The Fundamental Solution of the Hyperbolic Dirac Operator on $\mathbb{R}^{1,m}$: a new approach

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Abstract

In this paper, the fundamental solution of the Dirac equation on hyperbolic space will be calculated by means of the fundamental solution for the wave-operator in the $(m+1)$-dimensional Minkowski space-time of signature $(1, m)$. This leads to addition formulas for the fundamental solution in terms of the solution in a lower-dimensional Minkowski space-time. Certain identities between hypergeometric functions can then be used to obtain a closed form for the fundamental solution of the Dirac equation.

1 Introduction

In this paper the fundamental solution for the Dirac operator on a hyperbolic space is constructed within the framework of Clifford analysis. Clifford analysis may be regarded as a direct and elegant generalization to higher dimension of the theory of holomorphic functions of a complex variable, centered around the notion of a monogenic function, i.e. a null-solution of the Dirac operator. In the third section, a brief introduction to Clifford algebras and Clifford analysis will be given.

In the second half of the previous century Clifford analysis has grown out to an independent discipline; standard reference books are [1], [4] and [12]. For a nice overview of the basic results we refer the reader to [5]. In most of the literature concerned, the Dirac operator considered is the elliptic one acting in a flat Euclidean

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space. This setting was generalized by considering Dirac operators on manifolds (see e.g. [12] and [3]). For Clifford analysis on Riemannian spaces of constant positive or negative curvature we refer e.g. to [17], [15], [16], [13] and [9].

In reference [2] it was observed that these last generalizations concern the Dirac operator for Spin-1 fields and a way to generalize the Dirac operator for Spin-$\frac{1}{2}$ fields was introduced. In this paper, we will use the same model for hyperbolic space as in [2], though introduced in another way (see section 2).

In the fourth section the problem of finding the fundamental solution for the Dirac operator will be turned into a scalar problem, while in the fifth section some elementary notions concerning the hypergeometric function will be presented. In the last two sections (section 6 and 7) we will derive explicit formulas for the fundamental solution.

2 Hyperbolic Spaces

In this section, we will introduce a model for the m-dimensional hyperbolic space. For that purpose, consider the real orthogonal space $\mathbb{R}^{1,m}$ of signature $(1, m)$, with an orthonormal basis $(\epsilon, e_1, \cdots, e_m)$. We will denote space-time vectors as $X = \epsilon T + \vec{X}$, where we prefer to make a clear distinction between the spatial co-ordinates $(X_1, \cdots, X_m)$ and the time co-ordinate T. The quadratic form associated with the real orthogonal space $\mathbb{R}^{1,m}$ looks as follows :

$$Q(X) \quad = \quad T^2 - |\vec{X}|^2 \ , \quad \forall X \in \mathbb{R}^{1,m} \ .$$

The null cone NC is then defined as the set of all space-time vectors X satisfying $Q(X) = 0$, and this NC separates the *time-like* region (space-time vectors X for which $Q(X) > 0$) from the *space-like* region (space-time vectors X for which $Q(X) < 0$). The time-like region is the union of the future cone $FC = \{X : Q(X) > 0, T > 0\}$ and the past cone $PC = \{X : Q(X) > 0, T < 0\}$.

One obtains a model for the m-dimensional hyperbolic space by identifying the rays inside this $FC \subset \mathbb{R}^{1,m}$ with points on the hyperbolic unit ball. Notice that this approach is usually followed in literature when introducing models for spaces with constant (positive or negative) curvature (see e.g. [11]), and it provides us with a projective model. Other models for the m-dimensional hyperbolic space are then readily obtained by intersecting the manifold of rays inside the future cone FC with an arbitrary surface Σ inside the FC, such that each ray intersects Σ in a unique point. Two choices for Σ are of particular interest : one can choose Σ as the upper branch H^+ of the hyperboloid H defined by $Q(X) = 1$. The interesting fact about this surface H^+ is its invariance under the group $SO(1, m)$. One can also choose Σ as the hyperplane Π by putting $T = 1$. We will consider the latter surface in this paper.

3 The Clifford Setting

Let us introduce the Clifford algebra $\mathbb{R}_{1,m}$. It is defined as the real linear associative but non-commutative algebra generated by the orthonormal basis $(\epsilon, e_1, \cdots, e_m)$ for $\mathbb{R}^{1,m}$ and the following relations :

$$\begin{aligned}
&e_i e_j + e_j e_i = -2\delta_{ij},\ i,j = 1,\cdots,m \\
&\epsilon e_i + e_i \epsilon = 0,\ i = 1,\cdots,m \\
&\epsilon^2 = 1
\end{aligned}$$

Elements of $\mathbb{R}_{1,m}$ are called Clifford numbers and have the form

$$a = \sum_{A \subset M} a_A e_A,\ a_A \in \mathbb{R}$$

with $A = \{i_1, \cdots, i_k\} \subset M = \{0, \cdots, m\}$, where $i_1 < \cdots < i_k$ and $e_A = e_{i_1} \cdots e_{i_k}$ (here e_0 is to be replaced by ϵ). For $A = \emptyset$ we put $e_\emptyset = 1$. If A has k elements, e_A is a k-vector which will be denoted as $e_A \in \mathbb{R}^{(k)}_{1,m}$. If we denote the projection of a Clifford number a onto its k-vector part as $[a]_k$, we have

$$a = \sum_{k=0}^{1+m} [a]_k\ ,$$

with $[a]_0$ the scalar part of the Clifford number a. Notice also that space-time vectors X in $\mathbb{R}^{1,m}$ may be identified with 1-vectors in $\mathbb{R}_{1,m}$, where we choose to keep the notation X.

For two space-time vectors X and Y in $\mathbb{R}^{(1)}_{1,m}$, we have :

$$XY = X \cdot Y + X \wedge Y$$

where the inner product is defined as

$$X \cdot Y = \frac{XY + YX}{2}$$

and the outer product as

$$X \wedge Y = \frac{XY - YX}{2}\ .$$

On $\mathbb{R}_{1,m}$, the following involutory (anti-)automorphisms are of importance (in the following formulas, e_0 is again to be replaced by ϵ):

1. the main involution $a \mapsto a^*$

$$e_i^* = -e_i\ , \quad (a + \lambda b)^* = a^* + \lambda b^*\ , \quad (ab)^* = a^* b^*$$

2. the inversion $a \mapsto \tilde{a}$

$$\tilde{e}_i = e_i\ , \quad (a + \lambda b)\tilde{} = \tilde{a} + \lambda \tilde{b}\ , \quad (ab)\tilde{} = \tilde{b}\tilde{a}$$

3. the conjugation (also known as bar-map) $a \mapsto \bar{a}$

$$\begin{array}{l} \bar{e}_i = -e_i \ , \quad \overline{(a+\lambda b)} = \bar{a} + \lambda \bar{b} \ , \quad \overline{(ab)} = \bar{b}\bar{a} \\ a \ , \ b \in \mathbb{R}_{1,m} \ , \quad \lambda \in \mathbb{R} \ , \quad i = 0 \cdots m \ . \end{array}$$

Also, the following subgroups of the real Clifford algebra $\mathbb{R}_{1,m}$ are of interest : the Clifford group $\Gamma(1,m)$, the Pin group $Pin(1,m)$ and the Spin group $Spin(1,m)$. For a definition of these groups, we refer the reader to [4] and [14].

For each element $s \in Pin(1,m)$ the map $\chi(s) : \mathbb{R}^{1,m} \mapsto \mathbb{R}^{1,m} : X \mapsto sX\bar{s}$ induces a map from $\mathbb{R}^{1,m}$ onto itself. In this way, $Pin(1,m)$ defines a double covering of the orthogonal group $O(1,m)$ whereas $Spin(1,m)$ defines a double covering of the orthogonal group $SO(1,m)$.

Let $\partial_X = \epsilon\partial_T - \partial_{\vec{X}}$ be the Dirac operator on $\mathbb{R}^{1,m}$. In this paper, we would like to find the fundamental solution for this operator. However, since we are working with a *projective* model for hyperbolic space we need to look for a fundamental solution that forms an invariant object on the manifold of rays inside the future cone FC, which is our true hyperbolic model. This can be done by considering functions satisfying a fixed homogeneity condition of the form $f(\lambda X) = \lambda^\alpha f(X)$. Such functions are sections of homogeneous bundles over the manifold of rays issuing form the origin.

This means that the fundamental solution $E_{\alpha,m}(T,\vec{X})$ for the hyperbolic Dirac operator on $\mathbb{R}^{1,m}$, has to be an eigenfunction of the Euler-operator $\mathbb{E}_X = T\partial_T + \sum_i X_i\partial_{X_i}$ with eigenvalue $\alpha \in \mathbb{R}$ satisfying the following equation :

$$(\epsilon\partial_T - \partial_{\vec{X}})E_{\alpha,m}(T,\vec{X}) \;=\; T^{\alpha+m-1}H(T)\delta(\vec{X}) \ , \ \forall(\vec{X},T) \in FC \ . \qquad (1)$$

The Heaviside-function at the right-hand side indicates the fact that we will be looking for a fundamental solution in the future cone, whereas the factor $T^{\alpha+m-1}$ is due to the homogeneity condition. This can easily be seen as follows : each space-time vector $\epsilon T + \vec{X}$ may be written as $\lambda(\epsilon + \vec{x})$ with $\lambda = T$ and $\vec{x} = \vec{X}/T$. Imposing the following homogeneity condition :

$$E_{\alpha,m}(T,\vec{X}) \;=\; \lambda^\alpha E_{\alpha,m}(1,\vec{x}),$$

and using the co-ordinate transform $(T,\vec{X}) \longrightarrow (\lambda,\vec{x})$, it is easy to verify that, after projection on the hyperplane Π, the hyperbolic Dirac operator ∂_X becomes :

$$\epsilon\partial_T - \partial_{\vec{X}} \overset{(\vec{x},\lambda)}{\longrightarrow} -\frac{1}{\lambda}\bigg(\partial_{\vec{x}} + \epsilon(\mathbb{E}_{\vec{x}} - \lambda\partial_\lambda)\bigg) \ ,$$

where $\mathbb{E}_{\vec{x}}$ stands for the Euler operator with respect to the co-ordinates $\vec{x}$. Denoting $E_{\alpha,m}(1,\vec{x})$ as $E_{\alpha,m}(\vec{x})$, we thus have the following equation for the projection of $E_{\alpha,m}(T,\vec{X})$ on the hyperplane Π (see also [6]) :

$$(\partial_{\vec{x}} + \epsilon(\mathbb{E}_{\vec{x}} - \alpha))E_{\alpha,m}(\vec{x}) = -\delta(\vec{x}) \ .$$

This equation, which is the projection on the hyperplane $\Pi \leftrightarrow T = 1$ of the hyperbolic Dirac equation on $\mathbb{R}_{1,m}$ for an α-homogeneous fundamental solution, leads to equation (1) in space-time co-ordinates.

4 The Scalar Problem

Let us first introduce some definitions :

Definitions.

1. $\Box_m$ *is the wave-operator in* $(m+1)$*-dimensional Minkowski space-time :*

$$\begin{aligned}\Box_m &= \frac{\partial^2}{\partial T^2} - \Delta_m \\ &= \frac{\partial^2}{\partial T^2} - \sum_{i=1}^{m} \frac{\partial^2}{\partial X_i^2}\end{aligned}$$

2. $\mathcal{E}_m(T, \vec{X})$ *is the fundamental solution for this operator* $\Box_m$ *:*

$$\Box_m \mathcal{E}_m(T, \vec{X}) = \delta(\vec{X})\delta(T)$$

3. $\varphi_{\alpha,m}(T) = T^{\alpha+m-1}H(T)$ *is defined as a distribution in the* T*-variable with support* $[0, \infty[$

4. $\Phi_{\alpha,m}(T, \vec{X}) = (\mathcal{E}_m * \varphi_{\alpha,m})(T, \vec{X})$ *is the distribution in Minkowski space-time, satisfying*

$$\Box_m \Phi_{\alpha,m}(T, \vec{X}) = \varphi_{\alpha,m}(T)\delta(\vec{X})$$

Notice that $E_{\alpha,m}(T, \vec{X})$ can be expressed in terms of $\Phi_{\alpha,m}(T, \vec{X})$:

$$E_{\alpha,m}(T, \vec{X}) = (\epsilon\partial_T - \partial_{\vec{X}})\Phi_{\alpha,m}(T, \vec{X}) \ .$$

This means that the problem of finding the fundamental solution for the hyperbolic Dirac equation on $\mathbb{R}_{1,m}$ has been reduced to a convolution in the time-variable T :

$$E_{\alpha,m}(T, \vec{X}) = (\epsilon\partial_T - \partial_{\vec{X}}) \int_{\mathbb{R}} \mathcal{E}_m(T - S, \vec{X})\varphi_{\alpha,m}(S)dS \ .$$

Let us start with a few considerations concerning the fundamental solution $\mathcal{E}_m(T, \vec{X})$ for the wave-operator in $\mathbb{R}^{1,m}$. Since we restrict ourselves to the case of an odd spatial dimension $m \geq 3$, we start from the following formula, valid for all $m \in 2\mathbb{N}+3$ (see [18]) :

$$\begin{aligned}\mathcal{E}_m(T, \vec{X}) &= \frac{1}{(2\pi)^{\frac{m-1}{2}}} \left(\frac{1}{T}\frac{\partial}{\partial T}\right)^{\frac{m-3}{2}} \delta(T^2 - R^2) \\ &= \frac{1}{(2\pi)^{\frac{m-1}{2}}} \left(\frac{1}{T}\frac{\partial}{\partial T}\right)^{\frac{m-3}{2}} \left(\frac{\delta(T-R)}{2R} + \frac{\delta(T+R)}{2R}\right) ,\end{aligned}$$

where R stands for $|\vec{X}|$. Since that part of the fundamental solution which contains the delta-function $\delta(T+R)$ singles out a *negative* value for T - which obviously does not belong to the future cone FC - we will only work with the other part, containing the delta-function $\delta(T-R)$, in order to obtain a fundamental solution $E_{\alpha,m}(T,\vec{X})$ with support in the future cone.

So, from now on we will always refer to the following distribution if we mean the fundamental solution $\mathcal{E}_m(T,\vec{X})$ for the wave-operator in $\mathbb{R}^{1,m}$:

$$\mathcal{E}_m(T,\vec{X}) \quad = \quad \frac{1}{(2\pi)^{\frac{m-1}{2}}}\left(\frac{1}{T}\frac{\partial}{\partial T}\right)^{\frac{m-3}{2}}\frac{\delta(T-R)}{2R}\ , \quad \forall m \in 2\mathbb{N}+3\ , \quad R = |\vec{X}|\ .$$

In the fifth section, we will use the following Lemma :

Lemma 4.1 :

$$\textit{For all } m \in 2\mathbb{N}+3 \ : \ \mathcal{E}_m(T,\vec{X}) = \sum_{k=0}^{a} \frac{c_k^{(a)}}{(2\pi)^{\frac{m-1}{2}}}\frac{\delta^{(a-k)}(T-R)}{2R^{a+k+1}}$$

$$\textit{where} \quad a = \frac{m-3}{2}\ , \quad c_k^{(a)} \in \mathbb{N}_0$$

Proof : For $m=3$, the statement becomes trivial, and the rest can be proved by means of induction on the dimension m. ■

Notice that we are not interested in the exact value of $c_k^{(a)}$.

5 The Hypergeometric Function $F(a,b;c;z)$

Since we will frequently use hypergeometric functions, we devote a small section to them, in which we would like to list a few elementary properties.

The hypergeometric function, denoted as $F(a,b;c;z)$, is usually defined in terms of its series expansion around the origin :

$$F(a,b;c;z) \quad = \quad \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k\ , \ |z|<1\ ,$$

where the Pochammer symbol $(a)_k$ in terms of the Γ-function is given by

$$(a)_k \quad = \quad \frac{\Gamma(a+k)}{\Gamma(a)}\ , \quad k \in \mathbb{N}\ , \quad a \in \mathbb{C}\ .$$

The hypergeometric function is a solution of the hypergeometric differential equation

$$z(1-z)y'' + [c-(1+a+b)z]y' - aby = 0\ ,$$

which can be extended to a holomorphic function in the region $|\arg(1-z)| < \pi$. Since we will only use hypergeometric functions with a real argument $x \in [0,1[$, we can take the series expansion as a definition.

The following properties will be crucial (see e.g. [10]) :

Lemma 5.1 :

1. $$\frac{d^n}{dx^n}F(a,b;c;x) = \frac{(a)_n(b)_n}{(c)_n}F(a+n,b+n;c+n;x)$$

2. $$F(a,b;c-1;x) = \left(1+\frac{x}{c-1}\frac{d}{dx}\right)F(a,b;c;x)$$

3. $$F\left(a,a+\frac{1}{2};\frac{1}{2};x^2\right) = \frac{(1+x)^{-2a}+(1-x)^{-2a}}{2}$$

4. $$F\left(a,a+\frac{1}{2};\frac{3}{2};x^2\right) = \frac{(1+x)^{1-2a}-(1-x)^{1-2a}}{2(1-2a)x}$$

6 The 3-dimensional case

In this section, we will perform all calculations for the - rather trivial - case where $m=3$. We will need this in the next section, where the formulas for an arbitrary dimension $m \in 2\mathbb{N}_0+3$ will be proved by means of an induction argument.

By definition we have

$$\mathcal{E}_3(T,\vec{X}) = \frac{\delta(T-|\vec{X}|)}{4\pi|\vec{X}|} .$$

Hence,

$$\begin{aligned}\Phi_{\alpha,3}(T,\vec{X}) &= \int_{\mathbb{R}}\mathcal{E}_3(T-S,\vec{X})H(S)S^{\alpha+2}dS\\ &= \frac{T^{\alpha+2}(1-|\vec{x}|)^{\alpha+2}}{4\pi|\vec{X}|}\end{aligned}$$

where $\vec{x}=\frac{\vec{X}}{T}$. Notice that $\vec{x}\in B(1)$, for all $(T,\vec{X})\in FC$.

This function $\Phi_{\alpha,3}(T,\vec{X})$ can now be decomposed into a singular part $\mathcal{F}_{\alpha,3}(T,\vec{X})$ and a regular part $\mathcal{R}_{\alpha,3}(T,\vec{X})$ as follows :

$$\Phi_{\alpha,3}(T,\vec{X}) = \mathcal{F}_{\alpha,3}(T,\vec{X})+\mathcal{R}_{\alpha,3}(T,\vec{X}) ,$$

where

$$\begin{aligned}\mathcal{F}_{\alpha,3}(T,\vec{X}) &= \frac{T^{\alpha+2}}{4\pi|\vec{X}|}\frac{(1-|\vec{x}|)^{\alpha+2}+(1+|\vec{x}|)^{\alpha+2}}{2}\\ &= \frac{T^{\alpha+2}}{4\pi|\vec{X}|}\,F\left(-\frac{\alpha+2}{2},-\frac{\alpha+1}{2};\frac{1}{2};|\vec{x}|^2\right) ,\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_{\alpha,3}(T,\vec{X}) &= \frac{T^{\alpha+1}}{4\pi}\frac{(1-|\vec{x}|)^{\alpha+2}-(1+|\vec{x}|)^{\alpha+2}}{2|\vec{x}|}\\ &= -(\alpha+2)\frac{T^{\alpha+1}}{4\pi}\,F\left(-\frac{\alpha}{2},-\frac{\alpha+1}{2};\frac{3}{2};|\vec{x}|^2\right) .\end{aligned}$$

7 The case $m \in 2\mathbb{N}_0 + 3$

In this section, we will derive an addition formula for the fundamental solution $\Phi_{\alpha,m}(T,\vec{X})$ by means of an induction argument.

We start from the following observation :

$$\mathcal{E}_{m+2}(T,\vec{X}) \;=\; \frac{1}{2\pi}\frac{1}{T}\frac{\partial}{\partial T}\mathcal{E}_m(T,\vec{X}) \quad , \quad \forall m \in 2\mathbb{N}_0 + 3$$

which is a trivial consequence of the definition for $\mathcal{E}_m(T,\vec{X})$.

Hence,

$$\begin{aligned} \Phi_{\alpha,m+2}(T,\vec{X}) &= (\mathcal{E}_{m+2} * \varphi_{\alpha,m+2})(T,\vec{X}) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(\partial_T\mathcal{E}_m)(T-S,\vec{X})}{T-S} S^{\alpha+m+1} dS \ . \end{aligned}$$

With the aid of Lemma 4.1 it is easy to verify that $(\partial_T\mathcal{E}_m)(T-S,\vec{X}) = -(\partial_S\mathcal{E}_m)(T-S,\vec{X})$ and that the support of $\partial_T\mathcal{E}_m(T-S,\vec{X})$, if considered as a distribution in the S-variable, is $\{T-|\vec{X}|\}$. Since we are looking for a fundamental solution $E_{\alpha,m}(T,\vec{X})$ in the future cone FC, we know that $T > |\vec{X}|$. Hence there exists an infinitesimal $\eta \in \mathbb{R}^+$ such that $T - |\vec{X}| \in I_\eta$, where I_η is defined as the interval $[T-|\vec{X}|-\eta, T-|\vec{X}|+\eta]$.

Consequently,

$$\Phi_{\alpha,m+2}(T,\vec{X}) \;=\; \frac{1}{2\pi}\int_{I_\eta} \mathcal{E}_m(T-S,\vec{X}) \left\{ \frac{\alpha+m+1}{T-S} S^{\alpha+m} + \frac{1}{(T-S)^2} S^{\alpha+m+1} \right\} dS \ .$$

Because $S < T$ in I_η, both $(T-S)^{-1}$ and $(T-S)^{-2}$ can be expanded as a series in S/T, whence

$$\begin{aligned} \Phi_{\alpha,m+2}(T,\vec{X}) &= \frac{1}{2\pi}\int_{I_\eta}\sum_{k=0}^{\infty} \frac{\alpha+m+1}{T^{1+k}} \mathcal{E}_m(T-S,\vec{X}) S^{\alpha+m+k+1} dS \\ &+ \frac{1}{2\pi}\int_{I_\eta}\sum_{k=0}^{\infty} \frac{1+k}{T^{2+k}} \mathcal{E}_m(T-S,\vec{X}) S^{\alpha+m+k+2} dS \ . \end{aligned}$$

Bringing the integral inside the summation, we obtain :

$$\begin{aligned} \Phi_{\alpha,m+2}(T,\vec{X}) &= \frac{1}{2\pi}\sum_{k=0}^{\infty} \frac{\alpha+m+1}{T^{1+k}} (\mathcal{E}_m * \varphi_{\alpha+k+1,m})(T,\vec{X}) \\ &+ \frac{1}{2\pi}\sum_{k=0}^{\infty} \frac{1+k}{T^{2+k}} (\mathcal{E}_m * \varphi_{\alpha+k+2,m})(T,\vec{X}) \ . \end{aligned} \tag{2}$$

Theorem 7.1 : *For all $m \in 2\mathbb{N}+3$ we have :*

1. $$\Phi_{\alpha,m}(T,\vec{X}) = \mathcal{F}_{\alpha,m}(T,\vec{X}) + \mathcal{R}_{\alpha,m}(T,\vec{X}) \tag{3}$$

2. $$\mathcal{F}_{\alpha,m}(T,\vec{X}) = \frac{T^{\alpha+m-1}}{(m-2)A_m|\vec{X}|^{m-2}} F\left(\frac{1-\alpha-m}{2}, \frac{2-\alpha-m}{2}; 2-\frac{m}{2}; |\vec{x}|^2\right) \tag{4}$$

3. $$\mathcal{R}_{\alpha,m}(T,\vec{X}) = \frac{(-1)^{\frac{m-1}{2}}}{2^m \pi^{\frac{m}{2}-1}\Gamma\left(\frac{m}{2}\right)} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)} T^{1+\alpha} F\left(\frac{-\alpha}{2}, \frac{-1-\alpha}{2}; \frac{m}{2}; |\vec{x}|^2\right) \tag{5}$$

Proof :

The case $m = 3$ has been proved in the previous section. The rest of the proof will be based on the induction principle. This means that we will prove the theorem for an arbitrary dimension $(m+2) \in 2\mathbb{N}_0 + 3$, under the assumption that the theorem is true for $m \in 2\mathbb{N}+3$.

If the theorem holds for $m \in 2\mathbb{N}+3$, we have from (2) :

$$\begin{aligned}
\Phi_{\alpha,m+2}(T,\vec{X}) &= \frac{1}{2\pi}\sum_{k=0}^{\infty} \frac{\alpha+m+1}{T^{1+k}} \left\{\mathcal{F}_{\alpha+k+1,m} + \mathcal{R}_{\alpha+k+1,m}\right\}(T,\vec{X}) \\
&+ \frac{1}{2\pi}\sum_{k=0}^{\infty} \frac{1+k}{T^{2+k}} \left\{\mathcal{F}_{\alpha+k+2,m} + \mathcal{R}_{\alpha+k+2,m}\right\}(T,\vec{X}) \ .
\end{aligned}$$

Let us first calculate

$$\Sigma_1 = \frac{1}{2\pi}\sum_{k=0}^{\infty} \left\{ \frac{\alpha+m+1}{T^{1+k}}\, \mathcal{F}_{\alpha+k+1,m} + \frac{1+k}{T^{2+k}}\, \mathcal{F}_{\alpha+k+2,m} \right\} .$$

Using the induction hypothesis (7), we find for $\mathcal{F}_{\alpha+k+1,m}$:

$$\frac{\mathcal{F}_{\alpha+k+1,m}}{2\pi T^{1+k}} = \frac{T^{\alpha+m+1}|\vec{x}|^2}{(m-2)mA_{m+2}|\vec{X}|^m} F\left(\frac{-\alpha-k-m}{2}, \frac{1-\alpha-k-m}{2}; 2-\frac{m}{2}; |\vec{x}|^2\right) ,$$

and for $\mathcal{F}_{\alpha+k+2,m}$:

$$\frac{\mathcal{F}_{\alpha+k+2,m}}{2\pi T^{2+k}} = \frac{T^{\alpha+m+1}|\vec{x}|^2}{(m-2)mA_{m+2}|\vec{X}|^m} F\left(\frac{-1-\alpha-k-m}{2}, \frac{-\alpha-k-m}{2}; 2-\frac{m}{2}; |\vec{x}|^2\right) .$$

In order to perform the summation, we would like to rewrite the hypergeometric function appearing in the previous lines. For this purpose, we introduce a differential operator in the real variable u (representing the argument of the hypergeometric function on which this operator is supposed to act, i.e. $u = |\vec{x}|^2$) :

$$O_a = 1 - \frac{2u}{a}\frac{d}{du} \ , \ \forall a \in \mathbb{N} .$$

With the aid of Lemma 5.1.(2), we have :

$$F\left(a, a+\frac{1}{2}; 2-\frac{m}{2}; u\right) = O_{m-2}O_{m-4}\cdots O_3 O_1 F\left(a, a+\frac{1}{2}; \frac{1}{2}; u\right) .$$

Since $\vec{x} \in B(1)$ for all $(\vec{X}, T)$ in the future cone FC, the following expansions are valid :

$$\frac{1}{|\vec{x}|} = \frac{1}{1-(1-|\vec{x}|)} = \sum_{k=0}^{\infty}(1-|\vec{x}|)^k = -\sum_{k=0}^{\infty}(1+|\vec{x}|)^k$$

and

$$\frac{1}{|\vec{x}|^2} = \frac{1}{(1-(1-|\vec{x}|))^2} = \sum_{k=0}^{\infty}(1+k)(1-|\vec{x}|)^k = \sum_{k=0}^{\infty}(1+k)(1+|\vec{x}|)^k .$$

Using Lemma 5.1.(3), we thus find :

$$\sum_{k=0}^{\infty} F\left(\frac{-\alpha-k-m}{2}, \frac{1-\alpha-k-m}{2}; \frac{1}{2}; |\vec{x}|^2\right) = \frac{(1-|\vec{x}|)^{\alpha+m}-(1+|\vec{x}|)^{\alpha+m}}{2|\vec{x}|} ,$$

and

$$\sum_{k=0}^{\infty}(1+k)F\left(\frac{-1-\alpha-k-m}{2}, \frac{-\alpha-k-m}{2}; \frac{1}{2}; |\vec{x}|^2\right) = \frac{(1-|\vec{x}|)^{1+\alpha+m}+(1+|\vec{x}|)^{1+\alpha+m}}{2|\vec{x}|^2} .$$

Multiplying the first summation by $(\alpha+m+1)$ and adding it to the second summation, we find the following result for Σ_1, with the aid of Lemma 5.1 :

$$\Sigma_1 = \frac{T^{\alpha+m+1}|\vec{x}|^2}{(m-2)mA_{m+2}|\vec{X}|^m} O_{m-2}O_{m-4}\cdots O_3O_1 \frac{F\left(\frac{-1-\alpha-m}{2}, \frac{-\alpha-m}{2}; -\frac{1}{2}; |\vec{x}|^2\right)}{|\vec{x}|^2} .$$

Using the definition of the hypergeometric function, one can easily verify that

$$O_n \frac{nF\left(a, b; -\frac{n}{2}; u\right)}{u} = (n+2)\frac{F\left(a, b; -\frac{n}{2}-1; u\right)}{u} ,$$

whence

$$O_{m-2}O_{m-4}\cdots O_3O_1 \frac{F\left(a, b; -\frac{1}{2}; |\vec{x}|^2\right)}{|\vec{x}|^2} = (m-2)\frac{F\left(a, b; 1-\frac{m}{2}; |\vec{x}|^2\right)}{|\vec{x}|^2} .$$

This means that

$$\begin{aligned}\Sigma_1 &= \frac{T^{\alpha+m+1}}{mA_{m+2}|\vec{X}|^m} F\left(\frac{-1-\alpha-m}{2}, \frac{-\alpha-m}{2}; 1-\frac{m}{2}; |\vec{x}|^2\right) \\ &= \mathcal{F}_{\alpha,m+2}(T, \vec{X}) .\end{aligned}$$

Hence, we have the following result:

$$\mathcal{F}_{\alpha,m+2}(T, \vec{X}) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\left\{\frac{\alpha+m+1}{T^{1+k}}\, \mathcal{F}_{\alpha+k+1,m} + \frac{1+k}{T^{2+k}}\, \mathcal{F}_{\alpha+k+2,m}\right\}(T, \vec{X}) . \quad (6)$$

Next, let us calculate Σ_2

$$\Sigma_2 = \frac{1}{2\pi}\sum_{k=0}^{\infty}\left\{\frac{\alpha+m+1}{T^{1+k}}\ \mathcal{R}_{\alpha+k+1,m} + \frac{1+k}{T^{2+k}}\ \mathcal{R}_{\alpha+k+2,m}\right\} .$$

Before we use the induction hypothesis (7), we put the regular part $\mathcal{R}_{\alpha,m}(T,\vec{X})$ into a more useful form. For this purpose, we use Lemma 5.1.(1) :

$$F\left(\frac{-\alpha}{2},\frac{-1-\alpha}{2};\frac{m}{2};u\right) = \frac{\left(\frac{3}{2}\right)_n\left(\frac{d}{du}\right)^n}{\left(\frac{3-\alpha-m}{2}\right)_n\left(\frac{2-\alpha-m}{2}\right)_n}F\left(\frac{3-\alpha-m}{2},\frac{2-\alpha-m}{2};\frac{3}{2};u\right) ,$$

where $n=\frac{m-3}{2}$ and $u=|\vec{x}|^2$. Using the definition for the Pochammer symbol $(a)_n$, one easily finds :

$$\left(\frac{3-\alpha-m}{2}\right)_n\left(\frac{2-\alpha-m}{2}\right)_n = \frac{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{3-m-\alpha}{2}\right)\Gamma\left(\frac{2-m-\alpha}{2}\right)} .$$

Since $\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=2^{1-2x}\pi^{-1/2}\Gamma(2x)$ and $\Gamma(x)\Gamma(1-x)=\pi\sin(\pi x)^{-1}$, we obtain :

$$\begin{aligned}\frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)}\left\{\left(\frac{3-\alpha-m}{2}\right)_n\left(\frac{2-\alpha-m}{2}\right)_n\right\}^{-1} &= 2^{m-3}(\alpha+m-1)\frac{\sin((\alpha+m)\pi)}{\sin((\alpha+1)\pi)}\\ &= 2^{m-3}(\alpha+m-1) ,\end{aligned}$$

where we have used the fact that $m\in 2\mathbb{N}+3$ is odd. This means that $\mathcal{R}_{\alpha,m}(T,\vec{X})$ can also be written as follows :

$$\mathcal{R}_{\alpha,m}(T,\vec{X}) = \frac{(-1)^{\frac{m-1}{2}}T^{1+\alpha}}{4\pi^{\frac{m-1}{2}}}\left(\frac{d}{du}\right)^n(\alpha+m-1)F\left(\frac{3-m-\alpha}{2},\frac{2-m-\alpha}{2};\frac{3}{2};u\right) ,$$

where u stands for $|\vec{x}|^2$ and $n=\frac{m-3}{2}$.

Now using the induction hypothesis together with previous expressions, we find for $\mathcal{R}_{\alpha+k+1,m}$:

$$\frac{\mathcal{R}_{\alpha+k+1,m}}{2\pi T^{1+k}} = \frac{(-1)^{\frac{m-1}{2}}T^{1+\alpha}}{8\pi^{\frac{m+1}{2}}}\left(\frac{d}{du}\right)^n(\alpha+m+k)F\left(\frac{2-m-\alpha-k}{2},\frac{1-m-\alpha-k}{2};\frac{3}{2};u\right) ,$$

and for $\mathcal{R}_{\alpha+k+2,m}$:

$$\frac{\mathcal{R}_{\alpha+k+1,m}}{2\pi T^{1+k}} = \frac{(-1)^{\frac{m-1}{2}}T^{1+\alpha}}{8\pi^{\frac{m+1}{2}}}\left(\frac{d}{du}\right)^n(\alpha+m+k+1)F\left(\frac{1-m-\alpha-k}{2},\frac{-m-\alpha-k}{2};\frac{3}{2};u\right) .$$

Using Lemma 5.1.(4) and the expansions for $\frac{1}{|\vec{x}|} = \frac{1}{\sqrt{u}}$ and $\frac{1}{|\vec{x}|^2} = \frac{1}{u}$, we thus have :

$$\sum_{k=0}^{\infty} F\left(\frac{2-m-\alpha-k}{2}, \frac{1-m-\alpha-k}{2}; \frac{3}{2}; u\right) = -\frac{(1-\sqrt{u})^{\alpha+m} + (1+\sqrt{u})^{\alpha+m}}{2u} ,$$

and

$$\sum_{k=0}^{\infty}(1+k) F\left(\frac{1-m-\alpha-k}{2}, \frac{-m-\alpha-k}{2}; \frac{3}{2}; u\right) = \frac{(1+\sqrt{u})^{1+\alpha+m} - (1-\sqrt{u})^{1+\alpha+m}}{2u^{3/2}} .$$

Consequently, we find for Σ_2 (in terms of $u = |\vec{x}|^2$) :

$$\begin{aligned} \Sigma_2 &= \frac{(-1)^{\frac{m+1}{2}} T^{\alpha+1}}{4\pi^{\frac{m+1}{2}}} \left(\frac{d}{du}\right)^n \left\{ (\alpha+m+1) \frac{(1-\sqrt{u})^{\alpha+m} + (1+\sqrt{u})^{\alpha+m}}{4u} \right\} \\ &+ \frac{(-1)^{\frac{m+1}{2}} T^{\alpha+1}}{4\pi^{\frac{m+1}{2}}} \left(\frac{d}{du}\right)^n \left\{ \frac{(1-\sqrt{u})^{\alpha+m+1} - (1+\sqrt{u})^{\alpha+m+1}}{4u^{3/2}} \right\} . \end{aligned}$$

Comparing this with the expression for $\mathcal{R}_{\alpha,m+2}(T,\vec{X})$, we see that

$$\Sigma_2 = \mathcal{R}_{\alpha,m+2}(T,\vec{X})$$

provided the following equation holds :

$$\begin{aligned} \frac{d}{du} F\left(\frac{-m-\alpha}{2}, \frac{1-m-\alpha}{2}; \frac{3}{2}; u\right) &= \frac{(1-\sqrt{u})^{\alpha+m} + (1+\sqrt{u})^{\alpha+m}}{4u} \\ &+ \frac{(1-\sqrt{u})^{\alpha+m+1} - (1+\sqrt{u})^{\alpha+m+1}}{4(\alpha+m+1)u^{3/2}} \end{aligned}$$

One can easily verify this statement, using Lemma 5.1 and the definition for the hypergeometric function.

Hence, we also have the following result :

$$\mathcal{R}_{\alpha,m+2}(T,\vec{X}) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left\{ \frac{\alpha+m+1}{T^{1+k}} \mathcal{R}_{\alpha+k+1,m} + \frac{1+k}{T^{2+k}} \mathcal{R}_{\alpha+k+2,m} \right\} (\vec{X},T) \ . \quad (7)$$

Together with (2) and (6), this completes the proof. ■

Summarizing, we have for all odd $m \geq 3$:

$$E_{\alpha,m}(T,\vec{X}) = (\epsilon\partial_T - \partial_{\vec{X}})\Phi_{\alpha,m}(\vec{X},T) .$$

Let us now introduce the following definitions (see also [6]) :

Definitions. *For all $\vec{x} \in \mathbb{R}^m$ such that $|\vec{x}| < 1$:*

1. $$F_1(t) = F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}; k+\frac{m}{2}; t\right)$$

2. $$F_2(t) = F\left(\frac{1+k-\alpha}{2}, 1+\frac{k-\alpha}{2}; 1+k+\frac{m}{2}; t\right)$$

3. $$\mathrm{Mod}(\alpha, k, \vec{x}) = F_1(|\vec{x}|^2) + \frac{k-\alpha}{2k+m}\ \vec{x}\epsilon\ F_2(|\vec{x}|^2)$$

One can then easily verify that with

$$\mathcal{F}_{\alpha,m}(T,\vec{X}) \quad = \quad \frac{T^{\alpha+m-1}}{(m-2)A_m|\vec{X}|^{m-2}} F\left(\frac{1-\alpha-m}{2}, \frac{2-\alpha-m}{2}; 2-\frac{m}{2}; |\vec{x}|^2\right)$$

the singular part of the fundamental solution reduces to

$$(\epsilon\partial_T - \partial_{\vec{X}})\mathcal{F}_{\alpha,m}(T,\vec{X}) \quad = \quad T^\alpha \mathrm{Mod}(\alpha, 1-m, \vec{x}) \frac{\vec{x}}{A_m|\vec{x}|^m} \ .$$

This means that the singular part of the fundamental solution $E_{\alpha,m}(\vec{X},T)$ can be written as a modulated version of the Euclidian fundamental solution for the Dirac operator $\partial_{\vec{x}}$ on $\mathbb{R}_{0,m}$. This was already proved in [8].

On the other hand, one can verify that with

$$\mathcal{R}_{\alpha,m}(T,\vec{X}) \quad = \quad \frac{(-1)^{\frac{m-1}{2}}}{2^m \pi^{\frac{m}{2}-1}\Gamma\left(\frac{m}{2}\right)} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)} T^{1+\alpha} F\left(\frac{-\alpha}{2}, \frac{-1-\alpha}{2}; \frac{m}{2}; |\vec{x}|^2\right)$$

the regular part of the fundamental solution $E_{\alpha,m}(T,\vec{X})$ reduces to

$$(\epsilon\partial_T - \partial_{\vec{X}})\mathcal{R}_{\alpha,m}(T,\vec{X}) \quad = \quad \frac{(-1)^{\frac{m-1}{2}}}{2^m \pi^{\frac{m}{2}-1}\Gamma\left(\frac{m}{2}\right)} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+1)} T^\alpha \mathrm{Mod}(\alpha, 0, \vec{x})\epsilon \ .$$

This proves that the regular part of the fundamental solution $E_{\alpha,m}(T,\vec{X})$ is, up to a constant, a modulated version of a spherical monogenic $P_0(\vec{x})$. For more information concerning modulated versions of Euclidian monogenics, we refer to [6].

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