On the smallest abundant number not divisible by the first k primes

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Abstract

We say a positive integer n is abundant if $\sigma(n) > 2n$, where $\sigma(n)$ denotes the sum of the positive divisors of n. Number the primes in ascending order: $p_1 = 2$, $p_2 = 3$, and so forth. Let $A(k)$ denote the smallest abundant number not divisible by $p_1, p_2, \ldots, p_k$. In this paper we find $A(k)$ for $1 \le k \le 7$, and we show that for all $\epsilon > 0$, $(1-\epsilon)(k \ln k)^{2-\epsilon} < \ln A(k) < (1+\epsilon)(k \ln k)^{2+\epsilon}$ for all sufficiently large k.

1 Introduction

We say a positive integer n is *abundant* if $\sigma(n) > 2n$, where $\sigma(n)$ denotes the sum of the positive divisors of n. The smallest abundant number is 12, and the smallest odd abundant number is 945. With a computer search, Whalen and Miller [3] found $5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$ to be an odd abundant number not divisible by 3, and they raised the general question of how one goes about finding the smallest abundant number not divisible by the first k primes.

We number the primes in ascending order: $p_1 = 2$, $p_2 = 3$, and so forth. Let $A(k)$ denote the smallest abundant number not divisible by $p_1, p_2, \ldots, p_k$. Note that $A(1) = 945$. In this paper we devise an algorithm to find $A(k)$, and we apply it to find $A(k)$ for $1 \le k \le 7$. We shall also prove

Theorem 1. *For every $\epsilon > 0$ we have*

$$(1-\epsilon)(k \ln k)^{2-\epsilon} < \ln A(k) < (1+\epsilon)(k \ln k)^{2+\epsilon}$$

whenever k is sufficiently large.

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2 Preliminaries

For a positive integer n we define the *index* of n to be

$$\sigma_{-1}(n) = \frac{\sigma(n)}{n}.$$

Thus n is abundant if $\sigma_{-1}(n) > 2$. The function σ_{-1} is multiplicative, and for prime p and integer $a \geq 1$ we have

$$\sigma_{-1}(p^a) = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^a}.$$

Therefore $\sigma_{-1}(p^a)$ increases with a, and in fact

$$\frac{p+1}{p} \leq \sigma_{-1}(p^a) < \frac{p}{p-1}. \tag{1}$$

If $p < q$ are primes then $q/(q-1) < (p+1)/p$ and so for all integers $a \geq 1$, $b \geq 1$, we have

$$\sigma_{-1}(q^b) < \sigma_{-1}(p^a). \tag{2}$$

For each integer $k \geq 1$ let us define

$$V_t(k) = \prod_{j=k+1}^{t} \frac{p_j + 1}{p_j}$$

for integers $t > k$. By Theorem 19 in [1] and Theorem 3 of §28, Chapter VII in [2], $V_t(k)$ increases without bound as t increases, and therefore we may define

$$v(k) = \min\{\, t : V_t(k) > 2 \,\}.$$

Since $V_t(k) = \sigma_{-1}(p_{k+1}p_{k+2}\cdots p_t)$, we have

$$A(k) \leq p_{k+1}p_{k+2}\cdots p_{v(k)}. \tag{3}$$

We may also obtain a lower bound for $A(k)$. For each integer $k \geq 1$ we define

$$U_t(k) = \prod_{j=k+1}^{t} \frac{p_j}{p_j - 1}$$

for integers $t > k$. Since $p/(p-1) > (p+1)/p$, we have $U_t(k) > V_t(k)$ and so we may define

$$u(k) = \min\{\, t : U_t(k) > 2 \,\}. \tag{4}$$

Note that $u(k) \leq v(k)$. We can show that $A(k) \geq p_{k+1}p_{k+2}\cdots p_{u(k)}$; in fact we can show more:

Lemma 1. *$A(k)$ is divisible by $p_{k+1}p_{k+2}\cdots p_{u(k)}$.*

Proof. Let $M = p_{k+1}p_{k+2}\cdots p_{u(k)}$ and suppose $M \nmid A(k)$. Let $A(k)$ have the unique prime factorization given by $A(k) = \prod_{i=1}^{t} q_i^{a_i}$ for distinct primes $q_1 < q_2 < \cdots < q_t$, and positive integers a_i, $1 \le i \le t$. Note that $q_1 > p_k$. Hence $q_i \ge p_{k+i}$ for all i, $1 \le i \le t$.

We have $t \ge u(k) - k$. For, otherwise by (1), (2),

$$\sigma_{-1}(A(k)) < \frac{p_{k+1}}{p_{k+1}-1} \cdot \frac{p_{k+2}}{p_{k+2}-1} \cdots \frac{p_{u(k)-1}}{p_{u(k)-1}-1},$$

which implies $\sigma_{-1}(A(k)) \le 2$ by (4); this contradicts the abundance of $A(k)$.

Since $M \nmid A(k)$, we have $p_j \nmid A(k)$ for some j such that $k+1 \le j \le u(k)$. Therefore, since $t \ge u(k) - k$, at least one of the primes q_i dividing $A(k)$ must be greater than $p_{u(k)}$. Without loss of generality we may assume $q_1 > p_{u(k)}$. Then by (2),

$$\sigma_{-1}(p_j q_2^{a_2} \cdots q_t^{a_t}) > \sigma_{-1}(q_1^{a_1} q_2^{a_2} \cdots q_t^{a_t}) > 2.$$

But then,

$$p_j q_2^{a_2} \cdots q_t^{a_t} < q_1^{a_1} q_2^{a_2} \cdots q_t^{a_t} = A(k),$$

which contradicts the minimality of $A(k)$. ■

3 An Algorithm

From Lemma 1, we may devise an algorithm for finding $A(k)$:

(1) Find $u(k)$, as given by (4).

(2) Let $P_k = p_1 p_2 \cdots p_k$. Let m run through the positive integers which are relatively prime to P_k until we find

$$M(k) = \min_{(m,P_k)=1} \{\, m \;:\; \sigma_{-1}(m p_{k+1}p_{k+2}\cdots p_{u(k)}) > 2 \,\}\,.$$

It follows that

$$A(k) = M(k) p_{k+1}p_{k+2}\cdots p_{u(k)}\,.$$

Note that by (3) we have $M(k) \le p_{u(k)+1}p_{u(k)+2}\cdots p_{v(k)}$. Using the UBASIC software package, a computer search employing the algorithm was conducted to find $A(k)$ for $1 \le k \le 7$. In Table 1 is given the values for $M(k)$ and $A(k)$, along with those of $p_{u(k)}$ and $p_{v(k)}$, for $1 \le k \le 7$.

k	$p_{u(k)}$	$p_{v(k)}$	$M(k)$	$A(k)$
1	7	13	3^2	$3^3 \cdot 5 \cdot 7$
2	23	31	$5 \cdot 29$	$5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$
3	61	73	$7 \cdot 11 \cdot 67$	$7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdots 59 \cdot 61 \cdot 67$
4	127	149	$11 \cdot 13 \cdot 131 \cdot 137$	$11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdots 131 \cdot 137$
5	199	233	$13 \cdot 17 \cdot 211 \cdot 223 \cdot 227$	$13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdots 223 \cdot 227$
6	337	367	$17 \cdot 19 \cdot 23 \cdot 347 \cdot 349$	$17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdots 347 \cdot 349$
7	479	521	$19 \cdot 23 \cdot 29 \cdot 487 \cdot 491 \cdot 499$	$19^2 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 37 \cdots 491 \cdot 499$

TABLE 1. The values $A(k)$ for $1 \le k \le 7$.

4 Behavior of $A(k)$

In this section we estimate the growth of $A(k)$ by proving Theorem 1. We begin by stating a result due to Mertens (Theorem 429 in [1]),

$$\lim_{x\to\infty} \frac{e^{-\gamma}}{\ln x} \prod_{p\le x} \frac{p}{p-1} = 1\,, \tag{5}$$

where the product is taken over primes p and where γ denotes Euler's constant.

We now prove

Lemma 2.

$$\lim_{x\to\infty} \frac{\ln p_{u(k)}}{\ln p_k} = 2\,.$$

Proof. Let $0 < \epsilon < 2$ be given. Take $0 < \epsilon_1 < \epsilon/(2-\epsilon)$ (so that $2\epsilon_1/(1+\epsilon_1) < \epsilon$), and take $0 < \epsilon_2 < \epsilon_1/(2+\epsilon_1)$ (so that $(1+\epsilon_2)/(1-\epsilon_2) < 1+\epsilon_1$). By (5), there exists an integer k_1 such that for all $x \ge p_{k_1}$ we have

$$(1-\epsilon_2)e^{\gamma}\ln x < \prod_{p\le x}\frac{p}{p-1} < (1+\epsilon_2)e^{\gamma}\ln x\,.$$

Note that by (4) we have

$$2 < \prod_{p_k < p \le p_{u(k)}} \frac{p}{p-1} = \frac{\prod_{p\le p_{u(k)}} \frac{p}{p-1}}{\prod_{p\le p_k}\frac{p}{p-1}}\,.$$

Thus for all $k \ge k_1$ we have

$$2 < \frac{(1+\epsilon_2)e^{\gamma}\ln p_{u(k)}}{(1-\epsilon_2)e^{\gamma}\ln p_k} < (1+\epsilon_1)\frac{\ln p_{u(k)}}{\ln p_k}\,,$$

hence

$$\frac{\ln p_{u(k)}}{\ln p_k} > \frac{2}{1+\epsilon_1} = 2 - \frac{2\epsilon_1}{1+\epsilon_1} > 2-\epsilon\,.$$

Now take $0 < \epsilon_4 < (-3+\sqrt{9+4\epsilon})/2$ (so that $3\epsilon_4+\epsilon_4^2 < \epsilon$), take $0 < \epsilon_5 < \epsilon_4/(2+\epsilon_4)$ (so that $2/(1-\epsilon_5) < 2+\epsilon_4$), and take $0 < \epsilon_6 < \epsilon_5/(2-\epsilon_5)$ (so that $(1-\epsilon_6)/(1+\epsilon_6) < 1-\epsilon_5$). By (5) there exists an integer k_2 such that for all $k \ge k_2$ we have $1/(p_k-1) < \epsilon_4$ and such that for all $x \ge p_{k_2}$ we have

$$(1-\epsilon_6)e^{\gamma}\ln x < \prod_{p\le x}\frac{p}{p-1} < (1+\epsilon_6)e^{\gamma}\ln x\,.$$

By (4) we have

$$2\frac{p_{u(k)}}{p_{u(k)}-1} \ge \prod_{p_k < p \le p_{u(k)}} \frac{p}{p-1} = \frac{\prod_{p\le p_{u(k)}} \frac{p}{p-1}}{\prod_{p\le p_k}\frac{p}{p-1}}\,,$$

and so for $k \ge k_2$

$$2\frac{p_{u(k)}}{p_{u(k)}-1} \ge \frac{(1-\epsilon_6)e^{\gamma}\ln p_{u(k)}}{(1+\epsilon_6)e^{\gamma}\ln p_k} > (1-\epsilon_5)\frac{\ln p_{u(k)}}{\ln p_k}\,,$$

hence

$$\frac{\ln p_{u(k)}}{\ln p_k} < \frac{p_{u(k)}}{p_{u(k)}-1}\cdot\frac{2}{1-\epsilon_5} < (1+\epsilon_4)(2+\epsilon_4) = 2+3\epsilon_4+\epsilon_4^2 < 2+\epsilon\,.$$

Therefore if $k \ge \max\{k_1,k_2\}$ then $|\ln p_{u(k)}/\ln p_k - 2| < \epsilon$. ■

An almost identical proof (omitted here) gives

Lemma 3.

$$\lim_{x\to\infty} \frac{\ln p_{v(k)}}{\ln p_k} = 2\,.$$

The Prime Number Theorem (Theorem 8 in [1]) states that

$$\lim_{n\to\infty} \frac{p_n}{n \ln n} = 1\,. \tag{6}$$

An equivalent result (Theorem 420 in [1]) is

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1\,, \tag{7}$$

where θ denotes the function, defined for $x > 0$, given by

$$\theta(x) = \sum_{p\le x} \ln p\,,$$

the sum being taken over primes.

We may now begin proving Theorem 1. Let $\epsilon > 0$ be given. Take $0 < \epsilon_1 < \sqrt[4]{1+\epsilon} - 1$ (so that $(1+\epsilon_1)^4 < 1+\epsilon$), take $0 < \epsilon_2 < \epsilon_1$, and take $0 < \epsilon_3 < \min\{1, \epsilon\}$.

By (7), there exists an integer k_1 such that for all $k \ge k_1$ we have

$$\theta(p_{v(k)}) < (1+\epsilon_1) p_{v(k)}\,.$$

By (6) there exists an integer k_2 such that for all $k \ge k_2$ we have

$$p_k < (1+\epsilon_2) k \ln k\,.$$

By Lemma 3 there exists an integer k_3 such that for all $k \ge k_3$ we have

$$p_{v(k)} < p_k^{2+\epsilon_3}\,.$$

Then by (3), if $k \ge \max\{k_1, k_2, k_3\}$, we have

$$\ln A(k) \le \sum_{j=k+1}^{v(k)} \ln p_j < \theta(p_{v(k)})\,,$$

hence

$$\begin{aligned}
\ln A(k) &< (1+\epsilon_1) p_{v(k)} \\
&< (1+\epsilon_1) p_k^{2+\epsilon_3} \\
&< (1+\epsilon_1)(1+\epsilon_2)^{2+\epsilon_3} (k \ln k)^{2+\epsilon_3} \\
&< (1+\epsilon_1)^4 (k \ln k)^{2+\epsilon_3} \\
&< (1+\epsilon)(k \ln k)^{2+\epsilon}\,.
\end{aligned}$$

A similar proof (omitted here) shows that for sufficiently large k we have

$$\ln A(k) > (1-\epsilon)(k \ln k)^{2-\epsilon}\,,$$

and hence the proof of Theorem 1 is complete.

Bibliography

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