The Ricci Curvature of Totally Real 3-dimensional Submanifolds of the Nearly Kaehler 6-Sphere

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Abstract

Let M be a compact 3-dimensional totally real submanifold of the nearly Kaehler 6-sphere. If the Ricci curvature of M satisfies $Ric(M) \geq \frac{53}{64}$, then M is a totally geodesic submanifold (and $Ric(M) \equiv 2$).

1. Introduction

On a 6-dimensional unit sphere S^6, we can construct a nearly Kaehler structure J by making use of the *Cayley number* system (see [3] or [7]).

Let M be a compact 3-dimensional Riemannian manifold. M is called a totally real submanifold of S^6 if $J(TM) \subseteq T^{\perp}M$, where TM and $T^{\perp}M$ are the tangent bundle and the normal bundle of M in S^6, respectively. In [2], Ejiri proved that a 3-dimensional totally real submanifold of S^6 is orientable and minimal. In [1], Dillen-Opozda-Verstraelen-Vrancken proved the following sectional curvature pinching theorem

*This work is supported by Postdoctoral Foundation of China
Received by the editors June 1994.
Communicated by M. De Wilde.
1991 *Mathematics Subject Classification* : 53C42.

Theorem 1([1]). *Let M be a compact 3-dimensional totally real submanifold of S^6. If the sectional curvature K of M satisfies*

$$\frac{1}{16} < K \leq 1, \tag{1}$$

then M is a totally geodesic submanifold (i.e. $K \equiv 1$ on M).

In this paper, we prove the following Ricci curvature pinching Theorem

Theorem 2. *Let M be a compact 3-dimensional totally real submanifold of S^6. If the Ricci curvature of M satisfies*

$$Ric(M) \geq \frac{53}{64}, \tag{2}$$

then M is a totally geodesic submanifold (i.e. $Ric(M) \equiv 2$ on M).

2. Preliminaries

Suppose that M is an n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. We denote by UM the unit tangent bundle over M and by UM_p its fibre at $p \in M$. We denote by $<,>$ the metric of S^{n+p} as well as that induced on M. If h is the second fundamental form of M and A_ξ the Weingarten endomorphism associated to a normal vector ξ, we define

$$L : T_pM \longmapsto T_pM \quad and \quad T : T_p^\perp M \times T_p^\perp M \longmapsto R$$

by the expressions

$$Lv = \sum_{i=1}^{n} A_{h(v,e_i)}e_i \quad and \quad T(\xi,\eta) = trace A_\xi A_\eta,$$

where $T_p^\perp M$ is the normal space to M at p and $e_1, \ldots, e_n$ is an orthonormal basis of T_pM.

In [5], Montiel-Ros-Urbano proved the following results

Lemma 1([5]). *Let M be an n-dimensional compact minimal submanifold in S^{n+p}. We have*

$$\begin{aligned}
0 = &\frac{n+4}{3}\int_{UM} |(\nabla h)(v,v,v)|^2 dv + (n+4)\int_{UM} |A_{h(v,v)}v|^2 dv \\
&- 4\int_{UM} < Lv, A_{h(v,v)}v > dv - 2\int_{UM} T(h(v,v),h(v,v))dv \\
&+ 2\int_{UM} (< Lv, v > - |h(v,v)|^2)dv,
\end{aligned} \tag{3}$$

where dv denotes the canonical locally product measure on the unit tangent bundle UM over M.

Lemma 2([5]). *Let M be an n-dimensional compact minimal submanifold in S^{n+p}. Then, for any $p \in M$, we have*

$$\int_{UM_p} < Lv, A_{h(v,v)}v > dv_p = \frac{2}{n+2}\int_{UM_p} |Lv|^2 dv_p, \tag{4}$$

$$\int_{UM_p} |h(v,v)|^2 dv_p = \frac{2}{n+2}\int_{UM_p} < Lv, v > dv_p, \tag{5}$$

$$\int_{UM_p} < Lv, v > dv_p = \frac{1}{n}\int_{UM_p} |h|^2 dv_p, \tag{6}$$

$$\int_{UM_p} |A_{h(v,v)}v|^2 dv_p \geq \frac{2}{n+2}\int_{UM_p} < Lv, A_{h(v,v)}v > dv_p, \tag{7}$$

and the equality in (7) holds if and only if $Lv = \frac{n+2}{2}A_{h(v,v)}v$ for any $v \in UM_p$.

It is well-known that we can construct a nearly Kaehler structure J on a 6-dimensional unit sphere S^6 by making use of the Cayley system (see [3], [7] or [1] for details). Let G be the $(2,1)$-tensor field on S^6 defined by

$$G(X,Y) = (\overline{\nabla}_X J)Y, \tag{8}$$

where $X, Y \in T(S^6)$ and $\overline{\nabla}$ is the Levi-Civita connection on S^6. This tensor field has the following properties (see [1])

$$G(X,X) = 0, \tag{9}$$

$$G(X,Y) + G(Y,X) = 0, \tag{10}$$

$$G(X,JY) + JG(X,Y) = 0. \tag{11}$$

3. 3-dimensional totally real submanifolds of S^6

Let M be a 3-dimensional totally real submanifold of S^6. In [2], Ejiri proved that M is orientable and minimal, and that $G(X,Y)$ is orthogonal to M, i.e.

$$G(X,Y) \in T^{\perp}M, \qquad for \quad X, Y \in TM. \tag{12}$$

We denote the Levi-Civita connection of M by ∇. The formulas of Gauss and Weingarten are then given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{13}$$

$$\overline{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{14}$$

where X and Y are vector fields on M and ξ is a normal vector field on M. The second fundamental form h is related to A_ξ by

$$< h(X,Y), \xi >=< A_\xi X, Y > . \tag{15}$$

From (12)-(14), we find

$$D_X(JY) = G(X,Y) + J\nabla_X Y, \tag{16}$$

$$A_{JX}Y = -Jh(X,Y). \tag{17}$$

Since M is a 3-dimensional totally real submanifold of S^6, $JT^\perp M = TM$ and $JTM = T^\perp M$. We can easily verify that (17) is equivalent to

$$< h(X,Y), JZ >=< h(X,Z), JY >=< h(Y,Z), JX > . \tag{18}$$

Let S and R be the Ricci tensor and the scalar curvature of M. It follows from the Gauss equation that

$$S(X,Y) = 2 < X, Y > - < LX, Y >, \tag{19}$$

$$R = 6 - |h|^2, \tag{20}$$

where $|h|^2$ is the length square of h.

In order to prove our Theorem 2, we also need the following lemma which comes from lemma 2 and lemma 6 of [1]

Lemma 3([1]). *Let M be a compact 3-dimensional totally real submanifold of S^6. Then we have*

$$\int_{UM} |(\nabla h)(v,v,v)|^2 dv = \frac{9}{4}\int_{UM} < (\nabla h)(v,v,v), Jv >^2 dv + \frac{9}{28}\int_{UM} |h(v,v)|^2 dv. \tag{21}$$

4. Proof of Theorem 2

Let M be a 3-dimensional totally real submanifold of S^6. By Ejiri's result [2], we know that M is orientable and minimal. Let $n = 3$ in lemma 1, we get by use of (5) and (6)

$$\begin{aligned} 0 \quad = \quad & \frac{7}{3}\int_{UM} |(\nabla h)(v,v,v)|^2 dv + \frac{2}{5}\int_{UM} |h|^2 dv - 2\int_{UM} T(h(v,v), h(v,v)) dv \\ & +7\int_{UM} |A_{h(v,v)}v|^2 dv - 4\int_{UM} < Lv, A_{h(v,v)}v > dv. \end{aligned} \tag{22}$$

Let Q be the function which assigns to each point of M the infimum of the Ricci curvature of M at that point. Then $Ric(M) \geq Q$, at $p \in M$. ¿From (19) and (20), we have

$$0 \leq < Lv, v > \leq 2 - Q \tag{23}$$

for all $v \in UM$. If e_1, e_2, e_3 is an orthonormal basis of T_pM, $p \in M$ such that $Le_i = \lambda_i e_i$, we have $\lambda_i =< Le_i, e_i > \geq 0$ and

$$|Lv|^2 = \sum_{i=1}^{3} \lambda_i^2 < v, e_i >^2 \leq (2 - Q) \sum_{i=1}^{3} \lambda_i < v, e_i >^2 = (2 - Q) < Lv, v >, \tag{24}$$

where the equality implies that $\lambda_i = 2 - Q$ for all $i = 1, 2, 3$, i.e. the Ricci curvature of M is equal to Q at $p \in M$.

By (7), (4), (24) and (6), we have

$$\begin{aligned}
&7 \int_{UM_p} |A_{h(v,v)}v|^2 dv_p - 4 \int_{UM_p} < Lv, A_{h(v,v)}v > dv_p \\
&\geq -\frac{6}{5} \int_{UM_p} < Lv, A_{h(v,v)}v > dv_p \\
&= -\frac{12}{25} \int_{UM_p} |Lv|^2 dv_p \\
&\geq -\frac{12}{25}(2 - Q) \int_{UM_p} < Lv, v > dv_p \\
&= -\frac{4}{25}(2 - Q) \int_{UM_p} |h|^2 dv_p.
\end{aligned} \tag{25}$$

where the equality implies that M is Einsteinian.

Combining (25) with (22), we get

$$\begin{aligned}
0 \quad \geq \quad & \frac{7}{3} \int_{UM} |(\nabla h)(v, v, v)|^2 dv + \int_{UM} (\frac{2}{5} - \frac{4}{25}(2 - Q))|h|^2 dv \\
& -2 \int_{UM} T(h(v, v), h(v, v)) dv.
\end{aligned} \tag{26}$$

Let $h(v, v) = |h(v, v)|\xi$, for some unit normal vector ξ. From (18) and (23)

$$\begin{aligned}
T(h(v, v), h(v, v)) &= |h(v, v)|^2 T(\xi, \xi) \\
&= |h(v, v)|^2 < L(J\xi), J\xi > \\
&\leq (2 - Q)|h(v, v)|^2.
\end{aligned} \tag{27}$$

Putting (27) and (21) into (26), we obtain by use of (5) and (6)

$$
\begin{aligned}
0 \quad \geq \quad & \frac{21}{4}\int_{UM} < (\nabla h)(v,v,v), Jv >^2 dv + \frac{3}{4}\int_{UM} |h(v,v)|^2 dv \\
& + \int_{UM} (\frac{2}{5} - \frac{4}{25}(2-Q))|h|^2 dv - 2\int_{UM} (2-Q)|h(v,v)|^2 dv \\
= \quad & \frac{21}{4}\int_{UM} < (\nabla h)(v,v,v), Jv >^2 dv + \int_{UM} [\frac{2}{5} - \frac{4}{25}(2-Q)]|h|^2 dv \\
& + \int_{UM} (\frac{3}{4} - 2(2-Q))|h(v,v)|^2 dv \\
= \quad & \frac{21}{4}\int_{UM} < (\nabla h)(v,v,v), Jv >^2 dv + \frac{1}{150}\int_{UM} (64Q - 53)|h|^2 dv.
\end{aligned} \tag{28}
$$

Thus, under the hypothesis (2), (28) must be an equality, which implies that (24) and (25) are equalities. Hence, M is Einsteinian. It follows from (28) that either $|h|^2 = 0$, i.e. M is totally geodesic, in this case, $Ric(M) \equiv 2$; or

$$Ric(M) \equiv \frac{53}{64}$$

on M. In the latter case, since M is a 3-dimensional Einsteinian manifold, we know that the sectional curvature of M is

$$K \equiv \frac{53}{128}$$

on M, but by Theorem 1 of [1] or a result of Ejiri [2], this case can not happen. We conclude that M is totally geodesic. We complete the proof of Theorem 2.

Remark 1. By Myers' Theorem, we can assume "complete" instead of "compact" in Theorem 2.

Remark 2. For a compact minimal 3-dimensional submanifold M of $(3+p)$-dimensional unit sphere S^{3+p}, if the Ricci curvature of M satisfies $Ric(M) \geq 1$, the author [4] obtained a classification theorem.

Remark 3. F.Dillen, L.Verstraelen and L.Vrancken obtained a sharper result than Theorem 1 in their paper "Classification of totally real 3-dimensional submanifolds of S^6 with $K \geq 1/16$" , J. Math. Soc. Japan, 42(1990), 565-584.

Acknowlegement. I would like to thank referee for his useful suggestions and comments.

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