Relative normalizing extensions

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Abstract

The purpose of this note is to study relative normalizing ring homomorphisms and the functorial behaviour of the classical sheaves constructed in [19], with respect to these.

1 Introduction

In [20] we announced that the functoriality results proved for relative centralizing extensions could be generalized to ring morphisms, which are only assumed to be strongly normalizing with respect to a fixed symmetric biradical (λ, ρ). To prove this assertion, we need some "going up" and "going down" results, hence we included a section devoted to the development of the necessary techniques (similar to the ones used in [1]) which permit us to obtain analogous properties as in the relative centralizing case for relative normalizing extensions.

This note is organised as follows. In the first section, we recall some generalities on abstract localization. In section 2, we study relative normalizing extensions and in the last section we use the results proved before to show that the sheaves do indeed behave functorially in the classical case. As the results contained in section 3 of [20], i.e., the symmetric case, are easily translated to the present context we only give details in the classical case.

2 Some Background on Localization

2.1 We briefly recall here some of the notions and results about abstract localization which will be needed afterwards. For more details, we refer the reader to [3, 5, 7, 8].

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A radical σ in R-**mod** is a left exact subfunctor of the identity with the property that $\sigma(M/\sigma M) = 0$, for any left R-module M. A left R-module M is said to be *σ-torsion* or *σ-torsionfree*, whenever $\sigma(M) = M$ resp. $\sigma(M) = 0$. We let $\mathcal{T}_\sigma$ resp. $\mathcal{F}_\sigma$ denote the class of all σ-torsion resp. σ-torsionfree left R-modules. Each of these determines σ.

Denote by $\mathcal{L}(\sigma)$ the Gabriel filter associated to σ, i.e., the set of all left ideals L of R with the property that R/L is σ-torsion. It is well-known that $\mathcal{L}(\sigma)$ determines σ completely. Indeed, if M is a left R-module M and $m \in M$, then $m \in \sigma(M)$ if and only if $Ann_R(m) \in \mathcal{L}(\sigma)$.

2.2 If σ is a radical in R-**mod** and E is a left R-module, we say that E is *σ-injective* resp. *σ-closed* if the canonical morphism

$$Hom_R(M, E) \to Hom_R(N, E)$$

is surjective, resp. bijective for any morphism $N \to M$ with σ-torsion kernel and cokernel (such morphisms are usually called *σ-isomorphisms*). It is easy to see that E is σ-closed if and only if E is σ-injective and σ-torsionfree.

Associated to the radical σ, there is a localization functor $Q_\sigma(-)$ in R-**mod** which maps every left R-module M to a σ-closed left R-module $Q_\sigma(M)$, endowed with a σ-isomorphism $M \to Q_\sigma(M)$. One usually calls $Q_\sigma(M)$ the *module of quotients* of M at σ.

The module $Q_\sigma(R)$ has a canonical ring structure extending that of R and for any left R-module M, we have that $Q_\sigma(M)$ is a left $Q_\sigma(R)$-module.

We will also need the fact that if M is an R-bimodule, then so is $Q_\sigma(M)$. Hence, the localization functor $Q_\sigma(-)$ restricts to an internal functor in the category R-**mod**-R of R-bimodules. If M is an R-bimodule then $Q_\sigma(M)$ is canonically isomorphic to $Q_{\bar\sigma}(M)$, where $\bar\sigma$ is the induced radical in R-**mod**-R. Moreover, if M is σ-torsionfree then $Q_\sigma(M) \subseteq E^2(M)$, where $E^2(M)$ is the injective hull of M in R-**mod**-R.

2.3 Let $\mathcal{L}^2(\sigma)$ be the set of twosided ideals of R belonging to $\mathcal{L}(\sigma)$. We say that σ is *symmetric* if $\mathcal{L}^2(\sigma)$ is a filter basis for $\mathcal{L}(\sigma)$, i.e. , if for any $L \in \mathcal{L}(\sigma)$ we can find $I \in \mathcal{L}^2(\sigma)$ such that $I \subseteq L$. Let us denote by $K(\sigma)$ the set of all primes P in R with the property that $R/P \in \mathcal{F}_\sigma$. It is well known that if σ is symmetric or if the ring R is noetherian, given a prime ideal P of R, we have $P \in K(\sigma)$ if and only if $P \notin \mathcal{L}^2(\sigma)$.

We say that the ring R is *σ-noetherian*, if R satisfies the ascending chain condition for σ-closed subobjects or, equivalently, each left ideal of R is σ-finitely generated. A left R-module M is σ-finitely generated, if it contains a finitely generated left R-module N with the property that $M/N \in \mathcal{T}_\sigma$. The radical σ is said to be of *finite type*, if $\mathcal{L}(\sigma)$ has a basis consisting of finitely generated left ideals of R. It is clear that if R is σ-noetherian, then σ is of finite type.

2.4 A radical λ in R-**mod** is said to be a *biradical* (in the sense of [9]), if there exists a radical ρ in **mod**-R with the property that $\lambda(R/I) = \rho(R/I)$, for any twosided ideal I of R. We also call the couple (λ, ρ) a biradical over R. For example, if C is an Ore set in a noetherian ring R, then the radical λ_C in R-**mod** is a biradical.

In this note, we will work throughout with a symmetric biradical (λ, ρ) *with respect to which* R *is* (λ, ρ)*-noetherian and* (λ, ρ)*-closed.*

With these properties it is easily proved that for any left ideal L of R and any left R-module M, we have $LQ_\lambda(M) \subseteq Q_\lambda(LM)$ (see [19, (2.3)]). It also follows that $Q_\lambda(K) = Q_\rho(K)$ for any twosided ideal K of R (see [18, (2.6)]).

2.5 If μ is any radical in (R, λ)-**mod**, we can define a radical $\hat{\mu}$ in R-**mod** by letting $\hat{\mu}$ consist of all left R-modules M with the property that $Q_\lambda(M)$ is μ-torsion. We have typical examples of this situation when we take a multiplicatively closed filter $\mathcal{L} \supseteq \mathcal{L}^2(\lambda)$ of twosided ideals of R. Indeed, $\mathcal{L}$ defines a radical $\mu_{\mathcal{L}}$ in (R, λ)-**mod**, by letting a λ-closed R-module M be $\mu_{\mathcal{L}}$-torsion if and only if for any $m \in M$ there exists some $L \in \mathcal{L}$ such that $Lm = 0$. Let us denote by $\lambda_{\mathcal{L}}$ the radical $\hat{\mu_{\mathcal{L}}}$ induced by $\mu_{\mathcal{L}}$ in R-**mod**.

Let $Y \subseteq K(\lambda)$ be any generically closed subset of $K(\lambda)$ and denote by $\mathcal{L}(Y)$ the set of all twosided ideals I of R such that $I \not\subset P$ for all $P \in Y$. Then $\mathcal{L}(Y)$ defines the radical $\lambda_Y \overset{def}{=} \lambda_{\mathcal{L}(Y)}$ and it is easy to see that $K(\lambda_Y) = Y$. If I is a twosided ideal of R and

$$X(I, \lambda) = \{P \in K(\lambda)\, ;\, I \not\subseteq P\},$$

then we write λ_I for $\lambda_{X(I,\lambda)}$. Similarly, if $P \in K(\lambda)$ and $Y = \{P\}$, we write λ_{R-P} for $\lambda_{\{P\}}$.

2.6 All radicals defined before are examples of radicals which are *relatively symmetric with respect to* λ. Such a radical σ has the property that for any $L \in \mathcal{L}(\sigma)$, we can find $I \in \mathcal{L}^2(\sigma)$ such that $I \subseteq Q_\lambda(L)$. These radicals share most of the properties of the symmetric radicals in the absolute case. In particular, if σ is relatively symmetric with respect to λ, then $\sigma = \wedge_{P \in K(\sigma)} \lambda_{R-P}$. Also, if M is a λ-finitely generated, λ-torsionfree left R-module, then M is σ-torsion if and only if $Ann_R(M) \in \mathcal{L}^2(\sigma)$.

3 Relative normalizing extensions

3.1 Let $\varphi : R \to S$ be a ring homomorphism and let σ be a radical in R-**mod**. It is well-known that, in general, there is no ring homomorphism $\bar{\varphi} : Q_\sigma(R) \to Q_{\bar{\sigma}}(S)$ induced by φ. In [20] this problem is solved introducing relative centralizing extensions. Let us now consider a more general notion of ring morphisms, the so called relative strongly normalizing extensions.

First of all, let us define an R-bimodule M to be *left normalizing* (resp. *strongly left normalizing*) *with respect to* (λ, ρ) if M is (λ, ρ)-torsionfree and there exists a normalizing (resp. strongly normalizing) R-subbimodule $N \subseteq M$ such that $M/N \in T_\lambda$. Recall from [1] that an R-bimodule M is said to be normalizing (resp. strongly normalizing) if $M = RN_R(M)$ (resp. $M = RN_R^s(M)$), where $N_R(M)$ (resp. $N_R^s(M)$) is the set of all $m \in M$ with the property that $mR = Rm$ (resp. with the property that $Im = mI$, for all twosided ideals I of R.) Right normalizing and strongly right normalizing R-bimodules with respect to (λ, ρ) are defined similarly and we will say that an R-bimodule M is (λ, ρ)-normalizing (resp. (λ, ρ)- strongly normalizing) or normalizing with respect to (λ, ρ) (resp. strongly normalizing with respect to (λ, ρ)) if it has the property on both sides.

Let us say that $\varphi : R \to S$ is *normalizing* (resp. *strongly normalizing*) *with respect to* (λ, ρ) if S is (λ, ρ)-closed and (λ, ρ)-normalizing (resp. (λ, ρ)-strongly normalizing) as an R-bimodule.

3.2 In [20] it has been proved that if $\varphi : R \to S$ is centralizing with respect to (λ, ρ), we can define a map:

$$^a\varphi : K(\bar{\lambda}) \to K(\lambda) \, : \, Q \mapsto \varphi^{-1}(Q),$$

where $\bar{\lambda}$ is, as usual, the radical induced by λ in S-**mod**. The same result is easily proved for strongly normalizing extensions with respect to (λ, ρ). We want to know what happens for normalizing homomorphisms with respect to (λ, ρ).

3.3 Note that if $\varphi : R \to S$ is (λ, ρ)-normalizing (resp. strongly normalizing), then S/S' is (λ, ρ)-torsion for some normalizing (resp. strongly normalizing) R-bimodule S'. If we may choose S' of the form $S' = \sum_{i=1}^{n} Ra_i = \sum_{i=1}^{n} a_i R$ for some $\{a_1, \ldots, a_n\} \subseteq N_R(S)$ (resp. in $N_R^s(S)$), we will say that φ is a *finite* (λ, ρ)-normalizing homomorphism or a *finite* (λ, ρ)-strongly normalizing homomorphism. Suppose $\varphi : R \to S$ is a finite (λ, ρ)-normalizing homomorphism and canonically factorize φ as

$$R \xrightarrow{\pi} \bar{R} \overset{i}{\hookrightarrow} S$$

where $\bar{R} = R/ker(\varphi)$. Denote by λ'', resp. by ρ'', the radical induced by λ resp. by ρ in R-**mod**, resp. in **mod**-R. Then $i : \bar{R} \hookrightarrow S$ is an injective finite (λ'', ρ'')-normalizing extension with the property that $\bar{R}$ is (λ'', ρ'')-torsionfree.

Assume that R and S are endowed with symmetric biradicals (λ, ρ) resp. (λ', ρ'), with respect to which they are both noetherian and closed. Let $\varphi : R \to S$ be a ring homomorphism, which is a finite normalizing (λ, ρ)-homomorphism and such that $(\lambda', \rho') \geq (\bar{\lambda}, \bar{\rho})$. In this case, if we consider the injective component $\bar{R} \hookrightarrow S$ of φ, we are in the same situation as described above, with the difference however, that the ring $\bar{R}$ is only (λ'', ρ'')-torsionfree.

From now to the end of the section, we suppose R and S to be endowed with symmetric biradicals (λ, ρ) resp. (λ', ρ'), with respect to which they are both noetherian, S is closed and R is torsionfree. Let $\varphi : R \subseteq S$ be a ring monomorphism, which is normalizing with respect to (λ, ρ), of finite type, such that $(\lambda', \rho') \geq (\bar{\lambda}, \bar{\rho})$ and with the property that (λ, ρ) is φ-compatible.

3.4 In the situation described previously, we may define for all twosided ideals I of R and for all $1 \leq i \leq n$ twosided ideals

$$I_i = \{r \in R \, ; \, ra_i \in a_i I\}$$

and,

$$I_i' = \{r \in R \, ; \, a_i r \in I a_i\}.$$

We have that $a_iI = Ia_i$ and $Ia_i = a_iI_i'$. Assume that the following affirmations are equivalent:

1. $I \in \mathcal{L}^2(\lambda)$;
2. $I_i \in \mathcal{L}^2(\lambda)$;
3. $I_i' \in \mathcal{L}^2(\lambda)$.

Note that in the centralizing and the strongly normalizing cases we always have this property ($I = I_i = I_i'$).

Following the steps of [1, 2], we prove the following results.

Suppose that the S-module M is (λ, ρ)-closed as an R-bimodule. If $N \subseteq M$ is an R-submodule of M, we can define for all $1 \leq i \leq n$

$$a_i^{-1}N = \{m \in M\,;\, a_im \in N\}.$$

Lemma 3.5 *For ${}_RN \subseteq M$ and for all $1 \leq i \leq n$, we have $a_i^{-1}Q_\lambda(M) = Q_\lambda(a_i^{-1}N)$.*

Proof:

If $q \in a_i^{-1}Q_\lambda(N) \subseteq a_i^{-1}M$, there exists $L \in \mathcal{L}^2(\lambda)$ such that $La_iq = a_iL_i'q \subseteq N$. But in this case, $L_i'q \subseteq a_i^{-1}N$, so $q \in Q_\lambda(a_i^{-1}N)$.

For the other inclusion, take $q \in Q_\lambda(a_i^{-1}N)$ and $L \in \mathcal{L}^2(\lambda)$ such that $Lq \subseteq a_i^{-1}N$ or equivalently $a_iLq \subseteq N$. But $a_iLq = L_ia_iq$ with $L_i \in \mathcal{L}^2(\lambda)$, so $a_iq \in Q_\lambda(N)$ and $q \in a_i^{-1}Q_\lambda(N)$.□

Corollary 3.6 *If N is λ-closed, then $a_i^{-1}N$ is also λ-closed.*

For $N \subseteq M$, we let $b(N) = \bigcap_{i=1}^n a_i^{-1}N$ and call this the *bound* of N in M.

Lemma 3.7 *If N is λ-closed, then $b(N)$ is the biggest S-submodule of M inside N. It is also λ-closed.*

Proof:

Because of the last corollary, $b(N)$ is trivially λ-closed.

If we take $q \in b(N)$ and $s \in S$, we have to show that $sq \in b(N)$. If $s = a_i$ or $s \in S'$ the proof is similar to the absolute case. In general, if $s \in S$ there exists $L \in \mathcal{L}^2(\lambda)$ such that $Ls \subseteq S'$, so $Lsq \subseteq S'q \subseteq b(N)$ and this implies that $sq \in Q_\lambda(b(N)) = b(N)$.□

Lemma 3.8 *If $N = Q_\lambda(M)$ is R-essential in M, then $b(N)$ is R-essential in M.*

Proof: See [1].□

Argueing as in [1, 2], one easily proves:

Lemma 3.9 *Let $Q \lhd S$ and $Q \subseteq T \subseteq S$. Suppose Q verifies that for every $L \subseteq S$ with $Q \not\subseteq L$, we also have $Q \not\subseteq L \cap T$. If $Q \not\subseteq H$, then*

1. *$Q \not\subseteq H \cap a^{-1}T$ for every $a \in N_R(S)$.*

2. $Q \nsubseteq H \cap b(T)$

Lemma 3.10 *For any $Q \in K(\bar{\lambda})$, there exist prime ideals $P \in K(\lambda)$ and $\{P_1, \ldots, P_n\}$ in R such that*

1. $\bigcap_{i=1}^{n} P_i = Q \cap R$;
2. *if $P_i \neq R$, then $R/P \simeq R/P_i$ as rings (so $P_i \in K(\lambda)$).*

Proof:

Let $\mathcal{F} = \{{}_RY_S\,;\, b(Y) = Q\}$. As $\mathcal{F} \neq \emptyset$ is inductive, there exists ${}_RY_S \subseteq S$ maximal for the condition $b(Y) = Q$. Using the maximality of Y, it is clear that Y must be λ-closed. We have also that $Q = Q_\lambda(Q)$ because λ is i-compatible and so $S/Q \in \mathcal{F}_\lambda$.

Let $P := Y \cap R$, then $R/P \subseteq S/Y$ is λ-torsionfree. Take $r \in R$, $s \in S$ and suppose that $rRs \subseteq Y$. We are going to show that $r \in Y$ or $s \in Y$. If $r \notin Y$ and $s \notin Y$, we define

$$I := Q_\lambda(RrS) + Y$$
$$J := Q_\lambda(RsS) + Y.$$

It is clear that I and J are λ-closed (R, S) submodules of S, so $b(I)$ and $b(J)$ are twosided ideals of S. The maximality of Y permits us to say that $Q \nsubseteq b(I)$ and $Q \nsubseteq b(J)$. However,

$$\begin{aligned} b(I)b(J) &\subseteq Q_\lambda(RrS)b(J) + Yb(J) \subseteq Q_\lambda(Rrb(J)) + Y \\ &\subseteq Q_\lambda(RrQ_\lambda(RsS + Y)) + Y \subseteq Q_\lambda(Q_\lambda(Y) + Y) = Y. \end{aligned}$$

This proves in particular that $P \in Spec(R)$, so $P \in K(\lambda)$.

With $P_i = a_i^{-1}Y \cap R$ for any $1 \leq i \leq n$, it now follows that $R/P_i \subseteq S/a_i^{-1}Y \in \mathcal{F}_\lambda$ as $a_i^{-1}Y$ is λ-closed because of 3.6. We can define

$$\theta_i : R \to R/P_i$$

by $\theta_i(r) := r' + P_i$ where $ra_i = a_i r'$. Clearly θ_i is a ring morphism and $Ker\theta_i = P$, so if $P_i \neq R$ then $P_i \in K(\lambda)$.□

3.11 We can develop the same kind of results if we take ${}_RN_R \subseteq {}_SM_S$ and assume M to be (λ, ρ)-closed. We can define for all $i, j \in \{1, \ldots, n\}$

$$a_i^{-1}Na_j^{-1} = \{m \in M\,;\, a_i m a_j \in N\}$$

and $b(N) = \cap_{i,j} a_i^{-1}Na_j^{-1}$. If N is (λ, ρ)-closed, then $b(N)$ is the biggest S-subbimodule of M contained in N and it is (λ, ρ)-closed.

Remark 3.12 It is well known that if we work with a ring and a symmetric biradical (λ, ρ) with respect to which R is closed, then a prime P of R belongs to $K(\lambda) = K(\rho)$ if and only if it is (λ, ρ)-closed.

The problem appears when the ring R is not (λ, ρ)-closed. In our situation, the ring $\bar{R} = R/Ker\varphi$ is only torsionfree with respect to the symmetric biradical

(λ'', ρ''). However a prime P'' of $\bar{R}$ is in $K(\lambda'')$ if and only if $P'' = P/Ker\varphi$ with P in $K(\lambda)$. Nevertheless we will need that

$$Q_{\lambda''}(P/I) = Q_{\rho''}(P/I),$$

which is equivalent to

$$Q_{\lambda}(P/I) = Q_{\rho}(P/I).$$

We can suppose that $Q_\lambda(R/I) = Q_\rho(R/I)$, for any (λ, ρ)-closed twosided ideal I of R. For this it is enough to have that (λ, ρ) is a *strongly centralizing biradical*, which means that $Q_\lambda(M) = Q_\rho(M)$ for every centralizing R-bimodule M.

Lemma 3.13 *Suppose that (λ, ρ) is a strongly centralizing biradical. If $P \in K(\lambda)$, there exists $Q \in K(\lambda')$ such that P is minimal over $Q \cap R$.*

Proof:

Let us define

$$\mathcal{C} = \{I \subseteq S\,;\, I = Q_{\lambda'}(I)\,,\, I \cap R \subseteq P\}.$$

It is an easy exercise to prove that $\mathcal{C}$ is inductive and $\mathcal{C} \neq \emptyset$, so let Q be the maximal element in $\mathcal{C}$. Then $Q \cap R \subseteq P$ and $Q = Q_{\lambda'}(Q)$ so S/Q is λ'-torsionfree.

Let us show that $Q \in Spec(S)$. Suppose that I, J are twosided ideals of S such that $IJ \subseteq Q$, $I \nsubseteq Q$ and $J \nsubseteq Q$. The maximality of Q in $\mathcal{C}$ allows us to say that $Q_{\lambda'}(I+Q) \cap R \nsubseteq P$ and $Q_{\lambda'}(J+Q) \cap R \nsubseteq P$. But

$$\begin{array}{lcll}
(Q_{\lambda'}(I+Q) \cap R)(Q_{\lambda'}(J+Q) \cap R) & \subseteq & Q_{\lambda'}(I+Q)Q_{\lambda'}(J+Q) \cap R & \subseteq \\
Q_{\lambda'}(Q_{\lambda'}(I+Q)(J+Q)) \cap R & = & Q_{\lambda'}((I+Q)(J+Q)) \cap R & \subseteq \\
Q_{\lambda'}(Q) \cap R & \subseteq & P. &
\end{array}$$

and this contradicts the fact that P is prime in R.

Finally, let us verify that P is minimal over $Q \cap R$ or, equivalently, $P_1 = P/Q \cap R$ is minimal in $R_1 = R/Q \cap R$. If P_1 is not minimal in R_1, then P_1 would be essential in R_1, so $rank(R_1) = rank(P_1)$. Let $Q \subseteq K \subseteq S$ be an R-subbimodule of S maximal with respect to $K \cap (R+Q) \subseteq Q$, then $K = Q_\lambda(K) = Q_\rho(K)$ and

$$rank(S_1) = rank(S/Q) = rank((K_1 = K/Q) \oplus R_1) = rank(K_1 \oplus P_1)$$

so $(K \oplus P + Q)/Q$ is essential in S_1 and, (by 3.9) $b(K \oplus P + Q)/Q$ is also essential in S_1. If we prove that $Q_{\lambda'}(b(K \oplus Q_\lambda(P) + Q)) \cap R \subseteq P$, we will contradict the maximality of Q, because $b(K \oplus Q_\lambda(P) + Q)$ is a twosided ideal of S (3.11).

Let us show that $Q_{\lambda'}(b(K \oplus Q_\lambda(P) + Q)) \cap R \subseteq P$.

- $b(K \oplus Q_\lambda(P) + Q) \cap R \subseteq (K \oplus Q_\lambda(P) + Q) \cap R \subseteq Q_\lambda(P) \cap R \subseteq R$;

- If I is a twosided ideal of S such that $I \cap R \subseteq P$ then also $Q_{\lambda'}(I) \cap R \subseteq P$. If $q \in Q_{\lambda'}(I) \cap R$, then there exists $L \in \mathcal{L}^2(\lambda') \subseteq \mathcal{L}^2(\bar{\lambda})$ such that $Lq \subseteq I \cap R \subseteq P$. But as λ is φ-compatible we have that $L \cap R \in \mathcal{L}^2(\lambda)$ and, as $(I \cap R)q \subseteq Pq \subseteq Q_\lambda(P) \cap R \subseteq P$.□

Lemma 3.14 *If $s \in N_R^s(S)$ and S is prime, then $Ann_R(s) \in rad_\lambda(R) := \bigcap_{P \in K(\lambda)} P$.*

Proof:

First, note that $Ann_R^l(s) = Ann_R^r(s)$ is a twosided ideal of R, because $s \in N_R^s(S)$. For every $P \in K(\lambda)$, the last lemma permits us to find $Q \in K(\bar{\lambda})$ such that $Q \cap R \subseteq P$. As $\bar{s} = s + Q \in N_{R/P}^s(S/Q)$, we have that $Ann_{R/P}^l(\bar{s}) = 0$ as it is a twosided ideal of R/P, while S is prime and $Ann_{R/P}^l(\bar{s})\bar{s} = 0$. This means that $Ann_R^l(s) \subseteq Q \cap R \subseteq P$.□

4 Functorial Behaviour

From now on, we assume throughout R to satisfy the strong second layer condition with respect to (λ, ρ) ([19]).

Assume that the ring homomorphism $\varphi : R \to S$ is strongly normalizing with respect to (λ, ρ). Exactly as in the relative centralizing case, it can be proved that this yields a map:

$$^a\varphi : K(\bar{\lambda}) \to K(\lambda)\,;\; Q \mapsto \varphi^{-1}(Q)$$

It has been verified in [18], that the set $T(R, \lambda, \rho)$ consisting of all sets $X_R(I, \lambda)$ such that (λ_I, ρ_I) is a biradical, is a topology on $K(\lambda)$.

4.1 Let us recall the definition of the structure sheaf on $(K(\lambda), T(R, \lambda, \rho))$ constructed in [18] through classical localization.

To every open subset $X_R(I, \lambda)$ in $T(R, \lambda, \rho)$, we associate the radical

$$\lambda_{(I)} = \bigwedge_{P \in X_R(I,\lambda)} \lambda_{\mathcal{C}(P)}$$

If we define $\rho_{(I)}$ in a similar way, it has been proved in [18] that if M is an R-bimodule which is normalizing with respect to (λ, ρ), then we can associate to any open subset $X_R(I, \lambda)$ of $T(R, \lambda, \rho)$ the module of quotients $Q_{(I)}(M)$ of M at $\lambda_{(I)}$. This yields a sheaf of R-bimodules $\mathcal{Q}_M^{\lambda;\rho}$ on $(K(\lambda), T(R, \lambda, \rho))$, with global sections $\Gamma(K(\lambda), \mathcal{Q}_M^{\lambda;\rho}) = Q_\lambda(M)$.

4.2 In [18], we introduce a relative notion of Ore set with respect to λ. If $C \subseteq R$ is a multiplicatively closed subset of R, we will say that C is a *left Ore set with respect to* λ if λ_C is an internal radical in (R, λ)-**mod**. This means that $\lambda_C(M) \in (R, \lambda)$-**mod** for any $M \in (R, \lambda)$-**mod** and that there exist, for any $r \in R$ and $c \in C$, a left ideal $I \in \mathcal{L}(\lambda)$ and $d \in C$, with $Idr \subseteq Rc$ (or equivalently if $Q_\lambda(R/Rc)$ is λ_C-torsion for any $c \in C$).

If $C \subseteq R$ is a left Ore set with respect to λ and $M \in \mathcal{F}_\lambda$ then

$$\lambda_C(M) = \{m \in M\,;\, \exists c \in C\,,\, cm = 0\}.$$

4.3 For any subset $X \subseteq K(\lambda)$, we define $\lambda_{(X)} := \wedge_{P\in X}\lambda_{\mathcal{C}(P)}$, resp. $\rho_{(X)} := \wedge_{P\in X}\rho_{\mathcal{C}(P)}$. It is clear that $(\lambda_{(X)}, \rho_{(X)}) \geq (\lambda, \rho)$.

If X is link closed, satisfies the weak intersection property and the strong second layer condition with respect to (λ, ρ), then $(\lambda_{(X)}, \rho_{(X)})$ is a biradical [18, (3.11)].

Moreover, the weak intersection property guarantees that $\lambda_{(X)} = \lambda_{\mathcal{C}(X)}$, resp. $\rho_{(X)} = \rho_{\mathcal{C}(X)}$ where $\lambda_{\mathcal{C}(X)}$ resp. $\rho_{\mathcal{C}(X)}$ is the radical in R-**mod**, resp. in **mod**-R obtained by reflecting the radical $\lambda_{\mathcal{C}(X)}$ resp. $\rho_{(X)}$ in (R,λ)-**mod**, resp. in **mod**-(R,ρ) (see 2.5).

Definition 4.4 *If $M \in R$-**mod**, we will say that $m \in M$ is* weakly regular with respect to λ, *if for every $P \in K(\lambda)$ and any $r \in R$, one has $rm \in Pm$ if and only if $r \in P$.*

Lemma 4.5 *If $R \subseteq S$ is (λ,ρ)-strongly normalizing of finite type and $s \in N_R^s(S)$, then s is weakly regular with respect to λ.*

Proof:

Let $P \in K(\lambda)$ and $r \in R$. If $rs \in Ps$, then $rs = ps$, for some $p \in P$. This means that $(r - p) \in Ann_R^l(s) \subseteq P$ (see 3.14), so $r \in P$.□

Remark 4.6 If we have $R \subseteq S$ with S a prime ring and $s \in N_R^s(S)$, then s is weakly regular with respect to λ. This is clear because $Ann_R^l(s)$ is a twosided ideal of R and if $Js = 0$ for any twosided ideal J of R, then $J = 0$ or $s = 0$.

Proposition 4.7 *Let M be an R-bimodule and let $m \in N_R^s(M)$. Assume m to be weakly regular with respect to λ and let $r, r' \in R$ be linked through $r'm = mr$. Then we have:*

- *for any $P \in K(\lambda)$, $r \in P$ if and only if $r' \in P$;*
- *for any $X \subseteq K(\lambda)$, $r \in \mathcal{C}_R(X)$ if and only if $r' \in \mathcal{C}_R(X)$.*

Proof: Exactly the same as in [3, (II.7.4)].□

Lemma 4.8 *Let M be an R-bimodule generated by elements in $N_R^s(M)$ which are weakly regular with respect to λ. Let $X \subseteq K(\lambda)$ and assume $C = \mathcal{C}(X)$ is a left Ore set with respect to λ. Then:*

1. *for any $m \in M$ and any $c \in C$, there exist some $d \in C$ and some $J \in \mathcal{L}(\lambda)$ such that $Jdm \subseteq Mc$;*
2. *for any $m \in M$ and any $c \in C$ with $mc = 0$, there exists some $d \in C$ such that $dm \in \lambda(M)$.*

Proof:

Let $c \in C$ and write $m = \sum_{i=1}^t r_i m_i$ with $r_i \in R$ and $m_i \in N_R^s(M)$ weakly regular with respect to λ. By the previous result, we can find for every index i, some $c_i' \in C$

such that $c_i'm_i = m_i c$. If we apply the commom denominator property, there exist $d \in C$ and $J \in \mathcal{L}(\lambda)$ such that $Jdr_i \subseteq Rc_i'$ for any i. Let $j \in J$, then

$$jdm = \sum_{i=1}^{t} jdr_i m_i = \sum_{i=1}^{t} r_i^j c_i' m_i = \sum_{i=1}^{t} r_i^j m_i c \in Mc,$$

so $Jdm \subseteq Mc$.

To prove the second assertion, one argues as in [20, (4.3)].□

Taking into account the previous result, one proves as in [20, (4.4)]:

Corollary 4.9 *Let $X \subseteq K(\lambda)$ such that $C = \mathcal{C}(X)$ is a left Ore set with respect to λ and let N be an R-bimodule generated by elements in $N_R^s(M)$ which are weakly regular with respect to λ. Let $M = Q_\lambda(N)$, then:*

1. *for any $m \in M$ and any $c \in C$, there exist some $d \in C$ and some $J \in \mathcal{L}(\lambda)$ such that $Jdm \subseteq Mc$;*
2. *for any $m \in M$ and any $c \in C$ with $mc = 0$, there exists some $d \in C$ such that $dm \in \lambda(M)$.*

4.10 Our next goal will be to show that

$$Q_X^l(N) \stackrel{def}{=} Q_{\lambda(X)}(N) \simeq Q_{\rho(X)}(N) \stackrel{def}{=} Q_X^r(N)$$

for any (λ, ρ)-torsionfree strongly normalizing R-bimodule N and any subset $X \subseteq K(\lambda)$ which is link closed with respect to (λ, ρ) (see [3, 10] for definitions).

As we will need that $Q_\lambda(N) = Q_\rho(N)$ for any (λ, ρ)-torsionfree strongly normalizing R-bimodule N, we assume (λ, ρ) to be a strongly normalizing biradical. Note that this implies, in particular, that (λ, ρ) is a strongly centralizing biradical.

Proposition 4.11 *Let (λ, ρ) be a strongly normalizing biradical over R with the property that R is (λ, ρ)-closed, (λ, ρ)-noetherian and satisfies the strong second layer condition with respect to (λ, ρ). Let $X \subseteq K(\lambda) = K(\rho)$ be a link closed subset with respect to (λ, ρ), satisfying the weak intersection property with respect to (λ, ρ). Then $Q_{(X)}^l(N) = Q_{(X)}^r(N)$, for any (λ, ρ)-torsionfree R-bimodule N generated by elements in $N_R^s(N)$ which are weakly regular with respect to λ.*

Proof:

Since $(\lambda_{(X)}, \rho_{(X)}) \geq (\lambda, \rho)$ and since our hypotheses imply [20, (3.2)] also to hold true for torsionfree strongly normalizing R-bimodules, we have that $\lambda_{(X)}N = \rho_{(X)}N$. So, $N/\lambda_{(X)}N = N/\rho_{(X)}N$ and as $(\lambda_{(X)}, \rho_{(X)}) \geq (\lambda, \rho)$, we may suppose that N is $(\lambda_{(X)}, \rho_{(X)})$- torsionfree.

Let $M = Q_\lambda(N) = Q_\rho(N)$ and take $q \in Q = Q_X^l(N) = Q_X^l(M)$. As Q/M is $\lambda_{(X)}$-torsion and λ-torsionfree, there exists $c \in \mathcal{C}(X)$ such that $m = cq \in M$. Applying the right handed version of 4.9, we can find $J \in \mathcal{L}^2(\lambda) = \mathcal{L}^2(\rho)$ and $d \in \mathcal{C}(X)$ such that $mdJ \subseteq cM$. For any $j \in J$, let $m_j \in M$ be such that $mdj = cm_j$. Then $c(qdj - m_j) = 0$ and as $qdj - m_j \in Q_{(X)}^l(N)$ and this is $\lambda_{\mathcal{C}(X)}$-torsionfree as,

$$\lambda_{\mathcal{C}(X)} Q_X^l(N) = \widehat{\lambda_{\mathcal{C}(X)}} Q_X^l(N) = \lambda_{(X)} Q_X^l(N) = 0$$

, it follows that $qdj = m_j$ so $qd \in Q_\rho(M) = M$.

Taking into account 2.2, with $\bar{q} = q + M \in Q/M$, it follows that

$$\begin{aligned} &\bar{q} \in \rho_{\mathcal{C}(X)}(Q/M) \subseteq \rho_{\mathcal{C}(X)}(E^2(N)/M) = \\ &= (E^2(N)/M) \cap \rho_{\mathcal{C}(X)} Q_\rho(E^2(N)/M) = \\ &= (E^2(N)/M) \cap \rho_{(X)} Q_\rho(E^2(N)/M) = \\ &= \rho_{(X)}(E^2(N)/M), \end{aligned}$$

so $Q/M \in \mathcal{T}_{\rho(X)}$ and then $Q^l_X(N) = Q \subseteq Q^r_X(M) = Q^r_X(N)$. The other inclusion is obtained by symmetry.□

Theorem 4.12 *Let (λ, ρ) be a strongly normalizing biradical over R with the property that R is (λ, ρ)-closed, (λ, ρ)-noetherian and satisfies the strong second layer condition with respect to (λ, ρ). Assume that every (λ, ρ)-clique in $K(\lambda)$ satisfies the weak intersection property with respect to (λ, ρ). Then $Q^l_{(X)}(N) = Q^r_{(X)}(N)$, for any subset $X \subseteq K(\lambda) = K(\rho)$ which is link closed with respect to (λ, ρ) and any (λ, ρ)-torsionfree R-bimodule N generated by elements in $N^s_R(N)$ which are weakly regular with respect to λ.*

Proof: This may be proved along the lines of [20, (4.8)], taking into account the previous result.□

Corollary 4.13 *Let (λ, ρ) be a strongly normalizing biradical over R with the property that R is (λ, ρ)-closed, (λ, ρ)-noetherian and satisfies the strong second layer condition with respect to (λ, ρ). Let $\varphi : R \to S$ be a ring homomorphism. Assume one of the following properties is satisfied:*

1. *φ is a finite strongly normalizing homomorphism with respect to (λ, ρ);*
2. *φ is strongly normalizing with respect to (λ, ρ) and S is prime.*

For any subset $X \subseteq K(\lambda) = K(\rho)$ which is link closed with respect to (λ, ρ), the rings $Q^l_X(S)$ and $Q^r_X(S)$ are canonically isomorphic.

Proof:

If we factorize $\varphi : R \to S$ as $R \xrightarrow{\pi} \bar{R} \xhookrightarrow{i} S$, then π is centralizing so the result follows by [20, (4.8)]. For the injective part use the previous theorem and 4.5 resp. 4.6 for the finite case resp. for the prime case.□

4.14 To derive functoriality we will only need to study what happens at the sheaf level because under our hypotheses it can be proved as in [20, (3.4)] that the map

$${}^a\varphi : K(\lambda') \to K(\lambda)$$

constructed before, induces a continuous map

$${}^a\varphi : (K(\lambda'), T(S, \lambda', \rho')) \to (K(\lambda), T(R, \lambda, \rho)).$$

Note that the result [20, (2.9)] is easily translated to the strongly normalizing case.

If $\sigma \geq \lambda$ is a radical in R-**mod**, we can show (as in [3, (II.5.3)]) that $Q_\sigma(S)$ possesses a ring structure making the canonical morphism $j_{\sigma,S}S \to Q_\sigma(S)$ into a ring homomorphism if and only if $Q_\sigma(S)$ has a left S-module structure extending that of S. In this case, this structure is necessarily unique. Moreover $Q_\sigma(S) \simeq Q_{\bar{\sigma}}(S)$ and

$$Q_\sigma(\varphi) : Q_\sigma(R) \to Q_\sigma(S) \simeq Q_{\bar{\sigma}}(S)$$

is the unique ring homomorphism extending φ.

Theorem 4.15 *Let (λ, ρ) be a strongly normalizing biradical over R with the property that R is (λ, ρ)-closed, (λ, ρ)-noetherian and satisfies the strong second layer condition with respect to (λ, ρ). Assume that (λ', ρ') is a symmetric biradical over S such that S satisfies analogous properties with respect to (λ', ρ') and that every (λ, ρ)-clique in $K(\lambda)$ satisfies the weak intersection property with respect to (λ, ρ). Assume also that the ring homomorphism $\varphi : R \to S$ satisfies one of the following properties:*

1. *φ is a finite strongly normalizing homomorphism with respect to (λ, ρ);*
2. *φ is strongly normalizing with respect to (λ, ρ) and S is prime.*

If $(\lambda', \rho') \geq (\bar{\lambda}, \bar{\rho})$, then φ induces a morphism of ringed spaces

$$^a\varphi : (K(\lambda'), T(S, \lambda', \rho'), \mathcal{Q}_S^{\lambda', \rho'}) \to (K(\lambda), T(R, \lambda, \rho), \mathcal{Q}_R^{\lambda, \rho}).$$

Proof:

If I is any twosided ideal of R such that (λ_I, ρ_I) is a biradical then $Q^l_{(I)}(S) \simeq Q^r_{(I)}(S)$, so after the previous comments of 4.14 we have a ring homomorphism:

$$Q^l_{(I)}(R) \to Q^l_{(I)}(S) \simeq Q_{\overline{\lambda_{(I)}}}(S).$$

We can conclude the result from the fact that $\overline{\lambda_{(I)}} \leq \lambda_{Q_\lambda(SI)}$ in S-**mod** (see [20, (4.9)]).□

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