has been an interest in the Banach space geometry of non-selfadjoint operator algebras. Subjects of investigation there include the extreme point structure of the unit ball [3],[6] and the isometries between such algebras [1],[4],[7]. One possible application of a characterization of extreme points is in the study of isometries. R.V.Kadison [4] used the characterization of extreme points in his study of isometries on C^*–algebras. For the same end, R.L.Moore and T.T.Trent investigated the extreme points for nest algebras in [6]. Theorem 6 and its Corollary are the main part of [6]. The essence of Theorem 6's proof in [6] is to show that Property EP holds for nest algebras. In this short note, we will verify that Property EP holds only for nest algebras. This shows that the method used in Theorem 6 of [6] can not work for Alg$\mathcal{L}$ where $\mathcal{L}$ is not a nest. Thus, the remark behind the corollary of Theorem 6 in [6] is not definite and therefore how to characterize the extreme points of Alg$\mathcal{L}$ is still an open problem when $\mathcal{L}$ is not a nest.

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Let x and y be vectors in $\mathcal{H}$. Let $x \otimes y$ represent the rank-one operator defined by $(x \otimes y)z =< z, y > x$, for each $z \in \mathcal{H}$. Recall that a reflexive subspace lattice $\mathcal{L}$ is a subspace lattice which satisfies $\mathcal{L} = \mathrm{LatAlg}\mathcal{L}$.

Lemma 1 Let $\mathcal{L}$ be a reflexive subspace lattice. A rank-one operator $x \otimes y \in (\mathrm{Alg}\mathcal{L})_\perp$ if and only if there exists an element $E \in \mathcal{L}$ such that $x \in E$ and $y \in E^\perp$.

Proof. Necessity. Set $E = [(\mathrm{Alg}\mathcal{L})x]$. Since $\mathcal{L}$ is reflexive, we have $E \in \mathcal{L}$ and $x \in E$. For any $A \in \mathrm{Alg}\mathcal{L}$, we have

$$0 = tr(A(x \otimes y)) =< Ax, y >$$

since $x \otimes y \in (\mathrm{Alg}\mathcal{L})_\perp$. Hence $y \in [(\mathrm{Alg}\mathcal{L})x]^\perp = E^\perp$.

Sufficiency. For any $A \in \mathrm{Alg}\mathcal{L}$,

$$\begin{aligned} tr(A(x \otimes y)) &= tr(AE(x \otimes y)E^\perp) \\ &= tr(E^\perp AE(x \otimes y)) = 0. \end{aligned}$$

This shows that $x \otimes y \in (\mathrm{Alg}\mathcal{L})_\perp$. ∎

Note that the hypothesis of reflexivity of $\mathcal{L}$ is not needed in the sufficiency of Lemma 1.

Lemma 2 Let $\mathcal{L}$ be a subspace lattice and $E_0 \in \mathcal{L}$. Then $E_0(x \otimes y)E_0^\perp \in \mathrm{Alg}\mathcal{L}$ for any $x, y \in \mathcal{H}$ if and only if E_0 is a comparable element in $\mathcal{L}$ (that is, for any $F \in \mathcal{L}$, either $F \leq E_0$ or $F > E_0$).

Proof. Sufficiency. For any $F \in \mathcal{L}$, if $F \leq E_0$, we have

$$E_0(x \otimes y)E_0^\perp F = (0) \subseteq F;$$

if $F > E_0$, $E_0(x \otimes y)E_0^\perp F \subseteq E_0 \subseteq F$. So $E_0(x \otimes y)E_0^\perp \in \mathrm{Alg}\mathcal{L}$ for any $x, y \in \mathcal{H}$.

Necessity. For any $F \in \mathcal{L}$ and $F \not\leq E_0$, we only need to verify that $F > E_0$. Since $E_0(x \otimes y)E_0^\perp \in \mathrm{Alg}\mathcal{L}$ for any $x, y \in \mathcal{H}$, we have

$$E_0(x \otimes y)E_0^\perp F \subseteq F.$$

Since $F \not\leq E_0$, $FE_0^\perp \neq 0$ and so there exists $y_0 \in \mathcal{H}$ such that $FE_0^\perp y_0 \neq 0$. Thus

$$[E_0(x \otimes y_0)E_0^\perp F](FE_0^\perp y_0) \in F$$

and

$$\| FE_0^\perp y_0 \|^2 E_0 x \in F, \quad \text{for any } x \in \mathcal{H}.$$

Since $\| FE_0^\perp y_0 \| \neq 0$, this implies that $E_0 < F$. Hence E_0 is a comparable element in $\mathcal{L}$. ∎

Corollary 3 Suppose that $\mathcal{L}$ is a subspace lattice. Then $(\mathrm{Alg}\mathcal{L})_\perp \subseteq \mathrm{Alg}\mathcal{L}$ if and only if $\mathcal{L}$ is a nest.

Proof. Sufficiency. It is well known that if $\mathcal{L}$ is a nest,

$$(\mathrm{Alg}\mathcal{L})_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{W},$$

where $\mathcal{W} = \{T \in \mathcal{B}(\mathcal{H}) : TE \subseteq E_- \text{ for any } E \in \mathcal{L}\}$. Since $\mathcal{L}$ is a nest, $E_- \leq E$ for any $E \in \mathcal{L}$ and $\mathcal{W} \subseteq \mathrm{Alg}\mathcal{L}$. So $(\mathrm{Alg}\mathcal{L})_\perp \subseteq \mathrm{Alg}\mathcal{L}$.

Necessity. For any $E \in \mathcal{L}$, it follows from the sufficiency of Lemma 1 that $E(x \otimes y)E^\perp \in (\mathrm{Alg}\mathcal{L})_\perp$ for any $x, y \in \mathcal{H}$. Thus

$$E(x \otimes y)E^\perp \in \mathrm{Alg}\mathcal{L}, \quad \forall x, y \in \mathcal{H}.$$

By virtue of Lemma 2, this implies that E is a comparable element in $\mathcal{L}$. Since E is arbitrary, $\mathcal{L}$ is totally ordered and so it is a nest. ∎

Lemma 4 Suppose that $\mathcal{L}$ is a subspace lattice and $E_0 \in \mathcal{L}$. If $B = E_0 B E_0^\perp$, then $BXB = 0$ for any rank-one operator $X \in \mathrm{Alg}\mathcal{L}$.

Proof. It follows from [5] Lemma 3.1 that a rank-one operator $X = x \otimes y \in \mathrm{Alg}\mathcal{L}$ if and only if there exists $E \in \mathcal{L}$ such that $x \in E$ and $y \in E_-^\perp$, where $E_- = \vee\{F \in \mathcal{L} : F \not\geq E\}$. If $E \leq E_0$, $E_0^\perp E = 0$. Thus

$$BXB = E_0 B E_0^\perp E(x \otimes y) E_-^\perp B = 0.$$

If $E \not\leq E_0$, it follows from the definition of E_- that $E_0 \leq E_-$. So $E_-^\perp E_0 = 0$ and

$$BXB = BE(x \otimes y) E_-^\perp E_0 B E_0^\perp = 0.$$

This concludes the proof. ∎

If $\mathcal{L}$ is a completely distributive lattice, then $BXB = 0$ for any rank-one operator X in $\mathrm{Alg}\mathcal{L}$ if and only if there exists $E_0 \in \mathcal{L}$ such that $B = E_0 B E_0^\perp$. The necessity of this result is implied in the proof of Theorem 6 of [6].

Theorem 5 Suppose that $\mathcal{L}$ is a subspace lattice. Property EP holds for $\mathrm{Alg}\mathcal{L}$ if and only if $\mathcal{L}$ is a nest.

Proof. If $\mathcal{L}$ is a nest, the proof of Theorem 6 in [6] shows that Property EP holds for $\mathrm{Alg}\mathcal{L}$. This is the essence of Theorem 6.

Conversely, we suppose that Property EP holds for $\mathrm{Alg}\mathcal{L}$. If $\mathcal{L}$ is not a nest, there exists an element $E_0 \in \mathcal{L}$ which is not a comparable element of $\mathcal{L}$. Naturally $E_0 \neq (0), \mathcal{H}$. Since E_0 is not a comparable element in $\mathcal{L}$, it follows from Lemma 2 that there exist unit vectors $z \in E_0$ and $w \in E_0^\perp$ such that $z \otimes w \notin \mathrm{Alg}\mathcal{L}$. Set $B = z \otimes w$. Since $B = E_0 B E_0^\perp$, Lemma 4 implies that $BXB = 0$ for any rank-one operator X in $\mathrm{Alg}\mathcal{L}$. However,

$$\begin{aligned} B(w \otimes z)B &= (z \otimes w)(w \otimes z)(z \otimes w) \\ &= z \otimes w \notin \mathrm{Alg}\mathcal{L}. \end{aligned}$$

This shows that Property EP does not holds for $\mathrm{Alg}\mathcal{L}$. This contradicts the hypothesis, so $\mathcal{L}$ is a nest. ∎

Theorem 5 shows that the method used in Theorem 6 of [6] can't work for any wider class of subspace lattices, except for nests. So how to characterize the extreme point structure of the unit ball of $\mathrm{Alg}\mathcal{L}$ is still an open and challenging problem when $\mathcal{L}$ is not a nest.

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