Pretty pictures of geometries

B. Polster *

Abstract

We present four construction principles that allow us to produce many beautiful plane and spatial models of some of the most important small finite geometries.

1 Introduction

Did you ever ask yourself why there are only a handful of pictures that pop up in texts on incidence geometry? The pictures we have in mind here are the traditional pictures of the Fano plane, the affine plane of order 3, the Desargues and Pappus configurations etc. How many times have you drawn these pictures in your lectures and how many times have you drawn them to illustrate to somebody outside your field what the kinds of objects are we are dealing with in incidence geometry? Once you start asking these kinds of questions, you also immediately start wondering whether these are really the only pictures which are worth drawing and whether they are even the 'best' pictures of the geometries involved.

We are in the process of compiling a comprehensive collection of good plane and spatial pictures of small incidence geometries which, eventually, will appear in [7]. While working on this collection, we have come to the conclusion that there are many more amazing pictures and models of geometries that everybody interested in geometry should know about.

In this note we describe four of the most useful construction principles for constructing pictures of small incidence geometries which capture large parts of the

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abstract beauty of the geometries they depict. In order to illustrate these construction principles, we use them to construct models for some of the most important small geometries such as the Fano plane, the generalized quadrangle of order 2, the Desargues configuration and the projective space PG$(3, 2)$.

Why are good pictures important? Two of the main reasons that come to mind are the following:

- To convey some of the abstract beauty of the objects we study to people outside our field. This seems to be especially important today as it becomes more and more important to 'justify' and 'sell' the kind of research we are fascinated by.
- Many of us think in terms of pictures of various degrees of abstraction. The kind of pictures we want to concentrate on in this note are immediately accessible and can serve to lure students into studying incidence geometry and as a first step in teaching students pictorial thinking in geometry.

2 Construction principles for pictures of small geometries

Whenever we are trying to create an appealing model of an abstract geometry, we are trying to merge its abstract symmetries with spatial symmetries. After having created some 500+ pictures while working on [7], we found that most of these pictures, including the traditional ones, can be made up using a couple of simple rules. Here is a first such rule.

Construction principle: number right → everything right

Given a small, highly symmetrical geometry with n points, look for the same number of points arranged into a highly symmetrical spatial object. Try to merge the two structures such that the symmetries of the spatial object translate into symmetries of the geometry.

Examples for the symmetrical spatial objects that we have in mind here are the regular solids and regular polygons. This construction principle may sound rather naive, but yields attractive models for most small geometries. This should not come as too much of a surprise. Just think of the multitude of interconnections that exist between other small highly symmetrical mathematical structures.

Consider, for example, the various symmetrical sets of points associated with the tetrahedron: the 4 vertices, the 4 centers of the faces, the 6 centers of the edges and the set consisting of the center of the tetrahedron alone. By combining these four different sets in all possible ways, symmetrical sets of 4, 5, 6, 7, 8, 9, 10, 11, 14 and 15 points can be generated. The affine plane of order 2, the inversive plane of order 2 and the Laguerre plane of order 2 are geometries with 4, 5 and 6 points, respectively. The Fano plane, the one-point extension of the Fano plane, the Desargues configuration, the inversive plane of order 3 and the generalized quadrangle of order 2 are geometries with 7, 8, 10, 11 and 15 points, respectively. The point sets of all these geometries can be identified with the set of points associated with the tetrahedron having the respective number of points, such that all symmetries of the tetrahedron translate into automorphisms of the geometries.

Example 1 *The affine plane of order 2.* The points and lines are the vertices and edges of the tetrahedron on the left in Figure 1. Note that the 3 parallel classes in this affine plane correspond to the 3 pairs of opposite edges of the tetrahedron.

Example 2 *The inversive plane of order 2.* We take the vertices and the center of the tetrahedron as points and all triangles of such points as circles. See, again, Figure 1. Note that the derived geometry at the center of the model yields the model of the affine plane of order 2 on the left.

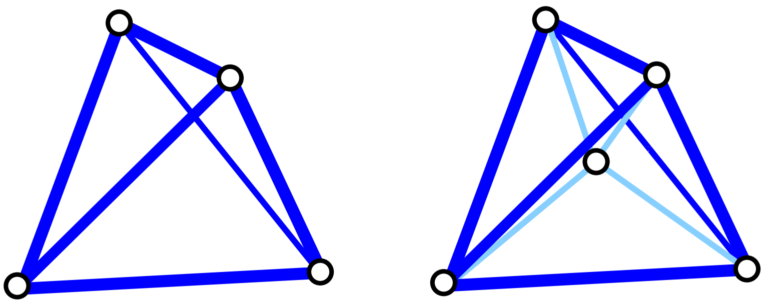

Figure 1: The affine and inversive planes of order 2

Example 3 *The Fano plane.* The points are the centers of the edges and the center of the tetrahedron. Take as the lines the 4 circles inscribed in the faces of the tetrahedron and the 3 line segments connecting opposite edges of the tetrahedron.

Figure 3 is a stereogram of this last geometry. It can be viewed with either the parallel or the cross-eyed technique, that is, one of the techniques that you had to master two or so years ago to be able to view some of the random-dot stereograms that did come into fashion around that time.

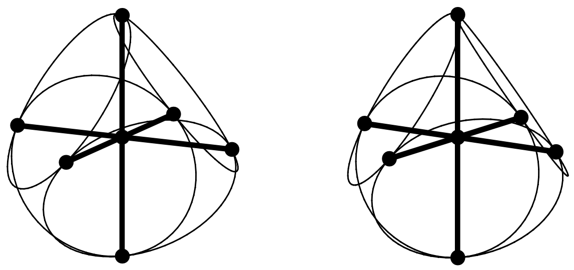

Figure 2: The Fano plane

The automorphism group of the model is the symmetry group of the tetrahedron and coincides with the point stabilizer in the center point of the automorphism group of the Fano plane. In this way, this model certainly captures more of the abstract beauty of the Fano plane than the traditional triangular model. In fact, we arrive at the traditional model by projecting the spatial model from one of the vertices of the tetrahedron onto the face opposite this vertex.

Example 4 *The Desargues configuration.* The points are the vertices plus the centers of the edges of the tetrahedron, the lines are the edges and the circles inscribed in the faces of the tetrahedron.

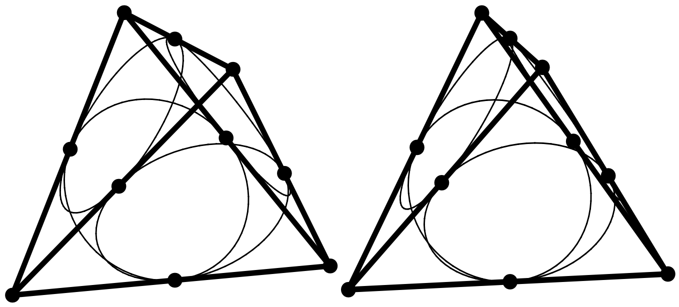

Figure 3: The Desargues configuration

Example 5 W(2)*, the generalized quadrangle of order 2.* The points are all 15 points that we associated with the tetrahedron. The lines are the 12 medians of the faces of the tetrahedron and the 3 line segments connecting opposite edges of the tetrahedron.

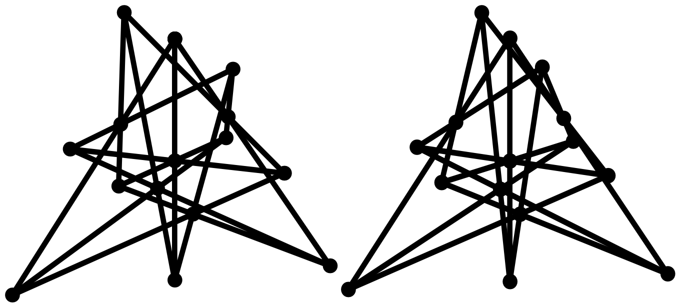

Figure 4: The generalized quadrangle of order 2

We call a geometry a *subset geometry* (with respect to the set O) if its lines can be identified with subsets of O and its points with sets of these distinguished subsets such that every permutation of O translates into an automorphism of the geometry.

Of course, a point is contained in a line if the set corresponding to the point contains the subset corresponding to the line. Many small geometries are subset geometries. The generalized quadrangle W(2), for example, has the following description as a subset geometry: Let O consist of 6 elements. The lines of the geometry are the 15 pairs of elements in O and the points are the 15 partitions of O into pairs.

Construction principle: subset geometries

Given a subset geometry on $|O| = n$ points, try to translate automorphisms of symmetrical arrangements of n points in the plane or in space into 'good' models of the geometry which exhibit as many of these automorphisms as possible.

We construct a picture for W(2) using this principle: One symmetrical arrangement of 6 points in the plane consists of the 5 vertices of a regular pentagon plus its center. Figure 5 shows the 3 essentially different partitions of the six points corresponding to 5 partitions each.

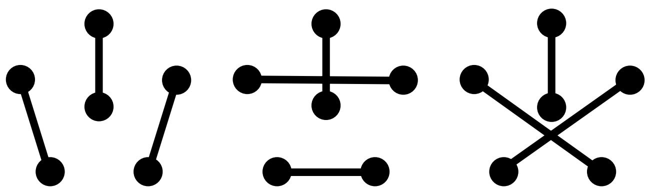

Figure 5: The different partitions of 6 points into pairs

In Figure 6 we arrange the 15 partitions on 3 regular pentagons having a common center and draw in the 15 lines of the geometry. Note that the order 5 automorphism apparent in the arrangement of the 6 points we started with translates into an automorphism of the resulting picture.

This picture (minus the labels) of the generalized quadrangle W(2) was first constructed by Payne (see the cover of [5]). He calls this picture the 'doily'. Other symmetrical arrangements of 6 points in space yield more beautiful models of this geometry. Further examples of subset geometries include the Desargues configuration and the Petersen graph.

Other important geometries can also be described with respect to a small set O of points such that at least some 'good' permutations of O translate into automorphisms of the geometry. By matching up this kind of automorphism with a similar one of a symmetrical spatial object, it is often possible to arrive at symmetrical models of the geometry that exhibit this automorphism, that is, the above construction principle can be applied modulo some fiddling. Let us call geometries that fall in the extended category *upset geometries.*

Example 6 *The small classical finite projective planes* are upset geometries. In a plane like this O can be chosen to be an oval or a hyperoval in the plane. To every

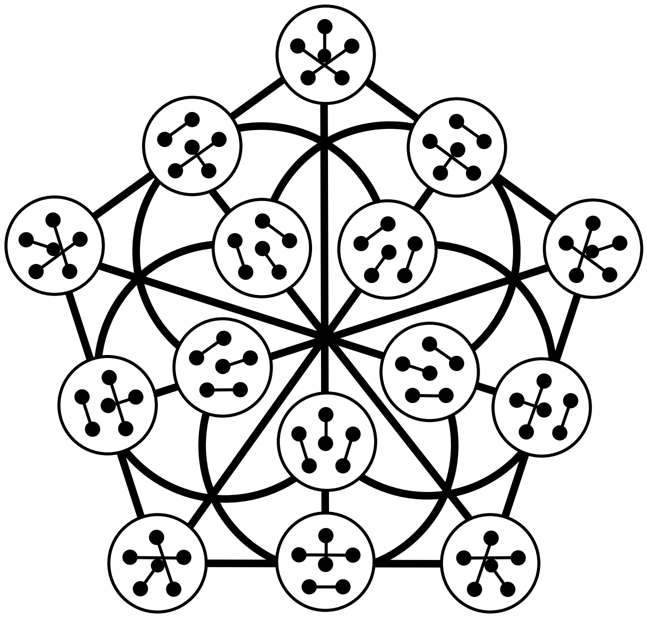

Figure 6: The 'doily'

point of the plane not contained in O there corresponds a bundle involution of O which has exactly 0 or 2 fixed points. In turn, this bundle involution corresponds to a partition of O into the fixed points and the pairs of points that get exchanged by the involution. In order to describe the projective plane with respect to the oval, we let the points of the plane be the elements of O plus all the partitions of O corresponding to the bundle involutions. The secants of O correspond, in the natural way, to pairs of elements of O, the tangents to the single points of O. For classical projective planes of small orders it is also not very hard to give an intrinsic definition of the exterior lines of O (see, for example, [2] for such a description in the case of the projective plane of order 5).

Example 7 *Small biplanes* are upset geometries. Given such a biplane, the set O can be chosen to be one of the blocks in the biplane and the description of the geometry with respect to this set that usually does the trick is the description via a set of Hussain graphs (see [3]).

Construction principle: subgeometry → full geometry

Try to extend 'good' models of subgeometries of a given geometry to a 'good' model of the full geometry.

The generalized quadrangle W(2), for example, is contained as a subgeometry in the generalized quadrangles of orders $(4, 2)$ and $(2, 4)$ (see [6, Chapter 6]), the projective plane of order 4 (see [1]), and the projective space $\mathrm{PG}(3, 2)$ (see, again, [6]). It turns out that most 'good' models of W(2) have extensions to good models of the other four geometries.

Example 8 *The projective space* PG$(3,2)$. This projective space has 15 points which coincide with the points of the generalized quadrangle and 35 lines of three points each. See [4] for detailed information about this space. Let S be a subset of points of W(2). Then $S^{\perp}$ denotes the set of points in W(2) which are collinear in W(2) to every single point in S. The lines of the projective space are the sets $\{x,y\}^{\perp}$, where x and y are different points. Note that, if x and y are collinear in W(2), then $\{x,y\}^{\perp}$ is the line in W(2) connecting x and y. In Figure 7 we extend the doily to a model of the projective space. The diagram on the right shows the 7 essentially different lines in the model. The other lines are constructed by successively rotating these lines. The 3 lines that generate the lines of the quadrangles are highlighted.

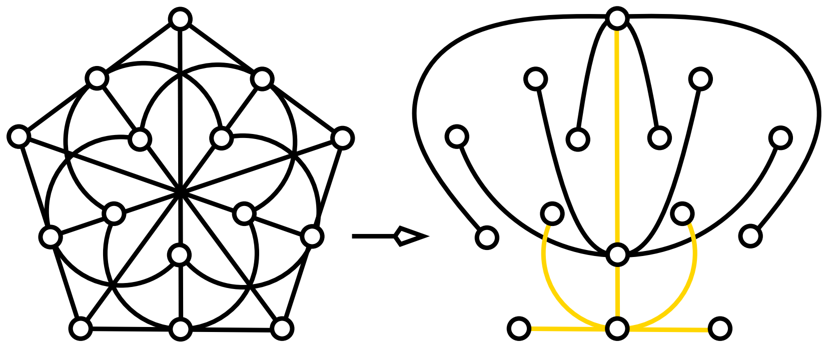

Figure 7: Constructing a projective space around the 'doily'

Construction principle: geometry → subgeometry

Try to find models of subgeometries of a given geometry 'right in the middle' of a good model of the geometry.

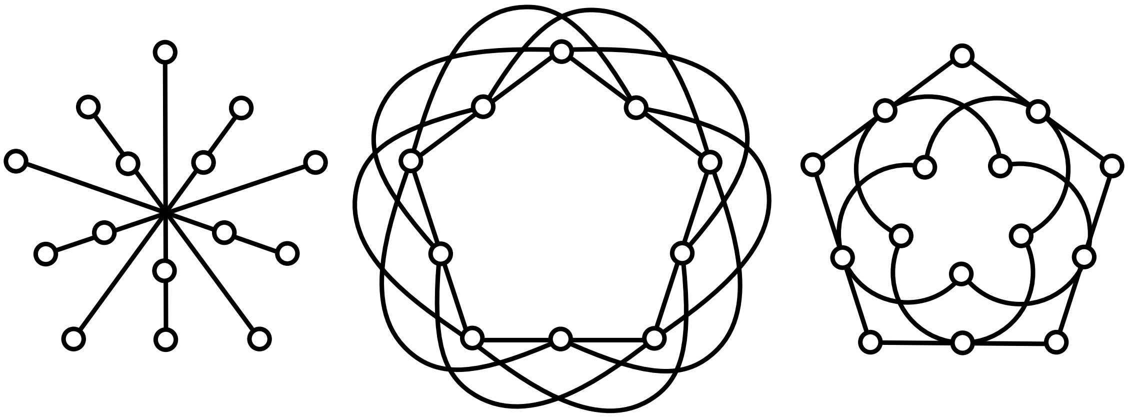

Figure 8: Spread, Desargues configuration and 2-spread

Figure 8 shows some examples of models of such subgeometries. The first one is a spread, the second one a Desargues configuration and the third one a 2-spread, that is, a set of lines such that every point is contained in exactly two of the lines in the set. Note that the union of the spread and the 2-spread is the generalized quadrangle we started with. Note also that the Desargues configuration is the union of two pentagons. See whether you can find some subgeometries with 15 points and lines, 3 points on every line and 3 lines through every point that are no generalized quadrangles.

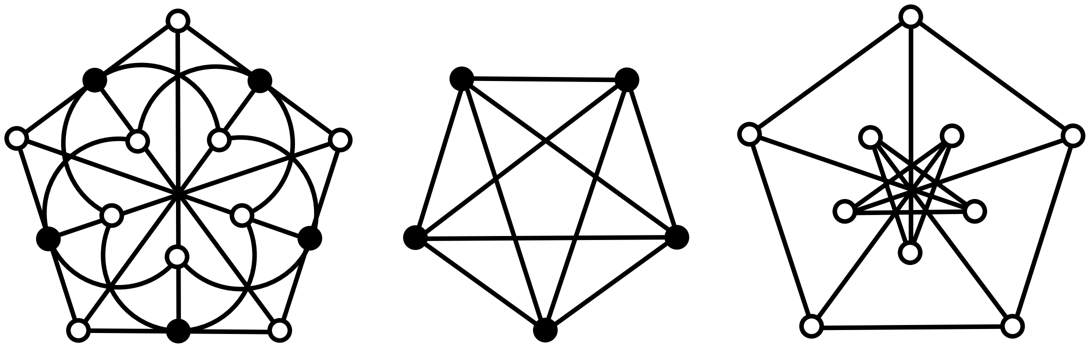

Figure 9: Elliptic quadric, inversive plane and Petersen graph

In Figure 9 the 5 solid points are the points of an elliptic quadric in the projective space as well as the points of an ovoid in the generalized quadrangle. The points of the quadric and the non-trivial plane sections of the quadric are the points and circles of an inversive plane of order 2. The middle diagram is a model for this geometry. Its points are the 5 solid points and its circles are the 10 triangles in the complete graph on the 5 points. By removing the solid points from the generalized quadrangle, we are left with the model of the Petersen graph on the right.

There are 56 spreads in our space. As a final example of the multitude of interesting substructures in $PG(3,2)$, we present an example of a packing of the line set of the space consisting of 7 spreads in Figure 10.

Finally, we remark that if we extend the spatial model of $W(2)$ in Example 5 to a model of $PG(3,2)$, we discover the spatial models of the Fano plane in Example 3 and the Desargues configuration in Example 4 'right in the middle' of this spatial model of $PG(3,2)$.

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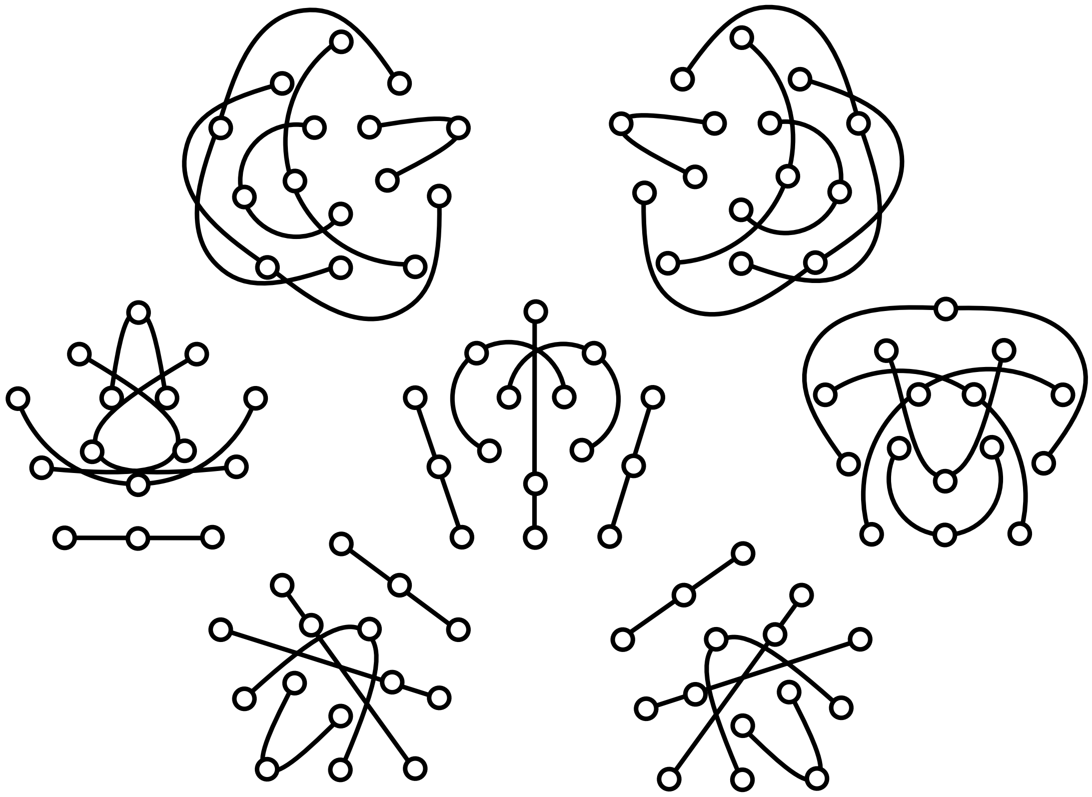

Figure 10: A packing of the projective space

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Burkard Polster
Department of Pure Mathematics
The University of Adelaide
South Australia 5005
AUSTRALIA
bpolster@maths.adelaide.edu.au
http://www.maths.adelaide.edu.au/Pure/bpolster