# On Certain Differential Subordination and its Dominant

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**Abstract**

Denote by $\mathcal{A}'$, the class of functions $f$, analytic in $E$ which satisfy $f(0) = 1$. Let $\alpha > 0, \beta \in (0, 1]$ be real numbers and let $\gamma$, $\mathrm{Re}\gamma > 0$, be a complex number. For $p, q \in \mathcal{A}'$, the authors study the differential subordination of the form

$$(p(z))^\alpha \left[1 + \frac{\gamma z p'(z)}{p(z)}\right]^\beta \prec (q(z))^\alpha \left[1 + \frac{\gamma z q'(z)}{q(z)}\right]^\beta, \ z \in E,$$

and obtain its best dominant. Its applications to univalent functions are also given.

## 1 Introduction

Let $\mathcal{A}$ be the class of functions $f$ which are analytic in $E$ and are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Denote by $\mathcal{A}'$, the class of functions $f$ which are analytic in the open unit disc $E$ and satisfy $f(0) = 1$ and $f(z) \neq 0$ in $E$.

A function $f$, analytic in the unit disc $E$, is said to be convex if it is univalent and $f(E)$ is a convex domain. It is well-known that $f$ is convex if and only if $f'(0) \neq 0$ and

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, \ z \in E.$$

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Denote by $K$, the class of convex functions in $E$.

An analytic function $f$, with $f(0) = 0, f'(0) \neq 0$, is said to be starlike (with respect to the origin) if $f(E)$ is a starlike domain with respect to $z = 0$ or, equivalently, if

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0,\ z \in E.$$

Let $St$ denote the class consisting of starlike functions.

Let the functions $f$ and $g$ be analytic in the unit disc $E = \{z : |z| < 1\}$. We say that $f$ is subordinate to $g$ in $E$, written as $f(z) \prec g(z)$ in $E$( or simply $f \prec g$), if $g$ is univalent in $E$, $f(0) = g(0)$ and $f(E) \subset g(E)$.

Let $\psi : C^2 \to C$ be an analytic function in a domain $D \subset C^2$ ($C$ being the complex plane), $p$ be an analytic function in $E$ with $(p(z), zp'(z)) \in D$ for $z \in E$, and let $h$ be a function analytic and univalent in $E$. The function $p$ is said to satisfy the first order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z),\ z \in E,\quad \psi(p(0), 0) = h(0). \tag{1}$$

A univalent function $q$ is said to be a dominant of the differential subordination (1) if $p \prec q$ for all $p$ satisfying (1). A dominant $\tilde{q}$ of (1) that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (1) is said to be the best dominant of the differential subordination (1). Several examples are available in literature where information about the range of a function is obtained from the information about its derivatives or a combination of derivatives. We list a few of them below.

In 1935, Goluzin [1] proved that if $h$ is convex, then

$$zp'(z) \prec h(z) \text{ in } E \Rightarrow p(z) \prec \int_0^z h(t)t^{-1}dt \text{ in } E. \tag{2}$$

Suffridge [17] improved it by showing that above result holds even if $h$ is starlike.

Robinson [15], in 1947, considered the differential subordination

$$p(z) + zp'(z) \prec h(z),\ z \in E, \tag{3}$$

and showed that if $h$ is univalent in $E$, then $q(z) = z^{-1}\int_0^z h(t)dt$ is the best dominant of the differential subordination (3), at least in $|z| < 1/5$.

Hallenbeck and Ruscheweyh [2] proved that the function $q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t)t^{\gamma-1}dt$ is the best dominant of the differential subordination

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z),\quad z \in E, \tag{4}$$

where $\gamma \neq 0$, $\mathrm{Re}\gamma \geq 0$ and $h$ is convex with $h(0) = 1$.

But the development of the theory of differential subordination gained momentum with the publication of an article by S.S. Miller and P. T. Mocanu [7] in 1981.

Since then, many authors have used it and obtained many interesting results. In [8], Miller Mocanu and Reade improved the result of Hallenbeck and Ruscheweyh by showing that $q$ is the best dominant of subordination (4),where $\gamma = 1$, even when

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -1/2.$$

In this paper, they proved a more general subordination. They proved that under certain conditions on the function $h$, the differential subordination

$$p(z)\left[\gamma + \beta\frac{zp'(z)}{p(z)}\right]^{1/\beta} \prec q(z)\left[\gamma + \beta\frac{zq'(z)}{q(z)}\right]^{1/\beta} = h(z),\ p(0) = q(0) = 0, \quad (5)$$

implies that

$$p(z) \prec q(z),\ z \in E,$$

where $\beta$ and $\gamma$ are complex numbers with $\mathrm{Re}\beta > 0$ and $\mathrm{Re}\gamma \geq 0$. In 1996, S. Kanas et.al. [3] generalized it further by considering the differential subordination of the form

$$p(z)\left[1 + \frac{zp'(z)}{p(z)}\phi(p(z))\right]^{\alpha} \prec q(z)\left[1 + \frac{zq'(z)}{q(z)}\phi(q(z))\right]^{\alpha},\ p(0) = q(0) = 0,\ z \in E, \quad (6)$$

where $\alpha \in [0, 1]$. With appropriate conditions on $q$ and $\phi$, they showed that the differential subordination (6) has $q$ as its best dominant.

Differential subordinations (5) and (6) provided us the motivation to study the differential subordination of the form

$$(p(z))^{\alpha}\left[1 + \frac{\gamma zp'(z)}{p(z)}\right]^{\beta} \prec (q(z))^{\alpha}\left[1 + \frac{\gamma zq'(z)}{q(z)}\right]^{\beta} = h(z),\ z \in E, \quad (7)$$

where $p(0) = q(0) \neq 0$ (note that in (5) and (6), $p(0) = q(0) = 0$ ), $\alpha$ and $\beta$ are suitably chosen real numbers and $\gamma$, $\mathrm{Re}\gamma > 0$, is a complex number. In this paper, we obtain the conditions which the function $q$ must satisfy so that it is the best dominant of the differential subordination (7). Section 4 gives some applications of this differential subordination to univalent functions wherein some new results have been obtained and few existing ones are derived as corollaries.

## 2 Preliminaries

**Definition 2.1.** A function $L(z, t), z \in E$ and $t \geq 0$ is said to be a subordination chain if $L(., t)$ is analytic and univalent in $E$ for all $t \geq 0$, $L(z, .)$ is continuously differentiable on $[0, \infty)$ for all $z \in E$ and $L(z, t_1) \prec L(z, t_2)$ for all $0 \leq t_1 \leq t_2$.

**Lemma 2.1.** [11, page 159] The function $L(z, t) : E \times [0, \infty) \to C$ of the form $L(z, t) = a_1(t)z + \ldots$ with $a_1(t) \neq 0$ for all $t \geq 0$, and $\lim_{t\to\infty} |a_1(t)| = \infty$, is said to be a subordination chain if and only if Re $\left[\frac{z\partial L/\partial z}{\partial L/\partial t}\right] > 0$ for all $z \in E$ and $t \geq 0$.

**Lemma 2.2.** [6,page 11] Let $n \geq 0$ be an integer and let $\gamma \in C$, with Re$\gamma > -n$. If $f(z) = \sum\limits_{m\geq n} a_m z^m$ is analytic in $E$ and $F$ is defined by

$$F(z) = \frac{1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt = \sum_{m\geq n}\frac{a_m z^m}{m+\gamma},$$

then $F$ is analytic in $E$.

**Lemma 2.3.** [10] Let $P(z)$ be an analytic function in $E$ such that Re$P(z) > 0, z \in E$. If $p \in A'$ satisfies the differential equation

$$zp'(z) + P(z)p(z) = 1,\ z \in E,$$

then

$$\text{Re}p(z) > 0,\ z \in E.$$

**Lemma 2.4.** Let $F$ be analytic in $E$ and let $G$ be analytic and univalent in $\overline{E}$ except for points $\zeta_0$ such that $\lim\limits_{z\to\zeta_0} F(z) = \infty$, with $F(0) = G(0)$. If $F \not\prec G$ in $E$, then there is a point $z_0 \in E$ and $\zeta_0 \in \partial E$ (boundary of $E$) such that $F(|z| < |z_0|) \subset G(E)$, $F(z_0) = G(\zeta_0)$ and $z_0F'(z_0) = m\zeta_0G'(\zeta_0)$ for $m \geq 1$.
Lemma 2.4 is due to Miller and Mocanu [7].

A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\alpha, \alpha > 0$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^\alpha,\ z \in E,$$

and let the class of all such functions be denoted by $S(\alpha)$.

**Lemma 2.5.** Let $f \in \mathcal{A}$ be such that

$$f'(z) \prec 1 + \lambda z,\ z \in E.$$

Then $f \in S(\alpha)$ where $\alpha$ is given by

$$0 < \lambda \leq \frac{2sin(\pi\alpha/2)}{\sqrt{5 + 4cos(\pi\alpha/2)}}.$$

Lemma 2.5 is a special case of Corollary 1.7 of [13].

## 3 Main Result

Before stating and proving our main result, we prove the following lemma. In what follows, all powers are chosen as principal ones.

**Lemma 3.1.** Let $\alpha > 0, \beta \in (0,1]$ be real numbers and let $\gamma, \text{Re}\gamma > 0$, be a complex number. Suppose that $h \in \mathcal{A}'$ satisfies

$$\text{Re}\left[\frac{zh'(z)}{h(z)} + \frac{\alpha}{\gamma}\right] > 0,\ z \in E.$$

Then the solution $q$ of the differential equation

$$(q(z))^{\alpha}\left[1 + \frac{\gamma z q'(z)}{q(z)}\right]^{\beta} = h(z),\ q(0) = 1, \tag{8}$$

is analytic in $E$, satisfies $\text{Re}\left[\frac{zq'(z)}{q(z)} + \frac{1}{\gamma}\right] > 0, z \in E$ and is given by

$$q(z) = \left[\frac{\alpha}{\beta\gamma z^{\frac{\alpha}{\beta\gamma}}}\int_0^z h^{\frac{1}{\beta}}(t)t^{\frac{\alpha}{\beta\gamma}-1}dt\right]^{\frac{\beta}{\alpha}}. \tag{9}$$

*Proof.* Let $h(z) = 1 + h_1 z + \ldots \in \mathcal{A}'$ and satisfies (8). Define

$$\phi(z) = \frac{1}{z^{\frac{\alpha}{\beta\gamma}}h^{\frac{1}{\beta}}(z)}\int_0^z h^{\frac{1}{\beta}}(t)t^{\frac{\alpha}{\beta\gamma}-1}dt \tag{10}$$

$$= \tfrac{\beta\gamma}{\alpha} + A_1 z + \ldots$$

Then, as $\alpha > 0, \beta > 0$ and $\text{Re}\gamma > 0$, we have $\text{Re}\left(\frac{\alpha}{\beta\gamma}\right) > 0$. Therefore, in view of Lemma 2.2 (with n=0) and the fact that $h^{\frac{1}{\beta}}(z) \neq 0$ in $E$, we conclude that $\phi$ is analytic in $E$. Differentiating (10), we get

$$z\phi'(z) + \phi(z)\left[\frac{\alpha}{\beta\gamma} + \frac{1}{\beta}\frac{zh'(z)}{h(z)}\right] = 1, \tag{11}$$

i.e.

$$z\phi'(z) + \phi(z)p(z) = 1,$$

where $p(z) = \frac{\alpha}{\beta\gamma} + \frac{1}{\beta}\frac{zh'(z)}{h(z)}$. Now $\text{Re}\left[\frac{zh'(z)}{h(z)} + \frac{\alpha}{\gamma}\right] > 0,\ z \in E$ and $\beta > 0$ implies that $\text{Re}p(z) > 0, z \in E$. So, in view of Lemma 2.3, we obtain $\text{Re}\phi(z) > 0, z \in E$. From (9) and (10), we get

$$q(z) = \left(\frac{\alpha}{\beta\gamma}\phi(z)\right)^{\frac{\beta}{\alpha}} h^{\frac{1}{\alpha}}(z) \tag{12}$$

$$= 1 + q_1 z + \ldots.$$

Since $h$ and $\phi$ are analytic, we conclude that $q$ is also analytic in $E$. Logarithmic differentiation of (12) leads to

$$\frac{\alpha}{\beta}\frac{zq'(z)}{q(z)} = \frac{1}{\beta}\frac{zh'(z)}{h(z)} + \frac{z\phi'(z)}{\phi(z)}$$

$$= \frac{1}{\phi(z)} - \frac{\alpha}{\beta\gamma},\quad (\text{using}(11)).$$

Thus

$$\frac{1}{\gamma} + \frac{zq'(z)}{q(z)} = \frac{\beta}{\alpha}\frac{1}{\phi(z)}.$$

Since Re$\phi(z) > 0, z \in E$ and $\alpha > 0, \beta > 0$, it follows that

$$\mathrm{Re}\left[\frac{zq'(z)}{q(z)} + \frac{1}{\gamma}\right] > 0, z \in E.$$

It is easy to verify that $q$ given by (9) is a solution of the differential equation (8). This completes the proof of Lemma 3.1.

**Theorem 3.1** Let $\alpha > 0, \beta \in (0, 1]$ be real numbers and let $\gamma$ be a complex number with Re$\gamma > 0$. Suppose that the differential equation

$$(q(z))^{\alpha}\left[1 + \frac{\gamma zq'(z)}{q(z)}\right]^{\beta} = h(z),\ z \in E,$$

where $q(0) = h(0) = 1, q(z) \neq 0$ in $E$, has an analytic and univalent solution $q$ which satisfies the following conditions:

(i) $\mathrm{Re}\left(\frac{1}{\gamma} + \frac{zq'(z)}{q(z)}\right) > 0$ in $E$ and

(ii) $\log q(z)$ is convex in $E$.

If $p \in A'$ satisfies the differential subordination

$$(p(z))^{\alpha}\left[1 + \frac{\gamma zp'(z)}{p(z)}\right]^{\beta} \prec h(z),\ z \in E, \tag{13}$$

then

$$p(z) \prec q(z)$$

in $E$, where $q$ is given by (9). Moreover, $q$ is the best dominant for the differential subordination (13).

*Proof.* Without any loss of generality, we assume that $q$ is univalent on $\overline{E}$ (closure of E). If not, then we can replace $p, q$, and $h$ by $p_r(z) = p(rz), q_r(z) = q(rz)$ and $h_r(z) = h(rz)$ respectively when $0 < r < 1$. These new functions satisfy the conditions of the theorem on $\overline{E}$. We would then prove that $p_r \prec q_r$, and by letting $r \to 1^-$, we obtain $p \prec q$.

We need to prove that $p \prec q$. If possible, suppose that $p \not\prec q$ in $E$. Then by Lemma 2.4, there exist points $z_0 \in E$ and $\zeta_0 \in \partial E$ such that $p(z_0) = q(\zeta_0)$ and $z_0p'(z_0) = m\zeta_0q'(\zeta_0), m \geq 1$. Then

$$(p(z_0))^{\alpha}\left[1 + \frac{\gamma z_0p'(z_0)}{p(z_0)}\right]^{\beta} = (q(\zeta_0))^{\alpha}\left[1 + \frac{\gamma m\zeta_0q'(\zeta_0)}{q(\zeta_0)}\right]^{\beta}. \tag{14}$$

Consider a function

$$L(z, t) = (q(z))^{\alpha}\left(1 + (1 + t)\frac{\gamma zq'(z)}{q(z)}\right)^{\beta} \tag{15}$$

$$= (q(z))^{\alpha}(1 + (1 + t)\gamma Q(z))^{\beta}$$

$$= 1 + a_1(t)z + \ldots$$

where $Q(z) = \frac{zq'(z)}{q(z)}$. Clearly $L(z,t)$ is analytic in $E$ for all $t \geq 0$ and is continuously differentiable on $[0,\infty)$ for all $z \in E$.

Now

$$a_1(t) = \left[\frac{\partial L(z,t)}{\partial z}\right]_{z=0} = q'(0)\left[\alpha + (1+t)\beta\gamma\right].$$

Thus, $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. A simple calculation yields

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = \operatorname{Re}\left[\frac{\alpha}{\beta\gamma}(1 + (1+t)\gamma Q(z)) + (1+t)\frac{zQ'(z)}{Q(z)}\right] \tag{16}$$

$$= \operatorname{Re}\left[\tfrac{\alpha}{\beta\gamma} + (1+t)\left(\tfrac{\alpha}{\beta}Q(z) + \tfrac{zQ'(z)}{Q(z)}\right)\right]$$

$$\geq \operatorname{Re}\left[\tfrac{\alpha}{\beta\gamma} + \tfrac{\alpha}{\beta}Q(z) + \tfrac{zQ'(z)}{Q(z)}\right] \qquad (t \geq 0)$$

$$= \operatorname{Re}\left[\tfrac{\alpha}{\beta}\left(\tfrac{1}{\gamma} + Q(z)\right) + \tfrac{zQ'(z)}{Q(z)}\right]$$

$$\geq 0,$$

in view of conditions (i) and (ii).Thus, $L(z,t)$ is a subordination chain and therefore, for $0 \leq t_1 \leq t_2$, we get

$$L(z,t_1) \prec L(z,t_2) \tag{17}$$

Since $L(z,0) = h(z)$, we deduce that the function $h$ is univalent in $E$ and hence, the subordination (13) is well-defined. Moreover,(17) implies that $L(\zeta_0,t) \notin h(E)$ for $|\zeta_0| = 1$ and $t \geq 0$. Now, in view of (14) and (15), we can write

$$(p(z_0))^\alpha \left[1 + \frac{\gamma z_0 p\prime(z_0)}{p(z_0)}\right]^\beta = L(\zeta_0, m-1), \tag{18}$$

where $z_0 \in E, |\zeta_0| = 1$ and $m \geq 1$. But $L(\zeta_0, m-1) \notin h(E)$ for $|\zeta_0| = 1$ and $m \geq 1$. This is a contradiction to (13). Hence $p \prec q$ in $E$. Since $p = q$ satisfies (13), the function $q$ is the best dominant of (13). This completes the proof of our theorem.

**Remark 3.1** From the proof of Theorem 3.1, we observe that the conditions (i) and (ii) can be replaced by a single condition that $(q(z))^{\alpha/\beta}$ is convex in $E$, since from (16), we have

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] \geq \operatorname{Re}\left[\frac{\alpha}{\beta\gamma} + \frac{\alpha}{\beta}Q(z) + \frac{zQ'(z)}{Q(z)}\right]$$

$$\geq \operatorname{Re}\left[\tfrac{\alpha}{\beta}Q(z) + \tfrac{zQ'(z)}{Q(z)}\right]$$

$$= \operatorname{Re}\left[\left(\tfrac{\alpha}{\beta} - 1\right)\tfrac{zq'(z)}{q(z)} + 1 + \tfrac{zq''(z)}{q'(z)}\right]$$

$$\geq 0,$$

provided $(q(z))^{\alpha/\beta}$ is convex in $E$.

**Remark 3.2.** Theorem 3.1 is also true for $\alpha = 0, \beta \in (0,1]$, though the best dominant $q$ in that case, is not an integral of the form (9). Therefore, we consider the case $\alpha = 0$ independently.

**Theorem 3.2.** Let $\beta \in (0,1]$ be real and let $\gamma$ be a complex number, with $\mathrm{Re}\gamma > 0$. Let $h(z), h(0) = 1$, be analytic in $E$. Let $q \in \mathcal{A}'$ be a univalent function for which $\log q(z)$ is convex in $E$. If an analytic function $p \in \mathcal{A}'$ satisfies the differential subordination

$$\left[1 + \frac{\gamma z p'(z)}{p(z)}\right]^{\beta} \prec \left[1 + \frac{\gamma z q'(z)}{q(z)}\right]^{\beta} = h(z), \ z \in E, \tag{19}$$

then

$$p(z) \prec q(z) = \exp \int_0^z \frac{h^{\frac{1}{\beta}}(t) - 1}{\gamma t} dt, \ z \in E. \tag{20}$$

*Proof.* Let us define a function

$$\begin{aligned} f(z,t) &= \left[1 + (1+t)\frac{\gamma z q'(z)}{q(z)}\right]^{\beta} \qquad (21)\\ &= [1 + \gamma(1+t)Q(z)]^{\beta} \\ &= 1 + a_1(t)z + \dots \end{aligned}$$

where $Q(z) = \frac{zq'(z)}{q(z)}$. The function $f(z,t)$ is analytic in $E$ for all $t \geq 0$ and is continuously differentiable on $[0,\infty)$for every $z \in E$. Moreover,

$$a_1(t) = \left[\frac{\partial f(z,t)}{\partial z}\right]_{z=0} = q'(0)\beta\gamma(1+t).$$

Thus $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. A simple calculation yields

$$\mathrm{Re}\left[\frac{z\partial f/\partial z}{\partial f/\partial t}\right] = \mathrm{Re}\left[(1+t)\frac{zQ'(z)}{Q(z)}\right] \geq 0$$

for $t \geq 0$ and for $\log q(z) \in K$. Thus, $f(z,t)$ is a subordination chain and, therefore, for $0 < t < s$, we have

$$f(z,t) \prec f(z,s) \tag{22}$$

in $E$. From (21) and (19), we have $f(z,0) = h(z)$. Thus $h$ is univalent in $E$ and hence, the subordination (19) is well-defined. We only need to show that $p(z) \prec q(z)$ in $E$.

First, we observe that (22) implies that $f(\zeta_0, t) \notin h(E)$ for $|\zeta_0| = 1$ and $t \geq 0$. We can assume that $q$ is univalent in $\overline{E}$. Now, suppose that $p \not\prec q$ in $E$. Then, by Lemma 2.4, there are points $z_0 \in E, \zeta_0 \in \partial E$ and an $m \geq 1$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$. Then

$$\left[1 + \frac{\gamma z_0 p'(z_0)}{p(z_0)}\right]^{\beta} = \left[1 + \frac{\gamma m \zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right]^{\beta}$$

$$= f(\zeta_0, m-1).$$

But $f(\zeta_0, m-1) \notin h(E)$ for $|\zeta_0| = 1$ and $m \geq 1$ which contradicts (19). Hence, $p(z) \prec q(z)$ in $E$. Since $p(z) = q(z)$ satisfies (19), it is clear that the function $q$, given by (20) is the best dominant . The proof of our theorem is now complete.

Now, we give some interesting applications of these theorems. Writing $\alpha = 1$ in Theorem 3.1, we obtain the following result (see also [3, Theorem 2.2]):

**Corollary 3.1.** Let $\beta \in (0,1]$ be real and let $\gamma$ be a complex number, where $\mathrm{Re}\gamma > 0$. Let $h \in \mathcal{A}'$. Suppose that the differential equation

$$q(z)\left[1 + \frac{\gamma z q'(z)}{q(z)}\right]^{\beta} = h(z),\ z \in E,$$

has an analytic solution $q$, $q(0) = 1, q(z) \neq 0$ in $E$, which satisfies the following conditions:

(i) $q$ is univalent in $E$ and

(ii)$(q(z))^{1/\beta}$ is convex in $E$.

If a function $p \in \mathcal{A}'$ satisfies the differential subordination

$$p(z)\left[1 + \frac{\gamma z p'(z)}{p(z)}\right]^{\beta} \prec h(z),\ z \in E,$$

then

$$p(z) \prec q(z) = \left[\frac{1}{\beta\gamma z^{\frac{1}{\beta\gamma}}} \int_0^z h^{\frac{1}{\beta}}(t) t^{\frac{1}{\beta\gamma}-1} dt\right]^{\beta} \tag{23}$$

in $E$.

Taking $\beta = 1$ in Theorem 3.1, we get the following result:

**Corollary 3.2.** Let $\alpha > 0$ be a real number and let $\gamma \in C$ where Re $\gamma > 0$. Let $h \in \mathcal{A}'$. Suppose that $q \in \mathcal{A}'$ is a univalent function for which $(q(z))^{\alpha}$ is convex in $E$. If an analytic function $p \in \mathcal{A}'$ satisfies the differential subordination

$$(p(z))^{\alpha} + \gamma(p(z))^{\alpha-1} z p'(z) \prec (q(z))^{\alpha} + \gamma(q(z))^{\alpha-1} z q'(z) = h(z),$$

then,

$$p(z) \prec q(z) = \left[\frac{\alpha}{\gamma z^{\frac{\alpha}{\gamma}}} \int_0^z h(t) t^{\frac{\alpha}{\gamma}-1} dt\right]^{\frac{1}{\alpha}}.$$

Let us take $\alpha = 1/2, \gamma = 1,\ q(z) = (1+az)^2, a \in (0,1]$, in above corollary. Then it is easy to check that the function $q$ is univalent in $E$ and $(q(z))^{1/2}$ is convex in $E$. Thus, we get

**Example 3.1.** If $p \in \mathcal{A}'$ satisfies

$$\sqrt{p(z)} + \frac{zp'(z)}{\sqrt{p(z)}} \prec 1 + 3az, \ a \in (0,1], \ z \in E,$$

then

$$p(z) \prec (1 + az)^2, \ z \in E.$$

Taking $\alpha = \gamma = 2$ in Corollary 3.2, we get the following result (see also [6, page 77]):

**Corollary 3.3.** Let $h \in \mathcal{A}'$. Let $q \in \mathcal{A}'$ be univalent in $E$. If $(q(z))^2$ is convex in $E$, then for $p \in \mathcal{A}'$

$$p^2(z) + 2p(z)zp'(z) \prec q^2(z) + 2q(z)zq'(z) = h(z), z \in E,$$

implies that

$$p(z) \prec q(z) = \left[\frac{1}{z}\int_0^z h(t)dt\right]^{1/2}$$

in $E$ and $q$ is the best dominant.

As mentioned in the introduction, D. J. Hallenbeck and St. Ruscheweyh [2] obtained the best dominant for the differential subordination (4) assuming that the superordinate function $h(z)$ in (4) is convex in $E$. Miller Mocanu and Reade obtained the same conclusion from the differential subordination (4) under much weaker conditions on $h(z)$. In fact they proved [8] that

$$p(z) + zp'(z) \prec q(z) + zq'(z) = h(z), z \in E,$$

implies $p(z) \prec q(z)$ provided Re $\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}$ in $E$.

Setting $\alpha = \beta = 1$ in Theorem 3.1, we get the following result which shows that the above-said result holds under much weaker hypothesis.

**Corollary 3.4.** Let $\gamma$, Re$\gamma > 0$, be a real number. Let $h, h(0) = 1$, be analytic in $E$. Suppose that $q, q(0) = 1$, is a convex function. If a function $p$ in $\mathcal{A}'$ satisfies the differential subordination

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z) = h(z), z \in E,$$

then,

$$p(z) \prec q(z) = \frac{1}{\gamma z^{\frac{1}{\gamma}}}\int_0^z h(t)t^{\frac{1}{\gamma}-1}dt$$

in $E$.

As an example, consider $q(z) = e^z$, which is a convex function in $E$. Then Corollary 3.4 becomes:

**Example 3.2.** Let $\gamma, \mathrm{Re}\gamma > 0$ be a complex number. If an analytic function $p \in \mathcal{A}'$ satisfies

$$p(z) + \gamma z p'(z) \prec e^z(1 + \gamma z), \ z \in E,$$

then

$$p(z) \prec e^z \text{ in } E.$$

We observe that this result cannot be handled by the differential subordination results proved in [2] and for $\mathrm{Re}\gamma > \frac{1}{2}$, it can be proved even by the result in [8].

Setting $\gamma = 1$ and $h(z) = \frac{1}{(1+bz)^{\alpha+\beta}}$ where $b \in C, |b| \le \frac{\beta}{\alpha} \le 1$, in Theorem 3.1, we get $q(z) = \frac{1}{1+bz}$. Clearly, $q(z)$ satisfies the conditions of Theorem 3.1. Thus, we obtain:

**Example 3.3.** Let $\alpha > 0$ and $\beta \in (0, 1]$ be real numbers and let $b \in C$ be such that $|b| \le \frac{\beta}{\alpha} \le 1$. If $p \in \mathcal{A}'$ satisfies the differential subordination

$$(p(z))^\alpha \left[1 + \frac{zp'(z)}{p(z)}\right]^\beta \prec \frac{1}{(1 + bz)^{\alpha+\beta}}, z \in E$$

then

$$p(z) \prec \frac{1}{1 + bz}$$

in $E$.

Setting $\beta = 1$ in Theorem 3.2, we obtain the following result of Miller and Mocanu [9]:

**Corollary 3.5.** Let $h$ be starlike in $E$, with $h(0) = 0$. If $p \in \mathcal{A}'$ and $q \in \mathcal{A}'$ satisfy

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)} = h(z), \ z \in E,$$

then,

$$p(z) \prec q(z) = \exp \int_0^z \frac{h(t)}{t} dt.$$

## 4 Applications to Univalent Functions

In 1932/33, Marx [5] and Strohhäcker [16] proved the following beautiful result:

**Theorem 4.1.** For a function $f \in \mathcal{A}$ and $z \in E$,

(i) $\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0 \Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z)} > 1/2.$

(ii) $\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0 \Rightarrow \mathrm{Re}\sqrt{f'(z)} > 1/2.$

(iii) $\mathrm{Re}\frac{zf'(z)}{f(z)} > 1/2 \Rightarrow \mathrm{Re}\frac{f(z)}{z} > 1/2.$

(iv) $\mathrm{Re}\sqrt{f'(z)} > 1/2 \Rightarrow \mathrm{Re}\frac{f(z)}{z} > 1/2$.

In this section, we obtain several interesting applications of Theorem 3.1 and Theorem 3.2 to univalent functions. We find that the results (ii), (iii) and (iv) in Theorem 4.1 follow from our theorems by giving different values to the function $p(z)$ and parameters $\alpha$, $\beta$ and $\gamma$.

**Theorem 4.2.** Let $\alpha > 0$ and $\beta \in (0,1]$. Let $\gamma$ be a complex number with $\mathrm{Re}\gamma > 0$. Let $h \in \mathcal{A}'$. Suppose that the differential equation

$$(g'(z))^{\alpha}\left[1+\frac{\gamma z g''(z)}{g'(z)}\right]^{\beta} = h(z),\ g(0)=0,$$

has a solution $g, g'(z) \neq 0$ in $E$, which satisfies the following conditions:

(i) $g'(z)$ is univalent in $E$ and

(ii) $(g'(z))^{\frac{\alpha}{\beta}}$ is convex in $E$.

If $f \in \mathcal{A}$, $f'(z) \neq 0$ in $E$, satisfies the differential subordination

$$(f'(z))^{\alpha}\left[1+\frac{\gamma z f''(z)}{f'(z)}\right]^{\beta} \prec h(z),\ z \in E, \tag{24}$$

then

$$f'(z) \prec g'(z) = \left[\frac{\alpha}{\beta\gamma z^{\frac{\alpha}{\beta\gamma}}}\int_0^z h^{\frac{1}{\beta}}(t)t^{\frac{\alpha}{\beta\gamma}-1}dt\right]^{\frac{\beta}{\alpha}}$$

in $E$.

*Proof.* Proof follows by writing $p(z) = f'(z)$ and $q(z) = g'(z)$ in Theorem 3.1.

Writing $\alpha = \beta = 1$ in Theorem 4.2, we get the following result:

**Corollary 4.1.** Let $\gamma$ be a complex number with $\mathrm{Re}\gamma > 0$. Let $h, h(0) = 1$, be an analytic function. Let $g \in \mathcal{A}$ be such that $g'(z)$ is a convex univalent function. If $f \in \mathcal{A}$ satisfies the differential subordination

$$f'(z) + \gamma z f''(z) \prec g'(z) + \gamma z g''(z) = h(z),\ z \in E,$$

then

$$f'(z) \prec g'(z) = \frac{1}{\gamma z^{1/\gamma}}\int_0^z h(t)t^{\frac{1}{\gamma}-1}dt$$

in $E$.

If we let $g'(z) = 1+az, a \in (0,1]$, which is convex univalent in $E$, then we obtain the following result (also see S. Ponnusamy and V. Singh [14]):

**Example 4.1.** For $\gamma \in C$, $\mathrm{Re}\gamma > 0$, if $f \in \mathcal{A}$ satisfies

$$f'(z) + \gamma z f''(z) \prec 1 + (\gamma+1)az,\ z \in E,\ a \in (0,1],$$

then

$$f'(z) \prec 1 + az \text{ in } E. \tag{25}$$

In view of Lemma 2.5, (25) implies that $f \in S(\alpha)$, where $\alpha$ is given by the inequality

$$0 < a \leq \frac{2sin(\pi\alpha/2)}{\sqrt{5 + 4cos(\pi\alpha/2)}}.$$

Writing $\alpha = \beta = 1/2$ and $\gamma = 1$ in Theorem 4.2, we get

**Corollary 4.2.** Let $g \in \mathcal{A}$ be such that $g'(z)$ is convex univalent in $E$. If $f \in \mathcal{A}$ satisfies

$$\sqrt{f'(z) + zf''(z)} \prec \sqrt{g'(z) + zg''(z)},\ z \in E,$$

then

$$f'(z) \prec g'(z),\ z \in E.$$

Writing $p(z) = \frac{f(z)}{z}$ and $q(z) = \frac{g(z)}{z}$ and $\gamma = 1$ in Theorem 3.1, we obtain:

**Theorem 4.3.**Let $\alpha > 0, \beta \in (0, 1]$. Let $h \in \mathcal{A}'$. Suppose that $g \in \mathcal{A}$ satisfies the following conditions:

(i) $g(z)/z$ is univalent in $E$ and

(ii) $(\frac{g(z)}{z})^{\frac{\alpha}{\beta}}$ is convex in $E$.

If $f \in \mathcal{A}$ satisfies the differential subordination

$$\left(\frac{f(z)}{z}\right)^{\alpha-\beta} (f'(z))^{\beta} \prec \left(\frac{g(z)}{z}\right)^{\alpha-\beta} (g'(z))^{\beta} = h(z),\ z \in E,$$

then

$$\frac{f(z)}{z} \prec \frac{g(z)}{z} = \left[\frac{\alpha}{\beta z^{\alpha/\beta}} \int_0^z h^{1/\beta}(t) t^{\frac{\alpha}{\beta}-1} dt\right]^{\beta/\alpha}$$

in $E$.

Taking $\beta = 1, 0 < \alpha \leq 1$ and $\frac{g(z)}{z} = \frac{1}{1-z}$ in Theorem 4.3, we get:

**Example 4.2.** For $0 < \alpha \leq 1$, if $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ in $E$, satisfies

$$\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z) \prec \frac{1}{(1-z)^{\alpha+1}},\ z \in E,$$

then

$$\frac{f(z)}{z} \prec \frac{1}{1-z} \text{ in } E.$$

For $\alpha = 1$, it reduces to case (iv) of Theorem 4.1:

**Example 4.3.** Let $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ in $E$ satisfy

$$f'(z) \prec \frac{1}{(1-z)^2}, \; z \in E.$$

Then

$$\frac{f(z)}{z} \prec \frac{1}{1-z} \text{ in } E.$$

Setting $\beta = \gamma = 1, p(z) = \frac{f(z)}{z}$ and $q(z) = \frac{g(z)}{z}$ in Theorem 3.2, we obtain the following result(also see [9]):

**Corollary 4.3.** Let $g \in \mathcal{A}$ be such that $g(z)/z$ is univalent in $E$ and $\log(g(z)/z)$ is convex in $E$. If an analytic function $f$, $\frac{f(z)}{z} \neq 0$ in $E$, satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}, \; z \in E,$$

then

$$\frac{f(z)}{z} \prec \frac{g(z)}{z}$$

in $E$.

Consider $g(z) = \frac{z}{1-z}$.Then $\frac{g(z)}{z}$ is univalent in $E$ and log $\frac{g(z)}{z} = -\log(1-z)$ is convex in $E$. Thus, we obtain the result (iii) in Theorem 4.1:

**Example 4.4.** For a function $f \in \mathcal{A}$

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1-z} \Rightarrow \frac{f(z)}{z} \prec \frac{1}{1-z} \text{ in } E.$$

Writing $\beta = \gamma = 1, p(z) = f'(z), q(z) = g'(z)$ in Theorem 3.2, we obtain

**Corollary 4.4.** Let $h, h(0) = 1$, be an analytic function. Let $g, g(0) = 0, g'(z) \neq 0$ in $E$, be an analytic function such that $g'(z)$ is univalent in $E$ and log $g'(z)$ is convex in $E$. If $f \in \mathcal{A}, f'(z) \neq 0$ in $E$, satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{zg''(z)}{g'(z)} = h(z), \; z \in E,$$

then

$$f'(z) \prec g'(z) = \exp \int_0^z \frac{h(t)-1}{t} dt, \; z \in E.$$

In particular, consider the function $h(z) = \frac{1+z}{1-z}$. Then $g'(z) = \frac{1}{(1-z)^2}$. Obviously, $g'(z)$ is univalent in $E$ and log $g'(z) = -2\log(1-z)$ is convex in $E$. Thus, we get case (ii) of Theorem 4.1:

**Example 4.5.** Let $f \in \mathcal{A}$, $f'(z) \neq 0$ in $E$, satisfy

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, \; z \in E.$$

Then,

$$f'(z) \prec \frac{1}{(1-z)^2}$$

in $E$.

Taking $\gamma = 1, p(z) = \frac{zf'(z)}{f(z)}$ and $q(z) = \frac{zg'(z)}{g(z)}$, where $f$ and $g$ are members of class $\mathcal{A}$, in Theorem 3.2, we get

**Theorem 4.4.** Let $\beta \in (0, 1]$ be a real number. Let $h \in \mathcal{A}'$ be analytic in $E$. Set $\frac{zg'(z)}{g(z)} = G(z)$. Assume that

(i)$G(z)$ is univalent in $E$ and

(ii)$\log\, G(z)$ is convex in $E$.

If an analytic function $f, f(0) = 0, \frac{f(z)f'(z)}{z} \neq 0$ in $E$, satisfies the differential subordination

$$\left[2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right]^{\beta} \prec \left[2 + \frac{zg''(z)}{g'(z)} - \frac{zg'(z)}{g(z)}\right]^{\beta} = h(z),\ z \in E,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)} = \exp \int_0^z \frac{h^{1/\beta} - 1}{t} dt$$

in $E$.

Taking $h(z) = \frac{1+(1-2\alpha)z}{1-z}, 1/2 \leq \alpha < 1$ and $\beta = 1$ in Theorem 4.3, we get $G(z) = \frac{zg'(z)}{g(z)} = \frac{1}{(1-z)^{2(1-\alpha)}}$. Clearly, $G(z)$ is univalent in $E$ and $\log\, G(z)$ is convex in $E$. Thus, we get the following result of S. Ponnusamy and V. Singh [12]:

**Corollary 4.7.** Let $f \in \mathcal{A}$ satisfy

$$2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{1 + (1-2\alpha)z}{1-z},\ 1/2 \leq \alpha < 1,\ z \in E.$$

Then,

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{(1-z)^{2(1-\alpha)}},\ z \in E,$$

i. e. $f$ is starlike in $E$.

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