A note on certain classes of transformation formulas involving several variables

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Abstract

This paper gives certain new classes of transformation formulas in the form of multiple-series identities involving several variables. The results obtained, besides being capable of unifying and providing extensions to various transformation and reduction formulas, also yield other new formulas. The applicability of the main results is treated briefly in the concluding section.

1 Introduction

Transformation and reduction (or summation) formulas relating to special functions of one or several variables are of utmost importance. These are invariably used in different applied branches of theoretical physics and engineering sciences. See, for example [1], [3], [7], [8], and [9].

While working on an alternative proof of a recently posed problem concerning a certain hypergoemetric identity, Grosjean and Srivastava [2] were lead to its multiple-series generalization and to its other related extensions. Several transformation and reduction formulas of hypergeometric functions have been deduced in [2] from the main multiple-series identities. Multiple-series identities have also recently been obtained in [5], [8], and were also recorded in [9]. Our motive in this paper is to obtain new classes of transformation formulas in the form of certain general

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multiple-series identities (Theorems 1-4, below). These results can be applied to substantially more general classes of special functions and orthogonal polynomials. Applications are discussed briefly in the concluding section of this paper.

Denoting by (a_p) the p-dimensional vector of complex numbers $(a_1, \ldots, a_p)$, and let the Pochhammer symbol $(\lambda)_n$ stand for

$$(1.1) \qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \left\{ \begin{array}{ll} 1, & \text{if } n=0, \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & \text{if } n \in \{1,2\ldots\} \end{array} \right\},$$

then the generalized hypergeometric series in r variables $z_1, \ldots, z_r$ is defined by ([9, p. 38, eqn. (24)])

$$(1.2) \quad F^{p:p_1;\ldots;p_r}_{q:q_1;\ldots;q_r} \left[\begin{array}{ll} (a_p):(b^1_{p_1});\ldots;(b^r_{p_r}); & \\ & z_1,\ldots,z_r \\ (c_q):(d^1_{q_1});\ldots;(d^r_{q_r}); & \end{array} \right] = \sum_{m_1,\ldots,m_r=0}^{\infty} f(m_1,\ldots,m_r) \prod_{i=1}^{r} \{z_i^{m_i}/m_i!\},$$

where

$$(1.3) \qquad f(m_1,\ldots,m_r) = \frac{\prod_{i=1}^{p}(a_i)_{m_1+\ldots m_r}}{\prod_{i=1}^{q}(c_i)_{m_1+\ldots+m_r}} \prod_{k=1}^{r} \left\{ \frac{\prod_{i=1}^{p_k}(b^k_i)_{m_k}}{\prod_{i=1}^{q_k}(d^k_i)_{m_k}} \right\}.$$

When $r=1$, (1.2) reduces to the generalized hypergeometric function of one variable, and for $r=2$, it reduces to the generalized Kampé de Fériet function of two variables.

For further details about these functions, see [9, Chapter 1]. In our sequel we shall also be using the triple hypergeometric series of Srivastava (see [9,p. 44]) $F^{(3)}[x,y,z]$ which provides unification of Lauricella's fourteen hypergeometric functions $F_1, F_2, \ldots, F_{14}$ and additional functions H_A, H_B and H_C.

2 Main results

For non-negative integers m_j and q_j we define

$$(2.1) \qquad A_i(m_i,q_i) = \sum_{k_i=0}^{[m_i/q_i]} (-m_i)_{q_i k_i} B_i(m_i,k_i), \quad (i \in \{1,\ldots,r\}),$$

where $[x]$ denotes the greatest integer in x, and $B_i(m_i,k_i)(m_i,k_i \geq 0$, $i \in \{1,\ldots,r\})$ are bounded sequences of real (or complex) parameters.

Consider the multiple-series

$$(2.2) \qquad I = \sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r) \prod_{i=1}^{r} \left\{ A_i(m_i,q_i) \frac{x_i^{m_i}}{m_i!} \right\},$$

where $\Delta(m_1,\ldots,m_r)$ is a bounded multiple sequence of real (or complex) parameters, and $|x_i| \le R_i (R_i > 0, \forall i \in \{1,\ldots,r\})$,

Using (2.1), and the elementary relation

$$(-m)_k = \begin{cases} (-1)^k m!/(m-k)!, & 0 \le k \le m, \\ 0, & k > m, \end{cases} \tag{2.3}$$

then (2.2) gives

$$I = \sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r) \cdot \prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{[m_i/q_i]} \frac{(-1)^{q_i k_i}}{(m_i - q_i k_i)!} B_i(m_i,k_i) x_i^{m_i} \right\}. \tag{2.4}$$

The absolute convergence of the series involved permits change in the order of summation, and with this assumption making an appeal to the series arrangement property [10, Lemma 3, p. 10], we are lead to the following multiple-series identity :

Theorem 1. *Corresponding to the bounded sequence $B_i(m_i,k_i)$, let $A_i(m_i,q_i)$ be defined by (2.1), $\forall i \in \{1,\ldots,r)$, where m_i, q_i are non-negative integers. Then*

$$\begin{aligned} \sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r) \prod_{i=1}^{r} \left\{ A_i(m_i,q_i) \frac{x_i^{m_i}}{m_i!} \right\} \\ = \sum_{m_1,k_1,\ldots,m_r,k_r=0}^{\infty} \Delta(m_1+q_1k_1,\ldots,m_r+q_rk_r) \\ \cdot \prod_{i=1}^{r} \left\{ (-1)^{q_ik_i} B_i(m_i+q_ik_i,k_i) \frac{x_i^{m_i+q_ik_i}}{m_i!} \right\}, \end{aligned} \tag{2.5}$$

where, $\Delta(m_1,\ldots,m_r)$ is a single-valued, bounded multiple sequence of real (or complex) parameters such that each of the series involved is absolutely convergent.

Following [2], we consider other interesting variations of Theorem 1. To this end, let us put the sequence

$$\Delta(m_1,\ldots,m_r) = \Omega(m_1,\ldots,m_r) \prod_{i=1}^{r} \{m_i\,!\}, \tag{2.6}$$

and

$$B_i(m_i,k_i) = c_i(k_i)/m_i\,!, \tag{2.7}$$

so that from (2.1), we write

$$A_i^*(m_i,q_i) = \sum_{k_i=0}^{[m_i/q_i]} (-m_i)_{q_ik_i}\, C_i(k_i)/m_i\,!, \quad (\forall i \in \{1,\ldots,r\}), \tag{2.8}$$

then Theorem 1 yields the following :

Theorem 2. *For non-negative integers $m_i, q_i (i = 1, \ldots, r)$ there exists the identity :*

$$(2.9)\qquad \sum_{m_1,\ldots,m_r=0}^{\infty} \Omega(m_1,\ldots,m_r)\prod_{i=1}^{r}\{A_i^*(m_i,q_i)x_i^{m_i}\}$$
$$= \sum_{m_1,k_1,\ldots,m_r,k_r=0}^{\infty} \Omega(m_1+q_1k_1,\ldots,m_r+q_rk_r)\cdot\prod_{i=1}^{r}\left\{(-1)^{q_ik_i}C_i(k_i)\frac{x_i^{m_i+q_ik_i}}{m_i!}\right\},$$

where, $A_i^*(m_i, q_i)$ is defined by (2.8), and $\Omega(m_1, \ldots, m_r)$ is a bounded multiple sequence of real (or complex) parameters such that each of the series is absolutely convergent.

If we set the sequence

$$(2.10)\qquad \Omega(m_1,\ldots,m_r) = C^*(m_1+\ldots+m_r),$$

and invoke the elementary identity

$$(2.11)\qquad \sum_{m_1,\ldots,m_r=0}^{\infty} f(m_1+\ldots+m_r)\prod_{i=1}^{r}\{z_i^{m_i}/m_i!\} = \sum_{m=0}^{\infty}(z_1+\ldots+z_r)^m\frac{f(m)}{m!},$$

then Theorem 2 gives the result :

Theorem 3. *Under the assumptions of Theorem 2, there exists the identity*

$$(2.12)\qquad \sum_{m_1,\ldots,m_r=0}^{\infty} C^*(m_1+\ldots+m_r)\prod_{i=1}^{r}\{A_i^*(m_i,q_i)x_i^{m_i}\}$$
$$= \sum_{k_1,\ldots,k_r,m=0}^{\infty} C^*(m+q_1k_1+\ldots+q_rk_r)$$
$$\cdot\frac{(x_1+\ldots+x_r)^m}{m!}\prod_{i=1}^{r}\{C_i(k_i)(-x_i)^{q_ik_i}\},$$

where $C^*(m)$ is an arbitrary bounded sequence such that the series involved in (2.12) converge absolutely.

Another variation of Theorem 2 can be contemplated if we put

$$(2.13)\qquad \Omega(m_1,\ldots,m_r) = C^*(m_1+\ldots+m_r)(\gamma_1)_{m_1}\ldots(\gamma_r)_{m_r},$$

and

$$(2.14)\qquad x_i = x \quad (i = 1,\ldots,r),$$

in Theorem 2, and make use of the identity [10, p. 61, eqn. (9)] :

$$(2.15)\qquad \sum_{m_1,\ldots,m_r=0}^{\infty} f(m_1+\ldots+m_r)\prod_{i=1}^{r}\{(\gamma_1)_{m_i}x_i^{m_i}/m_i!\}$$
$$= \sum_{m=0}^{\infty} f(m)(\gamma_1+\ldots+\gamma_r)_m\frac{x^m}{m!},$$

to simplify the multiple m-series on the right side of (2.9), and this way we are lead to the following :

Theorem 4. *Under the assumptions of Theorem 2, there exists the identity*

$$\sum_{m_1,\ldots,m_r=0}^{\infty} C^*(m_1+\ldots+m_r)\prod_{i=1}^{r}\{A_i^*(m_i,q_i)(\gamma_i)_{m_i}x^{m_i}\}$$
$$=\sum_{k_1,\ldots,k_r,m=0}^{\infty} C^*(m+q_1k_1+\ldots+q_rk_r)\frac{x^m}{m!}$$
$$\cdot(\gamma_1+q_1k_1+\ldots+\gamma_r+q_rk_r)_m\prod_{i=1}^{r}\{(\gamma_r)_{q_ik_i}C_i(k_i)(-x_i)^{q_ik_i}\},$$

provided that the series involved in (2.16) converge absolutely.

3 Applications

On account of the presence of arbitrary sequences, our Theorems 1-4 would widely be applicable and thus lead to numerous interesting transformation and reduction (or summation) formulas. We shall consider some cases of deductions, thereby revealing the usefulness of our main results.

We observe that if

$$q_i = 2 \quad (i=1,\ldots,r),$$

and

$$C_i(k_i) = \frac{2^{-2k_i}}{k_i!(\lambda_i+1/2)_{k_i}}, \tag{3.1}$$

then from (2.8) by Gauss summation theorem [10, p.30, eqn. (7)], we have

$$A_i^*(m_i,q_i) = \frac{2^{m_i}(\lambda_i)_{m_i}}{m_i!(2\lambda_i)_{m_i}}, \tag{3.2}$$

then Theorems 2, 3 and 4 give the recent known results of Grosjean and Srivasta [2, pp. 289-291].

It is worth noticing that most of the reduction formulas recorded in Srivasta-Karlsson's monograph [9, Chapter 1] would emerge from our Theorems 1 and 2 by suitably specializing the arbitrary sequences in accordance with the involvement of various summation theorems relating to the hypergeometric functions listed, for instance, in [4, Chapter 7]. To illustrate, we deduce one such reduction formula from Theorem 1.

For $q_i = 1$, we put

$$B_i(m_i,k_i) = \frac{(a_i)_{k_i}(b_i)_{k_i}}{(c_i)_{k_i}(1+a_i+b_i-c_i-m_i)_{k_i}k_i!}, \tag{3.3}$$

so that from [4, p. 539, entry 88] :

$$A_i(m_i,1) = \frac{(c_i-a_i)_{m_i}(c_i-b_i)_{m_i}}{(c_i)_{m_i}(c_i-a_i-b_i)_{m_i}}. \tag{3.4}$$

Theorem 1 gives then

$$\sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r)\prod_{i=1}^{r}\{\frac{(c_i-a_i)_{m_i}(c_i-b_i)_{m_i}x_i^{m_i}}{(c_i)_{m_i}(c_i-a_i-b_i)_{m_i}m_i!}\}$$
$$= \sum_{m_1,k_1,\ldots,m_r,k_r=0}^{\infty} \Delta(m_1+k_1,\ldots,m_r+k_r)$$
$$\cdot\prod_{i=1}^{r}\{\frac{(-1)^{k_i}(a_i)_{k_i}(b_i)_{k_i}}{(c_i)_{k_i}(1+a_i+b_i-c_i-m_i-k_i)_{k_i}m_i!k_i!}x_i^{m_i+k_i}\}. \tag{3.5}$$

If $r=1$, and

$$\Delta(m)=\frac{(\alpha_1)_m\ldots(\alpha_P)_m}{(\beta_1)_m\ldots(\beta_Q)_m}, \tag{3.6}$$

then (3.5) in conjunction with the definition (1.2) assumes the form

$$F^{P\ :1;2}_{Q+1\ :0;1}\left[\begin{array}{lcccc}(\alpha_P) & : & c-a-b & ; & a,b\ ; \\ & & & & & x,x \\ (\beta_Q),c-a-b & : & --- & ; & c\ ;\end{array}\right]$$
$$= {}_{P+2}F_{Q+2}\left[\begin{array}{cc}(\alpha_P),c-a,c-b; & \\ & x \\ (\beta_Q),c,c-a-b; & \end{array}\right], P\le Q \tag{3.7}$$

Next, for $q_i=2$, we set,

$$C_i(k_i)=(-1/4t_i^2)^{k_i}/k_i!, \tag{3.8}$$

in (2.8) to get

$$A_i^*(m_i,2)=(2t_i)^{m_i}H_{m_i}(t_i)/m_i!, \tag{3.9}$$

where $H_n(t)$ denotes the Hermite polynomials [10,p.40]. With the substitutions (3.8) and (3.9), Theorem 3 yields

$$\sum_{m_1,\ldots,m_r=0}^{\infty} C^*(m_1+\ldots+m_r)\prod_{i=1}^{r}\{(x_i/2t_i)^{m_i}H_{m_i}(t_i)/m_i!\}$$
$$= \sum_{k_1,\ldots,k_r,m=0}^{\infty} C^*(m+2k_1+\ldots+2k_r)$$
$$\cdot\frac{(x_1+\ldots+x_r)^m}{m!}\prod_{i=1}^{r}\{(-x_i^2/4t_i^2)^{k_i}/k_i!\}. \tag{3.10}$$

When $r=2$, (3.10) takes the form

$$\sum_{m,n=0}^{\infty} A(m+n)H_m(u)H_n(v)\frac{(x/2u)^m}{m!}\frac{(y/2v)^n}{n!}$$
$$= \sum_{m,n,p=0}^{\infty} A(m+2n+2p)\frac{(x+y)^m}{m!}\frac{(-x^2/4u^2)^n}{n!}\frac{(-y^2/4v^2)^p}{p!}. \tag{3.11}$$

Lastly, when $q_i = 1$, and

$$C_i(k_i) = \frac{(\lambda_i)_{k_i}}{(\mu_i)_{k_i} k_i!}, \tag{3.12}$$

in (2.8) then from Gauss summation theorem [10,p.30, eqn. (9)], we have

$$A_i^*(m_i, 1) = \frac{(\mu_i - \lambda_i)_{m_i}}{(\mu_i)_{m_i} m_i!}, \tag{3.13}$$

and choosing the sequence $C^*(m)$ same as in (3.6), we arrive at the following result from Theorem 4 :

$$\begin{aligned} F^{P:2;\ldots;2}_{Q:1;\ldots;1}&\left[\begin{matrix} (\alpha_P) & : & \gamma_1, \mu_1 - \lambda_1; \ldots, \gamma_r, \mu_r - \lambda_r & ; \\ (\beta_Q) & : & \mu_1; \ldots; \mu_r & ; \end{matrix}\ x, \ldots, x\right] \\ &= \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_Q)_m} (\gamma_1 + \ldots + \gamma_r)_m \frac{x^m}{m!} \\ &\cdot F^{P+1:2;\ldots;2}_{Q+1:1;\ldots;1}\left[\begin{matrix} (\alpha_P + m) & , & \sum\gamma_i + m & : & \gamma_1, \lambda_1; \ldots, \gamma_r, \lambda_r & ; \\ (\beta_Q + m) & , & \sum\gamma_i & : & \mu_1; \ldots; \mu_r & ; \end{matrix}\ -x, \ldots, -x\right], \end{aligned} \tag{3.14}$$

where $P \le Q$, and $\sum\gamma_i = \gamma_1 + \ldots + \gamma_r$.

If $P = Q = 0$, and $r = 2$, then in view of [9,p.28, eqn.(31)], (3.14) in terms of the triple hypergeometric series gives

$$\begin{aligned} {}_2F_1&\left[\begin{matrix} \alpha, \gamma - \beta & ; \\ \gamma & ; \end{matrix}\ x\right] {}_2F_1\left[\begin{matrix} \lambda, v - \mu & ; \\ v & ; \end{matrix}\ x\right] \\ &= F^{(3)}\left[\begin{matrix} \alpha + \lambda & :: -; & - & ; -; -; & \alpha, \beta & ; & \lambda, \mu & ; \\ - & :: -; & \alpha + \lambda & ; -; -; & \gamma & ; & v & ; \end{matrix}\ x, -x, -x\right] \end{aligned} \tag{3.15}$$

We conclude this paper by remarking that numerous other transformation and reduction (or summation) formulas involving various special functions can be deduced from Theorems 1-4. More importantly, as pointed out in the derivation of (3.11), the specialisations of arbitrary sequences in Theorems 1-4 can be set as in [6], yielding summation formulas for different classical orthogonal polynomials also.

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