Lie Algebras and Cotriangular Spaces

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Abstract

Let $\Pi = (P, L)$ be a partial linear space in which any line contains three points and let K be a field. Then by $\mathcal{L}_K(\Pi)$ we denote the free K-algebra generated by the elements of P and subject to the relations $xy = 0$ if x and y are noncollinear elements from P and $xy = z$ for any triple $\{x, y, z\} \in L$.

We prove that the algebra $\mathcal{L}_K(\Pi)$ is a Lie algebra if and only if K is of even characteristic and Π is a cotriangular space satisfying Pasch's axiom.

Moreover, if Π is a cotriangular space satisfying Pasch's axiom, then a connection between derivations of the Lie algebra $\mathcal{L}_K(\Pi)$ and geometric hyperplanes of Π is used to determine the structure of the algebra of derivations of $\mathcal{L}_K(\Pi)$.

1 Introduction

A *cotriangular space* is a partial linear space $\Pi = (P, L)$ with point set P and set of lines L such that each line contains 3 points and, given a point not on a line, the point is collinear with zero or two points of the line. We say that a cotriangular space satisfies *Pasch's axiom*, if any pair of intersecting lines generates a subspace isomorphic to the dual affine plane of order 2 (i.e., a Fano plane from which a point and the three lines through that point are removed).

Examples of cotriangular spaces can be found as follows. Let V be a vector space over the field $\mathbb{F}_2$ equipped with a symplectic form f. Let P be a subset of $V \setminus \{0\}$ with the property that for any two elements $v, w \in P$ with $f(v, w) = 1$, the element $v + w$ is also in P. If we denote by L the set of triples $\{u, v, w\}$ from P

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with $u+v+w=0$ and $f(v,w)=f(v,u)=f(u,w)=1$, then it is easily checked that (P,L) is a cotriangular space satisfying Pasch's axiom.

While Shult [8] and Hall [2] have studied abstract cotriangular spaces, Seidel [7] and Kaplansky [4] considered subsets P of a symplectic space (V,f) as described above in relation with Lie algebras. (A *Lie algebra* L is a vector space together with a product satisfying $xy=-yx$ and the so called *Jacobi identity* $(xy)z+(yz)x+(zx)y=0$ for all $x,y,z\in L$.) Seidel was motivated by a question of Hamelink [3] on Lie algebras and Kaplansky [4] used these subsets P to construct some classes of Lie algebras. Given such a set P, Kaplansky constructs a Lie algebra on the vector space over a field of even characteristic with basis P. The Lie product on the basis elements $v,w\in P$ equals

$$vw=f(v,w)\cdot(v+w).$$

More recently Rotman and Weichsel, [5, 6], also considered the connection between Kaplansky's Lie algebras, special subsets of vectors of a symplectic space and cotriangular spaces. Some questions and results arising from their work form the motivation for the work that led to this paper.

Let $\Pi=(P,L)$ be an arbitrary partial linear space of order 2 (i.e., each line consists of three points) and suppose K is a field.

On the vector space KP over the field K with basis P we define the algebra $\mathcal{L}=\mathcal{L}_K(\Pi)$ by the linear extension of the following product defined on the basis elements. If x and y are two collinear points in Π, then the product xy of x and y equals the third point of the unique line in L through x and y. If x and y are not collinear or $x=y$, then xy is defined to be $0\in\mathcal{L}$.

Our first result describes a connection between cotriangular spaces and Lie algebras.

Theorem 1.1. *Let $\Pi=(P,L)$ be an partial linear space of order 2 and K a field. The algebra $\mathcal{L}_K(\Pi)$ is a Lie algebra if and only if the field K has even characteristic and Π is a cotriangular space satisfying Pasch's axiom.*

So, if Π is a cotriangular space satisfying Pasch's axiom, then the product in $\mathcal{L}_K(\Pi)$ has the following properties for x,y and z in P:

(i) $x^2=0$;

(ii) $xy=0$ or $xy\in P$ and $(xy)x=y$;

(iii) $(xy)z+(yz)x+(zx)y=0$, the Jacobi identity.

We will characterize such a Lie algebra $\mathcal{L}$ by the existence of a class of generating elements satisfying the above.

Theorem 1.2. *Let $\mathcal{L}$ be a Lie algebra over a field K of even characteristic containing some class X of elements such that:*

(i) $\mathcal{L}$ is generated by X;

(ii) for all x and y in X we have either $xy=0$ or $xy\in X$ and $(xy)y=x$.

Then there is a cotriangular space Π *satisfying Pasch's axiom such that* $\mathcal{L}$ *is isomorphic to a quotient of* $\mathcal{L}_K(\Pi)$.

Moreover, if $\mathcal{L}$ *is simple, then it is isomorphic to* $\mathcal{L}_K(\Pi)$.

A *derivation* δ of a Lie algebra L is a linear map from L to itself satisfying $\delta(xy) = x\delta(y) + y\delta(x)$ for all $x, y \in L$. For all derivations δ_1 and δ_2 of L we define the product $\delta_1\delta_2$ to be the composition of δ_1 with δ_2. By $[\delta_1, \delta_2]$ we denote $\delta_1\delta_2+\delta_2\delta_1$, which is again a derivation of L. The product $[\cdot,\cdot]$ defines a Lie algebra structure on the space space $Der(L)$ of all derivations of L. For each $x \in L$ the map $ad\, x : L \to L$, defined by $ad\ x(y) = xy$ is a derivation of L. It is called an *inner* derivation of L. The inner derivations form a subalgebra of the Lie algebra L denoted by $IDer(L)$. Derivations not in $IDer(L)$ are called *outer* derivations.

In the proof of the above result, we use some properties of special derivations of the Lie algebra $\mathcal{L}$. In particular, we use that the map $(ad\ x)^2$, $x \in X$, is an outer derivation of $\mathcal{L}$, as was already noticed by Rotman and Weichsel, [6]. In [6] Rotman and Weichsel conjecture that for finite irreducible Π (see Section 2 for notation) the Lie algebra $Der(\mathcal{L})$ of derivations of $\mathcal{L}$ equals the direct sum of $IDer(\mathcal{L})$, the algebra of inner derivations, and the subalgebra generated by the outer derivations $(ad\ x)^2$, with $x \in X$.

This conjecture is not completely correct. In Section 4 of this paper, we show that each geometric hyperplane H of Π, i.e., a subset H of P meeting every line in one or all points, gives rise to a derivation δ_H of $\mathcal{L}$. Here $\delta_H(x) = x$ for all $x \in X \setminus H$ and $\delta(x) = 0$ for $x \in H$. The derivation $(ad\ x)^2$ is a derivation of the form δ_H, where H is the hyperplane $x^\perp$ consisting of x and all points of Π equal to or not collinear to x.

If H_1 and H_2 are two hyperplanes of Π, then their symmetric difference $H_1 + H_2$ is also a geometric hyperplane. In this way we can put an $\mathbb{F}_2$-vector space structure on the set of all geometric hyperplanes of Π. We denote this $\mathbb{F}_2$-vector space of all geometric hyperplanes of Π by $\mathcal{U}(\Pi)^*$. As $\delta_{H_1+H_2} = \delta_{H_1} + \delta_{H_2}$ and $[\delta_{H_1}, \delta_{H_2}] = 0$, we see that $Der(\mathcal{L}_K(\Pi))$ contains a commutative Lie subalgebra isomorphic to $K \otimes_{\mathbb{F}_2} \mathcal{U}(\Pi)^*$. With the notation introduced above, we can give the following description of the algebra $Der(\mathcal{L})$ of derivations.

Theorem 1.3. *Suppose* Π *is a finite irreducible cotriangular space. Then the algebra* $Der(\mathcal{L}_K(\Pi))$ *of derivations of* $\mathcal{L}_K(\Pi)$ *is isomorphic to*

$$(K \otimes_{\mathbb{F}_2} \mathcal{U}(\Pi)^*) \oplus IDer(\mathcal{L}_K(\Pi)).$$

The spaces $\mathcal{U}(\Pi)^*$ have been determined by Hall [2], see Section 2. As it turns out, the space $\mathcal{U}(\Pi)^*$ is not always generated by the set of hyperplanes of the form $x^\perp$, $x \in P$, which implies that the subalgebra of derivations generated by the outer derivations $(ad\ x)^2$, $x \in P$, is not always a complement to $IDer(\mathcal{L})$ in $Der(\mathcal{L})$, thereby disproving Rotman and Weichsel's conjecture.

The organization of the remainder of this paper is as follows. In Section 2 we discuss some results on cotriangular spaces due to Hall [2]. Section 3 is concerned with the proof of both Theorem 1.1 and Theorem 1.2. Then in the final section we determine the structure of the algebra of derivations of the Lie algebra $\mathcal{L}_K(\Pi)$ where K is a field of even characteristic and Π a finite irreducible cotriangular space. In particular, we prove Theorem 1.3.

2 Cotriangular Spaces

In this section we briefly discuss some results on cotriangular spaces. But first we give some definitions.

A *partial linear space* is a pair (P, L) consisting of a set P of *points* and a set L of *lines*, where a line is a subset of P of cardinality at least 2, such that any two points are contained in at most one line. A *subspace* of a partial linear space (P, L) is a subset X of the point set P closed under lines, i.e., if a line meets X in at least two points, then it is contained in X. A subspace X is often identified with the partial linear space induced on it by all lines meeting it in at least two points.

As the intersection of a collection of subspaces is again a subspace, we can define the subspace *generated* by some subset Y of P to be the smallest subspace containing Y.

A *cotriangular space* $\Pi = (P, L)$ is a partial linear space in which every line contains exactly 3 points and in which a point not on a line is collinear with zero or two points of that line. If in a cotriangular space the union of any two intersecting lines generates a subspace isomorphic the dual of an affine plane of order 2, then we say that the space satisfies *Pasch's axiom*.

If Π is a cotriangular space, then for each point $x \in P$ we denote by $x^\perp$ the set of points $y \in P$ that are equal to x or not collinear to x. By $x^{\not\perp}$ we denote the complement of $X^\perp$ in X. So, $x^{\not\perp}$ denotes the set of points $y \neq x$ that are collinear to x. The space Π is called *connected*, if its collinearity graph is connected and it is called *reduced*, if $x^\perp = y^\perp$ implies $x = y$ for all points $x, y \in P$. We call Π *irreducible* if it is connected and reduced.

We describe some examples of cotriangular spaces.

Suppose Ω is a set and P the set of subsets of Ω of size 2. By L we denote the set of all triples $\{p_1, p_2, p_3\}$ of P with $p_1 \cup p_2 \cup p_3$ of size three. The space $\mathcal{T}_\Omega = (P, L)$ is a cotriangular space. For $|\Omega| > 4$, this cotriangular space is irreducible.

If (V, f) is a symplectic space over $\mathbb{F}_2$, then $\mathcal{S}p(V, f)$ denotes the partial linear space (P, L) where P consists of all the vectors of V not in the radical of the form f. A line in L is the set of three nonzero vectors in a 2-dimensional subspace W of V on which f does not vanish. The space $\mathcal{S}p(V, f)$ is an irreducible cotriangular space if and only if the form f is nondegenerate.

Finally we consider the spaces $\mathcal{N}(V, Q)$, where (V, Q) is an orthogonal $\mathbb{F}_2$-space. Let f denote the symplectic form associated to Q with radical $Rad(f)$. Here the points are the vectors $v \in V \setminus Rad(f)$ with $Q(v) = 1$. A typical line is the set of three nonzero vectors in an elliptic 2-space, i.e., a 2-space in which $Q(v) = 1$ for any nonzero vector v contained in it. Clearly, $\mathcal{N}(V, Q)$ is contained in $\mathcal{S}p(V, f)$. The space $\mathcal{N}(V, Q)$ is irreducible if it contains lines and the form f is nondegenerate.

Now we have the following theorem.

Theorem 2.1. (J.I. Hall [2]) *Let Π be an irreducible cotriangular space. Then Π satisfies Pasch's axiom and is isomorphic to one of the following spaces:*

1. *the space $\mathcal{T}_\Omega$ for some set Ω of size at least 5;*

2. *the space $\mathcal{S}p(V, f)$ for some nondegenerate symplectic geometry (V, f) over $\mathbb{F}_2$;*

3. *the space $\mathcal{N}(V,Q)$ for some nondegenerate orthogonal geometry (V,Q) over $\mathbb{F}_2$.*

A *(geometric) hyperplane* H of a partial linear space $\Pi = (P,L)$ is a subset H of the point set P with the property that each line of L meets H in one point or is contained in H.

A proper *embedding* of a partial linear space $\Pi = (P,L)$ is a faithful map ϕ from P into the point set of a projective space $\mathbb{P}$, such that lines of Π are mapped onto lines of $\mathbb{P}$, and the image $\phi(P)$ of P generates $\mathbb{P}$.

If Π is embedded into a projective space via ϕ, then the pre-image of the intersection of $\phi(P)$ with a hyperplane of $\mathbb{P}$ yields a hyperplane of Π. Below we describe embeddings of the various irreducible cotriangular spaces and recall a result of Hall stating that all hyperplanes can be obtained as hyperplane sections from that embedding.

If $\Pi = \mathcal{T}_\Omega$, where $|\Omega| \geq 5$, then let $\mathcal{U}(\Pi)$ be the subspace $E\mathbb{F}_2\Omega$ of the $\mathbb{F}_2$ vector space $\mathbb{F}_2\Omega$ with basis Ω generated by the vectors $\omega_1 + \omega_2$, where $\omega_1, \omega_2 \in \Omega$. The map $\phi_\Pi : P \to \mathbb{P}(E\mathbb{F}_2\Omega)$ given by $\phi_\Pi(\{\omega_1,\omega_2\}) = \langle \omega_1 + \omega_2 \rangle$, where $\{\omega_1,\omega_2\}$ is a point of Π, is a proper embedding of Π. (Notice that the standard dot product on $\mathbb{F}_2\Omega$ induces a symplectic form on $E\mathbb{F}_2\Omega$.)

If $\Pi = (P,L)$ equals $\mathcal{S}p(V,f)$ for some nondegenerate symplectic space (V,f), then denote by $\mathcal{U}(\Pi)$ the space $\hat{V} = V \oplus \langle v_\infty \rangle$, where v_∞ is a vector not in V. Let $Q : V \to \mathbb{F}_2$ be a quadratic form on V with associated bilinear form equal to f. Then we can extend Q to a quadratic form $\hat{Q}$ on $\hat{V}$ by

$$\hat{Q}(v + \lambda v_\infty) = Q(v) + \lambda,$$

where $v \in V$ and $\lambda \in \mathbb{F}_2$.

The map $\phi_\Pi : P \to \mathbb{P}(\mathcal{U}(\Pi))$ defined by

$$\phi_\Pi(v) = \langle v + (1 - q(v))v_\infty \rangle,$$

where $v \in V \setminus \{0\}$, yields an embedding of Π into $\mathbb{P}(\mathcal{U}(\Pi))$.

For the space $\mathcal{N}(V,Q)$, where (V,Q) is a nondegenerate orthogonal $\mathbb{F}_2$-space with $\dim(V) \geq 6$, we define ϕ_Π to be the natural embedding into $\mathcal{U}(\Pi) = V$, except when (V,Q) is 6-dimensional of hyperbolic type. In that case $\mathcal{N}(V,Q)$ is isomorphic to $\mathcal{T}_{\{1,\dots,8\}}$ and $\mathcal{U}(\Pi)$ is 7-dimensional, as defined above.

With this notation, the following result of Hall classifies all geometric hyperplanes of reduced cotriangular space.

Theorem 2.2. (J.I. Hall, [2]) *Let $\Pi = (P,L)$ be an irreducible cotriangular space. Then every geometric hyperplane of Π can be obtained as the pre-image of the intersection of $\phi_\Pi(P)$ with a hyperplane of $\mathbb{P}(\mathcal{U}(\Pi))$.*

Since in a cotriangular space lines contain three points, the symmetric difference of any two hyperplanes is again a geometric hyperplane. So, the operation 'taking the symmetric difference' defines an $\mathbb{F}_2$-vector space structure on the set of all geometric hyperplanes of (P,L). If Π is an irreducible cotriangular space, then this space is isomorphic to the dual $\mathcal{U}(\Pi)^*$ of $\mathcal{U}(\Pi)$.

Let $\Pi = (P,L)$ be a finite irreducible cotriangular space. If a subset Y of P generates Π, its image $\phi_\Pi(Y)$ generates $\mathbb{P}(\mathcal{U}(\Pi))$. So Y contains at least as many points as the (vector space) dimension of $\mathcal{U}(\Pi)$. The following result is also due to Hall.

Proposition 2.3. (J.I. Hall, [1]) *Let $\Pi = (P, L)$ be a finite irreducible cotriangular space. Then Π can be generated by $\dim(\mathcal{U}(\Pi))$ points.*

3 Lie Algebras

The purpose of this section is to prove the Theorems 1.1 and 1.2.

Suppose K is a field. Let $\mathcal{L}$ be a commutative K-algebra generated by a set X of elements such that for all $x, y \in X$

(i) $x^2 = 0$;

(ii) $xy = 0$ or $xy \in X$ and $(xy)x = y$.

Let $L(X)$ be the set of triples $\{x, y, xy\}$ from X inside the subalgebras generated by elements x and y, where x and y are elements from X with $xy \neq 0$. The elements in X will be called *points*, those in $L(X)$ *lines*. As the product of any two of the three elements of a line is the third point of the line, $\Pi = (X, L(X))$ is a partial linear space with 3 points per line. Thus $\mathcal{L}$ is a quotient of the free commutative K-algebra $\mathcal{L}_K(\Pi)$ generated by the elements in X subject to the relations $x^2 = 0$ for all $x \in X$, $xy = 0$ for noncollinear $x, y \in X$, and $xy = z$ for all triples $\{x, y, z\} \in L$.

Now Theorem 1.1 follows from the following proposition when taking $\mathcal{L}$ to be equal to $\mathcal{L}_K(\Pi)$.

Proposition 3.1. *$\mathcal{L}$ is a Lie algebra if and only if Π is a cotriangular space satisfying Pasch's axiom and the characteristic of K is even.*

Proof. Suppose K has even characteristic. Let Π be a cotriangular space satisfying Pasch's axiom. We show that $\mathcal{L}$ is a Lie algebra. Let us check the Jacobi identity $(xy)z + (yz)x + (zx)y = 0$ for x, y and z in X.

Since we are working in characteristic 2, the identity is trivial if x, y and z are not all distinct. Thus assume that x, y and z are three distinct points in X. If there is a point, without loss of generality we can assume it to be z, not collinear to the other two, then $xz = yz = 0$, and either $xy = 0$ or z is not collinear to xy and $(xy)z = 0$. Hence $(xy)z + (yz)x + (zx)y = 0 + 0 + 0 = 0$. Thus we can assume that there is a point, again we may assume it to be z, collinear with the other two, x and y. If x, y and z are collinear, then $xy = z$, $yz = x$ and $zx = y$, so that $(xy)z + (yz)x + (zx)y = zz + xx + yy = 0$. Finally assume that x, y and z are not collinear. So they generate a dual affine plane π. If $xy = 0$, we find that y is collinear to xz and $(xz)y = (yz)x$ is the unique point of π not collinear to z. Hence $(xy)z + (yz)x + (zx)y = 0$. If $xy \neq 0$, then xz and yz are not collinear to y, respectively, x, and xy is the unique point of π not collinear to z. Hence $(xy)z = (xz)y = (yz)x = 0$, and $(xy)z + (yz)x + (zx)y = 0$.

Now suppose $\mathcal{L}$ is a Lie algebra. Then, as the product on $\mathcal{L}$ is assumed to be commutative, K has even characteristic. Let x, y and z be three noncollinear points in X. Suppose that $xy \neq 0$. If both xz and yz are 0, then also $(xy)z = 0$, as follows from the Jacobi identity.

Thus, assume that z is collinear with two points of the line through x and y, say the points x and y.

Then we have the identity:

$$\begin{aligned} 0 &= ((xy)z + (yz)x + (zx)y)z \\ &= ((xy)z)z + ((yz)x)z + ((zx)y)z \\ &= ((xy)z)z + (xz)(yz) + (z(yz))x + (yz)(zx) + (z(zx))y \\ &= ((xy)z)z. \end{aligned}$$

Hence, as $xy \neq 0$, we have $(xy)z = 0$. Similarly, $(yz)x$ and $(zx)y = 0$. By the Jacobi identity this implies that $(xy)(xz) = yz$, and the 6 points x, y, xy, z, xz, yz and $(yz)x$ are the 6 points of a subspace isomorphic to a dual affine plane in Π. Hence Π is a cotriangular space satisfying Pasch's axiom. ∎

The remainder of this section is devoted to the proof of Theorem 1.2. Let $\mathcal{L}$ be a Lie algebra over K, a field of characteristic 2, not necessarily simple, generated by a set of elements X satisfying the hypothesis of Theorem 1.2.

By Proposition 3.1, $\Pi = (X, L(X))$ is a cotriangular space satisfying Pasch's axiom.

As the identity map on X induces a morphism from $\mathcal{L}_K(\Pi)$ onto $\mathcal{L}$, we have proved the first part of Theorem 1.2.

Proposition 3.2. *The Lie algebra $\mathcal{L}$ is isomorphic to a quotient of $\mathcal{L}_K(\Pi)$.*

If $x \in X$, then by $x^\perp$ we denote the set of elements $y \in X$ with $xy = 0$. By $x^{\not\perp}$ we denote the set of all $y \in X$ with $x(xy) = y$.

Lemma 3.3. *If Π is irreducible, then X is a basis for $\mathcal{L}$ and $\mathcal{L}$ is simple.*

Proof. Assume that Π is irreducible. Suppose $\lambda_1 x_1 + \cdots + \lambda_k x_k = 0$ for $x_i \in X$, $\lambda_i \in K^*$ and $k > 1$ minimal.

Since Π is irreducible, we can assume that there is an element x in $x_1^\perp$ but not in $x_2^\perp$. Then

$$0 = x0 = x(\lambda_1 x_1 + \cdots + \lambda_k x_k) = 0 + \lambda_2 x x_2 + \cdots + \lambda_k x x_k.$$

Since $\lambda_2 x x_2 \neq 0$, this contradicts the minimality of k. Hence X is an independent set of vectors. As the vector space $\mathcal{L}$ is spanned by X, we have shown that X is a basis for $\mathcal{L}$.

It remains to prove simplicity of $\mathcal{L}$. The arguments are similar to the above.

Suppose I is a proper ideal of $\mathcal{L}$. Choose an element $y = \lambda_1 x_1 + \cdots + \lambda_k x_k$ in the ideal with $x_i \in X$, $\lambda_i \in K^*$ and k minimal. As I is a proper ideal, $k > 1$. Now suppose $x \in x_1^\perp$ but not in $x_2^\perp$. Then

$$xy = x(\lambda_1 x_1 + \cdots + \lambda_k x_k) = 0 + \lambda_2 x x_2 + \cdots + \lambda_k x x_k \in I.$$

Minimality of k implies that $xy = 0$. But, as $x x_2 \neq 0$, that contradicts X being a basis. ∎

Notice, that the above lemma also applies to $\mathcal{L}_K(\Pi)$. So, if Π is irreducible, then $\mathcal{L}_K(\Pi)$ is simple. To prove Theorem 1.2, it remains to check that Π is irreducible provided $\mathcal{L}$ is simple. But first we consider some derivations of $\mathcal{L}$.

Let $Der(\mathcal{L})$ be the algebra of derivations of $\mathcal{L}$ and $IDer(\mathcal{L})$ its subalgebra of inner derivations.

For all $x \in \mathcal{L}$ we have $\delta_x = (ad\ x)^2$ is a derivation. Indeed,

$$\begin{aligned}
\delta_x(yz) &= x(x(yz)) \\
&= x(y(zx) + z(xy)) \\
&= y((zx)x) + (zx)(xy) + z((xy)x) + (xy)(xz) \\
&= y(x(xz)) + z(x(xy)) \\
&= y\delta_x(z) + z\delta_x(y).
\end{aligned}$$

Then for $x, y, z \in \mathcal{L}$ we find

$$\begin{aligned}
\delta_{x+y}(z) &= (x+y)((x+y)z) \\
&= x(xz) + x(yz) + y(xz) + y(yz) \\
&= \delta_x(z) + (xy)z + \delta_y(z) \\
&= [\delta_x + \delta_y + (ad\ xy)](z).
\end{aligned}$$

Hence

$$\delta_{x+y} = \delta_x + \delta_y + ad\ xy.$$

By $\Delta : \mathcal{L} \to Der(\mathcal{L})/IDer(\mathcal{L})$ we denote the map $x \in \mathcal{L} \mapsto \delta_x + IDer(\mathcal{L})$. The above implies that Δ is a semi-linear map.

For a subset Y of X we denote by $\langle Y \rangle$ the linear span of Y in the vector space $\mathcal{L}$.

Lemma 3.4. *If $x \in X$, then $ad\ x : \mathcal{L} \to \mathcal{L}$ has kernel $\langle x^\perp \rangle$ and image $\langle x^{\not\perp} \rangle$ and $\mathcal{L} = \langle x^\perp \rangle \oplus \langle x^{\not\perp} \rangle$.*

Proof. Clearly, $\langle x^\perp \rangle$ is contained in the kernel of $ad\ x$ and $\langle x^{\not\perp} \rangle$ in its image. As $(ad\ x)^2$ is the identity on $\langle x^{\not\perp} \rangle$, the vector space $\mathcal{L}$ equals $\langle x^\perp \rangle \oplus \langle x^{\not\perp} \rangle$. We have proved the lemma. ■

Lemma 3.5. *Suppose $l \in \mathcal{L}$. Then for any element $x \in X$ we find $ad\ l(x) \neq x$.*

Proof. Let $l \in \mathcal{L}$. As X is closed under multiplication, it generates $\mathcal{L}$ as a vector space. So we can write l as $\Sigma_{y \in X} \lambda_y y$ with $\lambda_y \in K$. Now suppose $x \in X$ with $ad\ l(x) = x$. Then $x = ad\ l(x) = \Sigma_{y \in X} \lambda_y yx \in \langle x^{\not\perp} \rangle$, by Lemma 3.4. However, the same lemma implies that $x \notin \langle x^{\not\perp} \rangle$ and we have reached a contradiction. ■

Corollary 3.6. *If $x \in X$, then δ_x is not an inner derivation.*

Proof. Any element $y \in x^{\not\perp}$ is fixed by δ_x, so, by the above lemma, δ_x is not inner. ■

Lemma 3.7. *If $\mathcal{L}$ is simple, then Π is irreducible.*

Proof. Suppose $\mathcal{L}$ is simple. As $\mathcal{L}$ is generated by X, we find Π to be connected. Let I be the subspace of $\mathcal{L}$ generated by the elements $x + y$, where $x, y \in X$ with $x^\perp = y^\perp$. For each element $x + y$, where $x, y \in X$ with $x^\perp = y^\perp$ we find δ_{x+y} to be 0. Indeed, for all z in X we have $(zx)x = (zy)y$. Hence Δ maps the whole space I to 0.

On the other hand, $\Delta(x) \neq 0$ for all $x \in X$. Thus I is a proper subspace of $\mathcal{L}$. Suppose $x, y \in X$ with $x^\perp = y^\perp$ and $z \in x^{\not\perp} = y^{\not\perp}$. If $u \in (xz)^\perp$, then the dual affine plane spanned by x, y and z is either contained in $u^\perp$ or meets $u^\perp$ in xz, yz. So $(xz)^\perp$ is contained in $(yz)^\perp$ and by symmetry of the argument we have hence $(xz)^\perp = (yz)^\perp$. So if $x, y \in X$ with $x^\perp = y^\perp$, then $(x + y)z \in I$ for all $z \in X$. In particular, as X generates $\mathcal{L}$, we have proved that I is an ideal of $\mathcal{L}$. By simplicity of $\mathcal{L}$ the ideal I is equal to 0. Hence, for any two elements x and y with $x^\perp = y^\perp$ we have $x + y = 0$, so that $x = y$. Thus Π is irreducible. ■

Now suppose $\mathcal{L}$ is a simple Lie algebra as in the hypothesis of Theorem 1.2. Then Lemma 3.7 implies that Π is irreducible. So, also $\mathcal{L}_K(\Pi)$ is a simple Lie algebra by Lemma 3.3. But then Proposition 3.2 finishes the proof of Theorem 1.2.

We notice that for each $x \in X$ the map $\tau_x : \mathcal{L} \to \mathcal{L}$ defined by

$$\tau_x = Id + ad\ x + (ad\ x)^2,$$

where Id is the identity map on $\mathcal{L}$, is an automorphism of $\mathcal{L}$. Indeed, τ_x maps X to X, fixing x and all elements of X not collinear to x and permuting the two points distinct from x on any line through x. So, as is shown in [2], τ_x induces an automorphism on $\Pi(X)$, which clearly extends to an automorphism of $\mathcal{L}$.

The elements τ_x, with $x \in X$, form a set of 3-transpositions in $\mathrm{Aut}(\mathcal{L})$, see also [2].

4 Derivations

Let $\Pi = (X, L)$ be a cotriangular space, K a field of even characteristic and $\mathcal{L} = \mathcal{L}_K(\Pi)$ the corresponding Lie algebra over the field K. In this section we study the algebra $Der(\mathcal{L})$. In particular, we prove:

Theorem 4.1. *If Π is finite and irreducible, then $Der(\mathcal{L})$ is isomorphic to $(K \otimes_{\mathbb{F}_2} \mathcal{U}(\Pi)^*) \oplus IDer(\mathcal{L})$.*

As corollaries we obtain:

Corollary 4.2. *If Π is finite and irreducible, then $H^1(\mathcal{L}, \mathcal{L})$ is isomorphic to $K \otimes_{\mathbb{F}_2} \mathcal{U}(\Pi)^*$.*

Proof. This is immediate as $H^1(\mathcal{L}, \mathcal{L}) \simeq Der(\mathcal{L})/IDer(\mathcal{L})$. ■

Corollary 4.3. *If Π_1 and Π_2 are two finite irreducible cotriangular spaces, then $\mathcal{L}_K(\Pi_1) \simeq \mathcal{L}_K(\Pi_2)$ if and only if $\Pi_1 \simeq \Pi_2$.*

Proof. Suppose $\mathcal{L}_K(\Pi_1) \simeq \mathcal{L}_K(\Pi_2)$. Then Π_1 and Π_2 have the same number of points, and by Theorem 4.1 the same number of hyperplanes. Going over the list of finite irreducible cotriangular spaces as given in Section 2, one easily finds that $\Pi_1 \simeq \Pi_2$. ■

In the previous section we already encountered the derivations $ad\ x$ and δ_x where $x \in X$. It follows from Lemma 3.4 that the kernel of both $ad\ x$ and δ_x restricted to X is the geometric hyperplane $x^\perp$ of Π.

The linear subspace of $Der(\mathcal{L})$ generated by the elements δ_x with $x \in X$ is isomorphic to the subspace of the vector space $K \otimes_{\mathbb{F}_2} \mathcal{U}^*$ of geometric hyperplanes of Π generated by the elements of the form $x^\perp$, where x is a point in Π.

The remaining hyperplanes of Π, however, do also give rise to derivations of the corresponding Lie algebra $\mathcal{L}$ in the following way. If H is a geometric hyperplane, then define $\delta_H : \mathcal{L} \to \mathcal{L}$ by $\delta_H(x) = 0$ for $x \in H$ and $\delta_H(x) = x$ for all other other points of Π, and extend it linearly to $\mathcal{L}$. For $H = x^\perp$ it is just δ_x.

Lemma 4.4. *For each hyperplane H of Π, the map δ_H is an outer derivation of $\mathcal{L}$.*

Proof. Let $x, y \in X$. If $xy = 0$, then $x\delta_H(y)$ equals $x0 = 0$ or $xy = 0$, and similarly $y\delta_H(x) = 0$. So $\delta_H(xy) = 0 = x\delta_H(y) + y\delta_H(x)$.

If $xy = z \in X$, then either H contains $\{x, y, z\}$ and $\delta_H(x) = \delta_H(y) = \delta_H(z) = 0$ from which we easily deduce that $0 = \delta_H(xy) = x\delta_H(y) + y\delta_H(x)$, or H meets $\{x, y, z\}$ in a unique element. If $H \cap \{x, y, z\} = \{z\}$, then $\delta_H(z) = 0 = xy + yx = x\delta_H(y) + y\delta_H(x)$. If $H \cap \{x, y, z\} = \{x\}$ or $\{y\}$, then we have $z = \delta_H(z) = xy + y0 = x\delta_H(y) + y\delta_H(x)$ or $z = \delta_H(z) = x0 + yx = x\delta_H(y) + y\delta_H(x)$, respectively. In any case we find $\delta_H(xy) = x\delta_H(y) + y\delta_H(x)$. So, δ_H is a derivation. Moreover, as $\delta_H(z) = z$ for all $z \in X \setminus H$, Lemma 3.5 implies that δ_H is not inner. ■

Lemma 4.5. *Let δ be a derivation. Then the kernel of δ meets X in a subspace of Π.*

Proof. Suppose $x, y \in X$ are collinear points with $\delta(x) = \delta(y) = 0$. Then $\delta(xy) = x\delta(y) + y\delta(x) = 0 + 0 = 0$. ■

Lemma 4.6. *Suppose Π is irreducible. Let $\delta \in Der(\mathcal{L})$. If $x \in X$, then $\delta(x) \in \langle x^{\not\perp} \rangle \oplus \langle x \rangle$.*

Proof. Let x be a point of Π. Suppose $\delta(x) = a + b$ with $a \in \langle x^\perp \setminus \{x\} \rangle$ and $b \in \langle x^{\not\perp}, x \rangle$. If $a \neq 0$, we can find a point $y \in x^\perp$, with $ay \neq 0$. Notice that the element ay is in $\langle x^\perp \setminus \{x\} \rangle$, whereas the element by is in $\langle x^{\not\perp} \rangle$. But we also have $0 = \delta(xy) = \delta(x)y + \delta(y)x$. Hence $\delta(x)y = ay + by = \delta(y)x \in \langle x^{\not\perp} \rangle$, see Lemma 3.4. But this contradicts that $\mathcal{L} = \langle x^{\not\perp} \rangle \oplus \langle x^\perp \rangle$, again see Lemma 3.4. Hence $a = 0$. ■

Now let V be the subspace of $Der(\mathcal{L})$ generated by all inner derivations and derivations δ_H, where H runs through the set of geometric hyperplanes of Π. As the derivations δ_H, with H a hyperplane of Π, generate a commutative Lie subalgebra of $Der(\mathcal{L})$ isomorphic to $(K \otimes_{\mathbb{F}_2} \mathcal{U}^*)$, we find V to be isomorphic to $IDer(\mathcal{L}) \oplus (K \otimes_{\mathbb{F}_2} \mathcal{U}^*)$. We prove the following:

Proposition 4.7. $Der(\mathcal{L}) = V$.

Proof. Suppose $\delta \in Der(\mathcal{L})$. Let Y be a generating set for Π of size $\dim(\mathcal{U}(\Pi))$, see Proposition 2.3. With induction on the number of elements in Y but not in $\mathrm{Ker}(\delta)$ we will show that $\delta \in V$.

If all elements from Y are in the kernel of δ, then $\delta = 0$ and hence in V.

Now suppose Y meets the kernel of δ in a proper subset Y_0 of Y. Let H be a hyperplane of Π containing Y_0. Such a hyperplane exists. Indeed, since Y_0 contains less than $\dim(\mathcal{U}(\Pi))$ elements, we can take for H the pre-image of the intersection of $\phi_\Pi(X)$ with a hyperplane of $\mathbb{P}(\mathcal{U}(\Pi))$ containing $\phi_\Pi(Y_0)$.

Now suppose $x \in Y \setminus Y_0$. Write $\delta(x) = \lambda_0 x + \lambda_1 x_1 + \cdots + \lambda_n x_n$, where $x_i \in X \setminus \{x\}$, $\lambda_0 \in K$ and $\lambda_i \in K^*$. By Lemma 4.6 we find $x_i \in x^{\not\perp}$.

If $y \in Y_0 \cap x^\perp$, then $0 = \delta(xy) = x\delta(y) + y\delta(x) = y\delta(x)$. Hence $yx_i = 0$, for $1 \leq i \leq n$. In particular, x_i and hence also xx_i is contained in $y^\perp$ for all $1 \leq i \leq n$.

If $y \in Y_0 \cap x^{\not\perp}$, then $x = y(yx)$. Hence $\delta(x) = \delta(y(yx)) = y\delta(yx) + (yx)\delta(y) = y\delta(yx)$. In particular, for $i \in \{1, \ldots, n\}$ we find $x_i \in \langle y^{\not\perp} \rangle$ and again $xx_i \in y^\perp$. So, the derivation

$$\delta' = \delta + \lambda_0 \delta_H + \lambda_1 ad(xx_1) + \cdots + \lambda_n ad(xx_n)$$

is zero on x and Y_0. By induction we have $\delta' \in V$. But then also $\delta \in V$. ■

The above proposition finishes the proof of Theorem 4.1.

If Π is infinite, then the subalgebra of $Der(\mathcal{L})$ generated by the inner derivations and the derivations δ_H, where H runs through the set of hyperplanes of Π, only contains derivations whose kernel intersects X in a subspace X_0 with $\langle \phi_\Pi(X_0) \rangle$ having finite codimension in $\mathcal{U}(\Pi)$.

We notice that the above proof actually shows, that each derivation δ of $\mathcal{L}$ with X_0 with $\langle \phi_\Pi(X_0) \rangle$ having finite codimension in $\mathcal{U}(\Pi)$ is in this subalgebra. However, this is not true for all derivations of $\mathcal{L}$ as follows from the next example.

Let Π be $\mathcal{T}_\Omega$, where Ω is an infinite set. For K take a field of even characteristic and of cardinality at least $|\Omega|$. Inside K we take a set of distinct elements α_i, where $i \in \Omega$. Let $\mathcal{L} = \mathcal{L}_K(\Pi)$ and define $\delta : \mathcal{L} \to \mathcal{L}$ to be the linear map defined by

$$\delta(\{i,j\}) = (\alpha_i + \alpha_j)\{i,j\}$$

where $i, j \in \Omega$, $i \neq j$. (Notice that here $\{i,j\}$ is considered as an element from the Lie algebra $\mathcal{L}$.) Then δ is indeed a derivation. For, let $i, j, k \in \Omega$ be distinct, then

$$\begin{aligned} \delta(\{i,j\}\{j,k\}) &= \delta(\{i,k\}) \\ &= (\alpha_i + \alpha_k)\{i,k\} \\ &= (\alpha_i + \alpha_j)\{i,k\} + (\alpha_j + \alpha_k)\{i,k\} \\ &= \delta(\{i,j\})\{j,k\} + \{i,j\}\delta(\{j,k\}). \end{aligned}$$

However, as $\alpha_i + \alpha_j \neq 0$ for $i \neq j$, the kernel of δ meets X trivially.

We end this paper with some examples of not necessarily commutative Lie algebras closely related to the partial linear spaces and Lie algebras studied in this paper. They are all algebras containing a generating set X satisfying

(i) $x^2 = 0$ and

(ii) $xy = 0$ or $xy \in X$ and $(xy)x = y$ for all $x, y \in X$.

These examples are due to Jonathan Hall.

Suppose K is a field. Let Ω be a set and denote by Ω_2 the set of ordered 2-tuples (ω_1, ω_2) of Ω, with $\omega_1 \neq \omega_2$. By $\mathcal{L}_K(\Omega_2)$ we denote the K-vector space generated by the elements in Ω_2 subject to the relations $(\omega_1, \omega_2) = -(\omega_2, \omega_1)$ for all $\omega_1 \neq \omega_2 \in \Omega$. On $\mathcal{L}_K(\Omega_2)$ we define a product $\cdot$ by linear extension of the following.

$$\begin{aligned}(\omega_1, \omega_2) \cdot (\omega_3, \omega_4) \quad &= (\omega_1, \omega_4) \text{ if } \omega_1 \neq \omega_4 \text{ and } \omega_2 = \omega_3,\\ &= -(\omega_1, \omega_3) \text{ if } \omega_2 = \omega_4 \text{ and } \omega_1 \neq \omega_3,\\ &= 0 \text{ else.}\end{aligned}$$

One easily checks that the product $\cdot$ imposes a Lie algebra structure on $\mathcal{L}_K(\Omega_2)$. If the field K is of even characteristic, we find $\mathcal{L}_K(\Omega_2)$ to be equal to $\mathcal{L}_K(\mathcal{T}_\Omega)$. For all fields K, the set Ω_2 satisfies the conditions (i) and (ii) described above.

A second class of examples can obtained as follows. Let $\mathbb{P}$ be the Fano plane with point set $\{e_0, \ldots, e_6\}$ and with as lines the seven triples $\{e_i, e_{i+1}, e_{i+3}\}$, where indices are taken modulo 7. For any field K consider the vector space $\mathcal{L}_K(\mathbb{P})$ over K with as basis $\{e_0, \ldots, e_6\}$. The linear extension to $\mathcal{L}_K(\mathbb{P})$ of the product $\cdot$ defined by

$$e_i \cdot e_i = 0,$$

$$(e_i \cdot e_j) \cdot e_i = 0 \text{ or } e_i,$$

$$e_i \cdot e_{i+1} = -e_{i+1} \cdot e_i = e_{i+3},$$

with $i, j \in \{0, \ldots, 7\}$ and indices modulo 7, turns $\mathcal{L}_K(\mathbb{P})$ into an algebra. This algebra is a Lie algebra if and only if the field K is of characteristic 3. The set $X = \{\pm e_i \mid 1 \leq i \leq 7\}$ is a generating set for $\mathcal{L}_K(\mathbb{P})$ satisfying the conditions (i) and (ii) described above.

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