About an integral operator preserving meromorphic starlike functions

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Abstract

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane.Let Σ_k be the class of meromorphic functions f in $\mathcal{U}$ having the form:

$$f(z) = \frac{1}{z} + \alpha_k z^k + \cdots, 0 < |z| < 1, k \geq 0$$

A function $f \in \Sigma = \Sigma_0$ is called starlike if

$$\mathrm{Re}\left[-\frac{zf'(z)}{f(z)}\right] > 0 \text{ in } \mathcal{U}$$

Let denote by Σ_k^* the class of starlike functions inΣ_k and by A_n the class of holomorphic functions g of the form:

$$g(z) = z + a_{n+1}z^{n+1} + \cdots, z \in \mathcal{U}, n \geq 1$$

With suitable conditions on $k, p \in \mathbb{N}$,on $c \in \mathbb{R}$,on $\gamma \in \mathbb{C}$ and on the function $g \in A_{k+1}$, the author shows that the integral operator
$L_{g,c,\gamma} : \Sigma \to \Sigma$ defined by:

$$K_{g,c}(f)(z) \equiv \frac{c}{g^{c+1}(z)} \int_0^z f(t)g^c(t)\mathrm{e}^{\gamma t^p} dt, z \in \mathcal{U}, f \in \Sigma$$

maps Σ_k^* into Σ_l^*, where $l = \min\{p - 1, k\}$.

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1 Introduction

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane.We denote by Σ_k the class of meromorphic functions f in $\mathcal{U}$ having the form:

$$f(z) = \frac{1}{z} + \alpha_k z^k + \cdots \ , 0 < |z| < 1, k \geq 0$$

A function $f \in \Sigma = \Sigma_0$ is called starlike if:

$$\mathrm{Re}\left[-\frac{zf'(z)}{f(z)}\right] > 0 \ , z \in \mathcal{U}$$

Let denote by Σ_k^* the class of starlike functions in Σ_k.
Let A_n be the class of functions

$$g(z) = z + a_{n+1}z^{n+1} + \cdots \ , z \in \mathcal{U}, n \geq 1$$

that are holomorphic in $\mathcal{U}$.Let $k, p \in \mathbb{N}$, $c > 0$, $\gamma \in \mathbb{C}$ and $g \in A_{k+1}$ with $g(z)/z \neq 0$ in $\mathcal{U}$.Let us define the following integral operators:

$$I_{g,c} \ , \ J_{g,c} \ , \ K_{g,c} \text{ and } L_{g,c,\gamma} : \Sigma \to \Sigma$$

by the following equations:

$$I_{g,c}(f)(z) = \frac{c}{g^{c+1}(z)} \int_0^z f(t)g^c(t)g'(t)dt \ , z \in \mathcal{U}, \ f \in \Sigma \tag{1}$$

$$J_{g,c}(f)(z) = \frac{c}{g^{c+1}(z)} \int_0^z \frac{zf(t)g^{c+1}(t)}{t}dt \ , z \in \mathcal{U}, \ f \in \Sigma \tag{2}$$

$$K_{g,c}(f)(z) = \frac{c}{g^{c+1}(z)} \int_0^z f(t)g^c(t)dt \ \ , \ z \in \mathcal{U} \ , \ \ f \in \ \Sigma \tag{3}$$

$$L_{g,c,\gamma}(f)(z) = \frac{c}{g^{c+1}(z)} \int_0^z f(t)g^c(t) \ mathrm e^{\gamma t^p} dt, z \in \mathcal{U}, f \in \Sigma \tag{4}$$

In [1] and [2] the authors found sufficient conditions on c and g so that

$$I_{g,c}\left(\Sigma_k^*\right) \subset \Sigma_k^* \ , \ J_{g,c}\left(\Sigma_k^*\right) \subset \Sigma_k^* \text{ and } K_{g,c}\left(\Sigma_k^*\right) \subset \Sigma_k^*$$

The purpose of this article is to find sufficient conditions on g, c and γ so that $L_{g,c,\gamma}\left(\Sigma_k^*\right) \subset \Sigma_l^*$ where $l = \min\{p-1, k\}$. For $\gamma = 0$ we obtain **Theorem 1** from [2]. In section 4 we give also a new example of an integral operator that preserves meromorphic starlike functions.

2 Preliminaries

For proving our main result we will need the following definitions and lemmas.

If f and g are holomorphic functions in $\mathcal{U}$ and g is univalent, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

The holomorphic function f, with $f(0) = 0$ and $f'(0) \neq 0$ is starlike in $\mathcal{U}$ (i.e. f is univalent in $\mathcal{U}$ and $f(\mathcal{U})$ is starlike with respect to the origin) if and only if $\mathrm{Re}[zf'(z)/f(z)] > 0$ in $\mathcal{U}$.

Lemma 1 [6] *Let h be starlike in $\mathcal{U}$ and let $p(z) = 1 + p_n z^n + \cdots$ be holomorphic in $\mathcal{U}$.If*

$$\frac{zp'(z)}{p(z)} \prec h(z)$$

then $p \prec q$, where

$$q(z) = \exp \frac{1}{n} \int_0^z \frac{h(t)}{t} dt.$$

This result is due to T.J.Suffridge and the proof can be found in [6]

Lemma 2 [3] *Let the function $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$ satisfy the condition:*

$$\operatorname{Re} \psi[is, t; z] \le 0$$

for all real s and $t \le -n(1 + s^2)/2$
If $p(z) = 1 + p_n z^n + \cdots$ is holomorphic in $\mathcal{U}$ and

$$\operatorname{Re} \psi[p(z), zp'(z); z] > 0 \;, z \in \mathcal{U}$$

then $\operatorname{Re} p(z) > 0$ in $\mathcal{U}$.

Lemma 3 [4] *Let B and C be two complex functions in the unit disc $\mathcal{U}$ satisfying:*

$$|\operatorname{Im} C(z)| \le n \operatorname{Re} B(z) \;, z \in \mathcal{U} \;, n \in \mathbb{N}$$

If $p(z) = 1 + p_n z^n + \cdots$ is holomorphic in $\mathcal{U}$ and

$$\operatorname{Re} [B(z)zp'(z) + C(z)p(z)] > 0 \;\; , z \in \mathcal{U}$$

then $\operatorname{Re} p(z) > 0$ in $\mathcal{U}$.

We mention here that **Lemma 3** is a particular case of **Lemma 2**. More general forms of this two lemmas and proofs can be found in [5]

3 Main result

Theorem 1 *Let $\gamma \in \mathbb{C}$, $c > 0$ and let p and k be positive integers. If $g \in A_{k+1}$ is starlike and $g(z)/z \ne 0$ in $\mathcal{U}$ and if $G(z) = zg'(z)/g(z)$ satisfies:*

$$\left| \operatorname{Im} \left[(c+1)g'(z) - \frac{g(z)}{z} \right] \mathrm{e}^{-\gamma z^p} \right| \le (k+1) \operatorname{Re} \frac{g(z)}{z} \mathrm{e}^{-\gamma z^p} \;, z \in \mathcal{U} \tag{5}$$

$$[2 + (k+1)(c+1)] \operatorname{Re} G(z) > 2 [1 + pRe\gamma z^p] \;, z \in \mathcal{U} \tag{6}$$

$$\begin{aligned} &(c+1) [\operatorname{Im} zG'(z) - 2 \operatorname{Im} G(z) \operatorname{Re} (1 - G(z) + \gamma p z^p)]^2 \le \\ &\le \{[2 + (k+1)(c+1)] \operatorname{Re} G(z) - 2 [1 + p \operatorname{Re} \gamma z^p]\} \cdot \\ &\cdot \left\{ \left[k+1+2(c+1) |G(z)|^2 \right] \operatorname{Re} G(z) + 2(c+1) \operatorname{Re} zG'(z)\overline{G(z)} - 2(c+1) |G(z)|^2 (1 + p \operatorname{Re} \gamma z^p) \right\} \end{aligned} \tag{7}$$

then $L_{g,c,\gamma}(\Sigma_k^*) \subset \Sigma_l^*$ where the integral operator $L_{g,c,\gamma}$ is defined by (4) and $l = \min\{p-1, k\}$.

Proof Let $f \in \Sigma_k^*$ and let $F = L_{g,c,\gamma}(f)$. From (4) we deduce:

$$zF'(z) + (c+1)G(z)F(z) = \frac{czf(z)e^{\gamma z^p}}{g(z)} \tag{8}$$

Let $\phi(z) = zf(z) = 1 + \alpha_k z^{k+1} + \cdots$. Since $f \in \Sigma_k^*$ we deduce:

$$\mathrm{Re}\,\frac{z\phi'(z)}{\phi(z)} = Re\left(1 + \frac{zf'(z)}{f(z}\right) < 1$$

and thus

$$\frac{z\phi'(z)}{\phi(z)} \prec \frac{2z}{1+z}$$

By **Lemma 1** we obtain that $\phi(z) \prec (1+z)^{2/(k+1)}$ where the power is considered with its principal branch. Since $k+1 \geq 2$ we deduce:

$$\mathrm{Re}\,\phi(z) = Rezf(z) > 0 \text{ in } \mathcal{U}$$

Let now $P(z) = zF(z)$. From (8) we obtain:

$$e^{-\gamma z^p}\left\{\frac{g(z)}{z}zP'(z) + \left[(c+1)g'(z) - \frac{g(z)}{z}\right]P(z)\right\} = czf(z)$$

Hence:

$$\mathrm{Re}\left\{e^{-\gamma z^p}\frac{g(z)}{z}zP'(z) + e^{-\gamma z^p}\left[(c+1)g'(z) - \frac{g(z)}{z}\right]P(z)\right\} > 0 \text{ in } \mathcal{U}$$

Then, from (5) and **Lemma 3** it follows immediately that:
$\mathrm{Re}\,P(z) = \mathrm{Re}[zF(z)] > 0$ in $\mathcal{U}$. Hence, the function

$$p(z) = -\frac{zF'(z)}{F(z)} = 1 + q_{l+1}z^{l+1} + \cdots$$

is holomorphic in $\mathcal{U}$ and (8) becomes:

$$F(z)\left[(c+1)G(z) - p(z)\right] = \frac{czf(z)e^{\gamma z^p}}{g(z)}$$

Taking the logarithmic derivative, we obtain:

$$p(z) + \frac{zp'(z) - (c+1)zG'(z)}{(c+1)G(z) - p(z)} + 1 - G(z) + \gamma p z^p = -\frac{zf'(z)}{f(z)}$$

Because $f \in \Sigma_k^*$, we deduce:

$$\mathrm{Re}\left[p(z) + \frac{zp'(z) - (c+1)zG'(z)}{(c+1)G(z) - p(z)} + 1 - G(z) + \gamma p z^p\right] > 0 \text{ in } \mathcal{U} \tag{9}$$

Let now define $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$ by

$$\psi[u, v; z] = u + \frac{v - (c+1)zG'(z)}{(c+1)G(z) - u} + 1 - G(z) + \gamma p z^p$$

From (9) we have:

$$\operatorname{Re}\psi[p(z), zp'(z); z] > 0 \text{ in } \mathcal{U} \tag{10}$$

In order to show that (10) implies $\operatorname{Re} p(z) > 0$ in $\mathcal{U}$ it is sufficient to check the inequality:

$$\operatorname{Re}\psi[is, t; z] = \operatorname{Re}\frac{t-(c+1)zG'(z)}{(c+1)G(z)-is} + 1 - \operatorname{Re} G(z) + \operatorname{Re}\gamma p z^p \le 0 \tag{11}$$

for all real s and $t \le -(k+1)(c+1)/2$ and then to apply **Lemma 2**.
If we denote:

$$D = |(c+1)G(z)-is|^2 = (c+1)^2 |G(z)|^2 - 2(c+1)\operatorname{Im} G(z) + s^2 \tag{12}$$

then we have:

$$\begin{aligned}\operatorname{Re}\psi[is,t;z] = \frac{1}{D}\operatorname{Re}\{&t(c+1)\overline{G(z)} + ist - (c+1)^2 zG'(z)\overline{G(z)} - (c+1)iszG'(z) + \\ &+(1-G(z)+\gamma p z^p)\left[(c+1)\overline{G(z)}+is\right]\left[(c+1)G(z)-is\right]\}\end{aligned}$$

Because $t \le -(k+1)(1+s^2)/2$ and g is starlike(i.e. $\operatorname{Re} G(z) > 0$ in $\mathcal{U}$), we have:

$$\begin{aligned}2D\operatorname{Re}\psi[is,t;z] \le -\{&s^2\left[(2+(k+1)(c+1))\operatorname{Re} G(z) - 2\,(1+p\operatorname{Re}\gamma z^p)\right] - \\ &-2s(c+1)[\operatorname{Im} zG'(z) - 2\operatorname{Im} G(z)\operatorname{Re}(1-G(z)+\gamma p z^p)] + \\ &+(c+1)\left[\left(k+1+2(c+1)\,|G(z)|^2\right)\operatorname{Re} G(z) + 2(c+1)\operatorname{Re} zG'(z)\overline{G(z)}\right] - \\ &-2(c+1)^2 |G(z)|^2 (1+p\operatorname{Re}\gamma z^p)\}\end{aligned}$$

Then, from (6) and (7) it follows immediately that $\operatorname{Re}\psi[is,t;z] \le 0$ for all real s and $t \le -(k+1)(s^2+1)/2$
Hence,by **Lemma 2** we obtain that p has positive real part in $\mathcal{U}$, and thus $F \in \Sigma_k^*$ and the theorem is proved.

4 Some particular cases

1. If we let $\gamma = 0$, by applying **Theorem 1** we obtain the result from [2].
2. If we let $c = k = p-1 = 1$, $g(z) = z\exp\frac{\lambda z^2}{2}$ and $\gamma = -\lambda/2$, then $G(z) = 1+\lambda z^2$ and for $|\lambda| < 1$ we have immediately that $\operatorname{Re} G(z) > 0$ in $\mathcal{U}$. Hence, g is starlike in $\mathcal{U}$ for $|\lambda| < 1$
Let $\lambda z^2 = \rho e^{i\theta}, 0 < \rho < 1, \theta \in \mathbb{R}$ and let $\tau = \rho\sin\theta \in (-1, 1)$.
Condition (5) is equivalent to:

$$|2\rho\sin(\theta+\tau) + \sin\tau| \le 2\cos\tau$$

It is easy to show that this inequality holds for all $\theta \in \mathbb{R}$ and $\rho \le (\sqrt{2}-1)/2$.
Condition (6) is equivalent to:

$$4(1+\rho\cos\theta) > 0$$

which is true for all $\rho \in (0,1)$.
Condition (7) is equivalent to:

$$\rho^4 \sin^2 2\theta - 4\rho^3 \cos^3\theta - 3\rho^2(2\cos^2\theta + 1) - 6\rho\cos\theta - 1 \leq 0$$

It is easy to show that this last inequality holds for all $\rho \leq (\sqrt{2}-1)/2)$. Hence, by applying **Theorem 1** we deduce the following result:

Corollary 1 *If $\lambda \in \mathbb{C}$ with $|\lambda| \leq (\sqrt{2}-1)/2 = 0.2071\ldots$ and if L is the integral operator defined by $F = L(f)$, where*

$$F(z) = \frac{1}{z^2 e^{\lambda z^2}} \int_0^z t f(t) dt$$

then $L(\Sigma_1^*) \subset \Sigma_1^*$.

References

[1] H.Al–Amiri, P.T.Mocanu, *Integral operators preserving meromorphic starlike functions*, Mathematica (Cluj-Romania) (to appear).

[2] E.Draghici, *An integral operator preserving meromorphic starlike functions*, (to appear).

[3] S.S.Miller, P.T.Mocanu, *Second order differential inequalities in the complex plane*, J.Math.Anal.Appl. 65(1978), pp.289–305.

[4] S.S.Miller, P.T.Mocanu, *Differential subordinations and inequalities in the complex plane*, J. of Diff.Equs., 67, 2(1987), pp 199–211.

[5] S.S.Miller, P.T.Mocanu, *The theory and applications of second order differential subordinations*, Studia Univ. Babeş-Bolyai, Math., 34, 4(1989), pp. 3–33.

[6] T.J.Suffridge, *Some remarks on convex maps of the unit disc*, Duke Math. J. , 37(1979), pp. 775–777.

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