Thin geometries for the Suzuki simple group Sz(8)

Dimitri Leemans*

Abstract

We classify, up to isomorphism, all the thin residually connected geometries on which the group Sz(8) acts flag-transitively. This paper is a sequel of [12] where all the firm residually connected geometries on which Sz(8) acts flag-transitively and residually weakly primitively are classified, up to isomorphism. We obtain 183 thin geometries, all of rank 3 (147 of them appearing already in [12]). We compute all their rank 2 truncations. When the Neumaier construction is applyable, we give the geometries obtained from this construction and we mention whether they are regular or chiral. Most of the results obtained here rely on computer algebra.

1 Introduction

An observation that arises while classifying all firm, residually connected geometries on which a group G acts flag-transitively and fulfils some primitivity condition (as for example PRI, RPRI or RWPRI) is that when we get thin geometries, we generally get lots of them.

The classification of all the firm, residually connected geometries on which Sz(8) acts flag-transitively and residually weakly primitively (RWPRI) [12] gave us a lot of thin geometries (147 out of the 151 rank 3 ones).

One of our projects is to classify all geometries satisfying these conditions for a

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Suzuki simple group $\mathrm{Sz}(q)$, with q an odd power of 2. Such a classification is already accomplished for the rank 2 geometries [13], for which of course, no thin geometries appeared. But for the higher ranks, we must be prepared to get lots of them. The classification of all the thin geometries of Sz(8) might help us to guess what is happening in the general case.

This work has first been done "by hand", that is by looking at the dihedral subgroups of Sz(8) and seeing which of these subgroups could give rise to thin geometries, using MAGMA [1]. During a stay at the University of Sydney, we made a series of MAGMA programs that classify all thin residually connected geometries on which a group G acts flag-transitively. This permitted us to cross-check the results. And this is also the reason why we prefer to mention our results as a *fact* instead of a *theorem*, because no proof is given for the classification itself.

The paper is organized as follows. In section 2, we give the basic definitions and we fix some notation. In section 3, we list all the thin residually connected geometries on which Sz(8) acts flag-transitively. We also give their correlation group and their rank 2 truncations. Some of these geometries can be derived from others by using a construction described in [14]. We mention when such relations occur. In section 4, we give some observations we made while looking at the results. We define a new diagram with more structure and we introduce a new property that seems interesting to impose. In section 5, we apply the Neumaier construction to obtain other geometries from those given in section 3. We mention whether they are chiral or regular. The geometries mentioned in this section can be seen as abstract polytopes which are a particular case of thin geometries. Much work has already been accomplished on abstract regular polytopes. The spherical case has been well detailed by Coxeter [8]. The toroidal case has been studied by McMullen and Shulte in a series of papers (see [16] for references on the subject). Chiral polytopes have been studied recently by Shulte and Weiss [17, 18] and Nostrand and Schulte [16].

This paper is also a prelude to a project quite similar to [6], [5], that is, to build a new Atlas of thin geometries [11].

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Electronic Availability. A MAGMA file, that contains the maximal parabolic subgroups of the 183 thin geometries given in this paper, is available at the address `http://cso.ulb.ac.be/~dleemans/abstracts/sz8thin.html`

2 Definitions and notation

The basic concepts about geometries constructed from a group and some of its subgroups are due to Tits [19] (see also [4], chapter 3).

Let G be a group together with a finite family of subgroups $(G_i)_{i\in I}$. We define the *pre-geometry* $\Gamma = \Gamma(G, (G_i)_{i\in I})$ as follows. The set X of *elements* of Γ consists of all cosets gG_i, $g \in G$, $i \in I$. We define an *incidence relation* $*$ on X by :

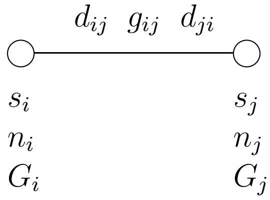
$$g_1G_i * g_2G_j \text{ iff } g_1G_i \cap g_2G_j \text{ is non-empty in } G.$$

The *type function* t on Γ is defined by $t(gG_i) = i$. The *type* of a subset Y of X is the set $t(Y)$; its *rank* is the cardinality of $t(Y)$ and we call $\mid I \mid$ the *rank* of Γ. The subgroups G_i's are called the *maximal parabolic subgroups*. The *Borel subgroup* of the pre-geometry is the subgroup $B = \cap_{i\in I} G_i$. A *flag* is a set of pairwise incident elements of X and a *chamber* of Γ is a flag of type I. An element of type i is also called an *i-element*.
The group G acts on Γ as an automorphism group, by left translation, preserving the type of each element.
As in [9], we call Γ a *geometry* provided that every flag of Γ is contained in some chamber and we call Γ *flag-transitive* provided that G acts transitively on all chambers of Γ, hence also on all flags of any type J, where J is a subset of I.
Assuming that Γ is a flag-transitive geometry and that F is a flag of Γ, the *residue* of F is the pre-geometry

$$\Gamma_F = \Gamma(\cap_{j\in t(F)} G_j, (G_i \cap (\cap_{j\in t(F)} G_j))_{i\in I\setminus t(F)})$$

and we readily see that Γ_F is a flag-transitive geometry.
Let J be a subset of I. The *J-truncation* of Γ, denoted Γ^J, is the geometry consisting of the elements of type $j \in J$, together with the restricted type-function and induced incidence relation. In group-geometry terms, the J-truncation of $\Gamma(G, (G_i)_{i\in I})$ is the geometry $\Gamma(G, (G_j)_{j\in J})$.
We call Γ *firm* (resp. *thick*, *thin*) provided that every flag of rank $\mid I \mid -1$ is contained in at least two (resp. at least three, exactly two) chambers. We call Γ *residually connected* provided that the incidence graph of each residue of rank ≥ 2 is a connected graph. We call Γ *primitive* (PRI) provided that G acts primitively on the set of i-elements of Γ, for each $i \in I$.
As in [6], we call Γ *residually primitive* (RPRI) if each residue Γ_F of a flag F is primitive for the group induced on Γ_F by the stabilizer G_F of F.
We call Γ *weakly primitive* (WPRI) provided there exists some $i \in I$ such that G acts primitively on the set of i-elements of Γ and we call Γ *residually weakly primitive* (RWPRI) provided that each residue Γ_F of a flag F is weakly primitive for the group induced on Γ_F by the stabilizer G_F of F.
If Γ is a geometry of rank 2 with $I = \{0, 1\}$ such that each of its 0-elements is incident with each of its 1-elements, then we call Γ a *generalized digon*.
Following [2] and [3], the *diagram* of a firm, residually connected, flag-transitive geometry Γ is a graph together with additional structure, whose vertices are the elements of I, which is further described as follows. To each vertex $i \in I$, we attach the *order* s_i which is $\mid \Gamma_F \mid -1$, where F is any flag of type $I\setminus\{i\}$, the *number* n_i *of varieties of type* i, which is the index of G_i in G, and the subgroup G_i. Elements i, j of I are not joined by an edge of the diagram provided that a residue Γ_F of type $\{i, j\}$ is a generalized digon. Otherwise, i and j are joined by an edge endowed with three positive integers d_{ij}, g_{ij}, d_{ji} where g_{ij} (the *gonality*) is equal to half the girth of the incidence graph of a residue Γ_F of type $\{i, j\}$ and d_{ij} (resp. d_{ji}), the *i-diameter* (resp. *j-diameter*) is the greatest distance from some fixed i-element (resp. j-element) to any other element in the incidence graph of Γ_F.

On a picture of the diagram, this structure will often be depicted as follows.



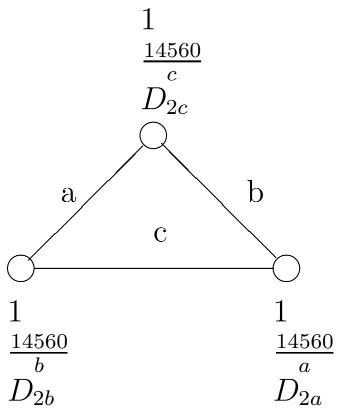
If $g_{ij} = d_{ij} = d_{ji} = n$, then Γ_F is called a *generalized n-gon* and on a picture, we do not write d_{ij} and d_{ji}.
We denote by $\sigma_0(F)$ the 0-shadow of a flag F, that is the set of elements of type 0 that are incident with the flag F. We say that a geometry Γ, over a set of type I, satisfies the *intersection property* (IP) when for every types $0, 1 \in I$, for every 1-element x, and for every flag F, $\mid \sigma_0(x) \cap \sigma_0(F) \mid > 1$ implies that there exists a flag F' such that x and F are incident with F' and $\sigma_0(F') = \sigma_0(x) \cap \sigma_0(F)$. We say that Γ satisfies the *intersection property of rank 2* $(IP)_2$ is every rank 2 residue of Γ satisfies (IP).
The ordered pairs (Γ, G) and (Γ', G) are *isomorphic* (resp. *conjugate*) if there exists an automorphism (resp. internal automorphism) of G mapping Γ onto Γ'. The group *Cor*(Γ, G) (resp. *Aut*(Γ, G)) is the group of automorphisms (resp. type-preserving automorphisms) of the pair (Γ,G).

As to notation for groups, we follow the conventions of the Atlas [7] up to slight variations. The symbol ":" stands for split extensions, the "hat" symbol "$\hat{:}$"stands for non split extensions and the symbol $\times$ stands for direct products.

Following ideas developped in [18], we say that a group G acts *chirally* on a geometry Γ if the chambers of Γ are divided in two orbits of the same length on which G acts transitively. We also say that the group G acts *regularly* on Γ if G is transitive on the set of chambers of Γ.

3 The thin geometries of $\mathrm{Sz}(8)$

Fact 3.1 *Up to isomorphism, there are 183 thin residually connected geometries on which the group* $\mathrm{Sz}(8)$ *acts flag-transitively.*



B = 1

Table 1: The thin geometries of Sz(8)

Nr.	a	b	c	#CC	$Cor(\Gamma, G)$	Rwpri
1	2	5	7	3		Yes
2	2	5	13	3		No
3	2	7	7	3	Sz(8)	Yes
4-6	2	7	13	3		Yes
7	2	13	13	3	Sz(8)	No
8	5	5	5	1	Sz(8):3	No
9-12	5	5	7	3	Sz(8)	Yes
13	5	5	7	3	Sz(8)×2	Yes
14-17	5	5	13	3	Sz(8)	No
18	5	5	13	3	Sz(8)×2	No
19-31	5	7	7	3	Sz(8)	Yes
32	5	7	7	3	Sz(8)×2	Yes
33-58	5	7	13	3		Yes
59-71	5	13	13	3	Sz(8)	No
72	5	13	13	3	Sz(8)×2	No
73-83	7	7	7	3	Sz(8)	Yes
84-85	7	7	7	3	Sz(8)×2	Yes
86	7	7	7	1	Sz(8):3	Yes
87-125	7	7	13	3	Sz(8)	Yes
126-128	7	7	13	3	Sz(8)×2	Yes
129-166	7	13	13	3	Sz(8)	Yes
167-169	7	13	13	3	Sz(8)×2	Yes
170-179	13	13	13	3	Sz(8)	No
180-181	13	13	13	3	Sz(8)×2	No
182-183	13	13	13	1	Sz(8):3	No

Table 1 lists, up to isomorphism, the thin geometries on which Sz(8) acts flag-transitively. The columns a, b, and c correspond to those a, b and c appearing in the diagram given above. The column "#CC" gives the number of conjugacy classes of geometries. These conjugacy classes are fused under $Aut(G)$. The correlation groups are given in the column $Cor(\Gamma, G)$. We do not mention the automorphism groups because $Aut(\Gamma, G) = G$ for every thin geometry Γ. We mention also when a geometry satisfies the Rwpri condition. If this condition is satisfied, the corresponding geometry was already mentioned in [12].

In [14], a concept of derived geometry is defined. It would be too long to recall that definition here. Roughly speaking, the construction is applyable to geometries having points and pairs of points as some of their varieties. If these geometries satisfy some additional conditions, we can replace the pairs of points by a copy of the points to obtain a new geometry. This process goes in the opposite way of the Neumaier construction (see section 5) which is a linearization process. Table 2 gives for each geometry having one of a, b or $c = 2$, the number of the derived geometry in the sense of [14].

If we decide to group geometries with their derived geometries and just count the "primitive" ones, then we have 176 geometries instead of 183.

Table 2: The derived geometries

Γ	Γ'
1	13 (replacing 2 by 5) and 32 (replacing 2 by 7)
2	18 (replacing 2 by 5) and 72 (replacing 2 by 13)
3	84 and 85 (replacing 2 by one of the two 7)
4-6	126-128 (replacing 2 by 7) and 167-169 (replacing 2 by 13)
7	180 and 181 (replacing 2 by one of the two 13)

Table 3 lists the rank 2 geometries appearing as truncations of the thin geometries. The first column gives a number to each of them. The next two columns give the structure of the maximal parabolic subgroups. And the last three columns give the diameters and the gonality of these geometries.

Table 3: Rank 2 geometries of Sz(8) appearing as truncations of the thin geometries

Nr.	G_0	G_1	d_{01}	g_{01}	d_{10}
1	D_4	D_{10}	16	7	15
2	D_4	D_{14}	14	5	13
3	D_4	D_{26}	10	5	10
4-5	D_{10}	D_{10}	9	4	10
6			10	5	10
7	D_{10}	D_{14}	8	2	8
8			8	3	8
9-11			8	4	8
12			10	3	9
13	D_{10}	D_{26}	6	3	6
14			8	2	8
15			8	3	8
16			8	4	8
17	D_{14}	D_{14}	7	2	7
18-22			7	3	7
23			7	4	7
24-26	D_{14}	D_{26}	6	2	6
27-29			6	3	6
30-31	D_{26}	D_{26}	6	2	6
32			6	3	6

Table 4 gives, for each of the 183 thin geometries, its rank 2 truncations. The first column contains the number of the geometry that is analysed. The next three columns give the three truncations $\Gamma^{ij} = \Gamma(G; G_i, G_j)$. Sometimes, two truncations that are non-isomorphic have the same parameters. For these, when they appear as truncations in table 4, we precise their number. For example, geometry 4 of table 4 has a truncation Γ^{12} isomorphic to geometry number 25 of table 3.

A $*$ at the end of the information given in one of these three columns means that the corresponding truncation is isomorphic to the dual of the geometry given in table 3. For example, geometry 3 of table 4 has a truncation Γ^{12} which is isomorphic to the dual of geometry 17 appearing in table 3.

Table 4: The truncations of the thin geometries

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
1	$16-7-15$	$14-5-13$	$8-2-8$
2	$16-7-15$	$10-5-10$	$8-2-8$
3	$14-5-13$	$14-5-13$	$7-2-7*$
4	$14-5-13$	$10-5-10$	$6-2-6^{(25)}$
5	$14-5-13$	$10-5-10$	$6-2-6^{(26)}$
6	$14-5-13$	$10-5-10$	$6-2-6^{(24)}$
7	$10-5-10$	$10-5-10$	$6-2-6^{(31)}$
8	$9-4-10^{(4)}*$	$9-4-10^{(4)}$	$9-4-10^{(4)}*$
9	$9-4-10^{(4)}*$	$8-4-8^{(9)}$	$10-3-9$
10	$9-4-10^{(4)}*$	$8-4-8^{(9)}$	$8-4-8^{(9)}$
11	$9-4-10^{(4)}*$	$8-4-8^{(9)}$	$8-4-8^{(11)}$
12	$9-4-10^{(4)}*$	$8-4-8^{(10)}$	$8-2-8$
13	$10-5-10$	$8-2-8$	$8-2-8$
14	$9-4-10^{(4)}$	$6-3-6$	$8-2-8$
15	$10-5-10$	$6-3-6$	$6-3-6$
16	$9-4-10^{(4)}*$	$6-3-6$	$8-2-8$
17	$9-4-10^{(4)}$	$8-2-8$	$8-2-8$
18	$10-5-10$	$8-2-8$	$8-2-8$
19	$8-3-8$	$8-3-8$	$7-3-7^{(19)}$
20	$8-4-8^{(10)}$	$8-3-8$	$7-3-7^{(20)}*$
21	$8-4-8^{(10)}$	$8-3-8$	$7-3-7^{(21)}$
22	$8-4-8^{(11)}$	$8-4-8^{(11)}$	$7-4-7$
23	$8-4-8^{(10)}$	$8-4-8^{(11)}$	$7-2-7*$
24	$8-2-8$	$8-4-8^{(11)}$	$7-2-7$
25	$8-4-8^{(10)}$	$8-4-8^{(11)}$	$7-3-7^{(21)}$
26	$10-3-9$	$8-4-8^{(11)}$	$7-3-7^{(18)}$
27	$8-4-8^{(9)}$	$8-4-8^{(9)}$	$7-3-7^{(18)}$
28	$8-4-8^{(10)}$	$8-4-8^{(9)}$	$7-3-7^{(18)}$
29	$8-4-8^{(10)}$	$8-4-8^{(9)}$	$7-4-7$
30	$10-3-9$	$8-2-8$	$7-3-7^{(20)}$
31	$8-2-8$	$8-2-8$	$7-2-7$
32	$8-2-8$	$8-2-8$	$7-4-7$
33	$8-4-8^{(9)}$	$8-2-8$	$6-2-6^{(24)}$

Table 4: The truncations of the thin geometries

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
34	$8-3-8$	$6-3-6$	$6-2-6^{(24)}$
35	$8-2-8$	$6-3-6$	$6-2-6^{(24)}$
36	$8-4-8^{(11)}$	$8-2-8$	$6-2-6^{(24)}$
37	$8-3-8$	$8-2-8$	$6-2-6^{(24)}$
38	$8-2-8$	$8-2-8$	$6-2-6^{(24)}$
39	$8-4-8^{(9)}$	$8-2-8$	$6-2-6^{(24)}$
40	$8-4-8^{(11)}$	$8-3-8$	$6-2-6^{(24)}$
41	$8-4-8^{(10)}$	$8-2-8$	$6-3-6^{(27)}$
42	$8-4-8^{(10)}$	$8-3-8$	$6-3-6^{(27)}$
43	$8-4-8^{(11)}$	$6-3-6$	$6-3-6^{(27)}$
44	$8-2-8$	$6-3-6$	$6-3-6^{(27)}$
45	$8-4-8^{(9)}$	$6-3-6$	$6-3-6^{(27)}$
46	$8-3-8$	$6-3-6$	$6-3-6^{(27)}$
47	$10-3-9$	$8-2-8$	$6-3-6^{(27)}$
48	$8-3-8$	$6-3-6$	$6-2-6^{(25)}$
49	$8-4-8^{(9)}$	$8-2-8$	$6-2-6^{(25)}$
50	$8-4-8^{(11)}$	$6-3-6$	$6-2-6^{(25)}$
51	$8-3-8$	$8-4-8$	$6-2-6^{(25)}$
52	$8-3-8$	$8-3-8$	$6-2-6^{(26)}$
53	$8-4-8^{(10)}$	$6-3-6$	$6-2-6^{(26)}$
54	$8-4-8^{(10)}$	$6-3-6$	$6-2-6^{(26)}$
55	$8-3-8$	$8-3-8$	$6-3-6^{(28)}$
56	$8-3-8$	$8-3-8$	$6-3-6^{(28)}$
57	$8-4-8^{(11)}$	$8-3-8$	$6-3-6^{(28)}$
58	$8-4-8^{(10)}$	$8-3-8$	$6-3-6^{(29)}$
59	$6-3-6$	$8-3-8$	$6-2-6^{(31)}$
60	$8-2-8$	$6-3-6$	$6-2-6^{(31)}$
61	$8-2-8$	$8-2-8$	$6-2-6^{(31)}$
62	$6-3-6$	$8-2-8$	$6-2-6^{(31)}$
63	$8-3-8$	$8-2-8$	$6-2-6^{(31)}$
64	$6-3-6$	$8-2-8$	$6-2-6^{(31)}$
65	$8-2-8$	$8-4-8$	$6-2-6^{(31)}$
66	$6-3-6$	$8-2-8$	$6-2-6^{(31)}$
67	$6-3-6$	$6-3-6$	$6-2-6^{(31)}$
68	$8-4-8$	$8-3-8$	$6-2-6^{(31)}$
69	$6-3-6$	$8-3-8$	$6-2-6^{(31)}$
70	$8-4-8$	$6-3-6$	$6-3-6$
71	$6-3-6$	$8-3-8$	$6-3-6$
72	$8-2-8$	$8-2-8$	$6-3-6$
73	$7-3-7^{(22)}$	$7-3-7^{(19)}*$	$7-3-7^{(20)}$
74	$7-3-7^{(20)}$	$7-3-7^{(19)}*$	$7-3-7^{(21)}$
75	$7-3-7^{(21)}$	$7-3-7^{(19)}*$	$7-2-7*$
76	$7-3-7^{(21)}*$	$7-3-7^{(19)}*$	$7-3-7^{(21)}$

Table 4: The truncations of the thin geometries

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
77	$7-3-7^{(20)}*$	$7-3-7^{(19)}*$	$7-3-7^{(20)}*$
78	$7-3-7^{(22)}$	$7-3-7^{(20)}*$	$7-3-7^{(21)}$
79	$7-3-7^{(20)}*$	$7-2-7*$	$7-3-7^{(20)}$
80	$7-3-7^{(21)}*$	$7-4-7$	$7-3-7^{(20)}$
81	$7-2-7*$	$7-4-7$	$7-3-7^{(18)}$
82	$7-3-7^{(21)}$	$7-3-7^{(18)}$	$7-2-7$
83	$7-3-7^{(21)}*$	$7-3-7^{(18)}$	$7-3-7^{(18)}$
84	$7-2-7$	$7-4-7$	$7-2-7*$
85	$7-2-7*$	$7-4-7$	$7-2-7$
86	$7-3-7^{(19)}$	$7-3-7^{(19)}*$	$7-3-7^{(19)}$
87	$7-3-7^{(19)}*$	$6-2-6^{(26)}$	$6-2-6^{(24)}$
88	$7-2-7*$	$6-2-6^{(26)}$	$6-3-6^{(29)}$
89	$7-3-7^{(19)}$	$6-3-6^{(29)}$	$6-2-6^{(24)}$
90	$7-3-7^{(21)}*$	$6-2-6^{(24)}$	$6-2-6^{(24)}$
91	$7-3-7^{(19)}$	$6-3-6^{(28)}$	$6-2-6^{(24)}$
92	$7-3-7^{(20)}$	$6-3-6^{(27)}$	$6-2-6^{(24)}$
93	$7-3-7^{(20)}*$	$6-3-6^{(28)}$	$6-2-6^{(24)}$
94	$7-3-7^{(22)}$	$6-2-6^{(26)}$	$6-2-6^{(24)}$
95	$7-2-7*$	$6-2-6^{(25)}$	$6-2-6^{(24)}$
96	$7-2-7*$	$6-3-6^{(27)}$	$6-2-6^{(24)}$
97	$7-4-7$	$6-3-6^{(28)}$	$6-2-6^{(24)}$
98	$7-3-7^{(20)}*$	$6-3-6^{(27)}$	$6-2-6^{(24)}$
99	$7-3-7^{(19)}*$	$6-3-6^{(27)}$	$6-2-6^{(25)}$
100	$7-3-7^{(19)}*$	$6-2-6^{(25)}$	$6-2-6^{(25)}$
101	$7-2-7*$	$6-2-6^{(25)}$	$6-2-6^{(25)}$
102	$7-3-7^{(19)}*$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
103	$7-3-7^{(20)}$	$6-3-6^{(27)}$	$6-2-6^{(25)}$
104	$7-3-7^{(21)}$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
105	$7-3-7^{(20)}*$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
106	$7-3-7^{(20)}$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
107	$7-3-7^{(19)}$	$6-2-6^{(26)}$	$6-2-6^{(26)}$
108	$7-2-7*$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
109	$7-3-7^{(19)}*$	$6-3-6^{(28)}$	$6-3-6^{(28)}$
110	$7-3-7^{(19)}$	$6-3-6^{(27)}$	$6-3-6^{(28)}$
111	$7-2-7*$	$6-2-6^{(26)}$	$6-3-6^{(28)}$
112	$7-3-7^{(18)}$	$6-2-6^{(26)}$	$6-3-6^{(28)}$
113	$7-2-7$	$6-3-6^{(28)}$	$6-3-6^{(28)}$
114	$7-3-7^{(19)}*$	$6-3-6^{(27)}$	$6-3-6^{(28)}$
115	$7-3-7^{(21)}$	$6-3-6^{(29)}$	$6-3-6^{(28)}$
116	$7-3-7^{(19)}$	$6-3-6^{(28)}$	$6-3-6^{(28)}$
117	$7-3-7^{(21)}$	$6-2-6^{(26)}$	$6-3-6^{(28)}$
118	$7-3-7^{(20)}*$	$6-3-6^{(29)}$	$6-3-6^{(28)}$
119	$7-3-7^{(21)}$	$6-3-6^{(27)}$	$6-3-6^{(28)}$

Table 4: The truncations of the thin geometries

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
120	$7-3-7^{(21)}$	$6-3-6^{(29)}$	$6-3-6^{(28)}$
121	$7-3-7^{(18)}$	$6-3-6^{(29)}$	$6-3-6^{(27)}$
122	$7-3-7^{(21)}$	$6-2-6^{(26)}$	$6-3-6^{(27)}$
123	$7-4-7$	$6-3-6^{(27)}$	$6-3-6^{(27)}$
124	$7-3-7^{(21)}*$	$6-2-6^{(26)}$	$6-3-6^{(27)}$
125	$7-3-7^{(20)}*$	$6-2-6^{(26)}$	$6-3-6^{(27)}$
126	$7-4-7$	$6-2-6^{(24)}$	$6-2-6^{(24)}$
127	$7-4-7$	$6-2-6^{(25)}$	$6-2-6^{(25)}$
128	$7-4-7$	$6-2-6^{(26)}$	$6-2-6^{(26)}$
129	$6-2-6^{(25)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
130	$6-3-6^{(29)}$	$6-3-6^{(27)}$	$6-2-6^{(30)}$
131	$6-3-6^{(28)}$	$6-2-6^{(24)}$	$6-2-6^{(30)}$
132	$6-3-6^{(27)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
133	$6-2-6^{(25)}$	$6-2-6^{(24)}$	$6-2-6^{(30)}$
134	$6-2-6^{(24)}$	$6-2-6^{(24)}$	$6-2-6^{(30)}$
135	$6-2-6^{(25)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
136	$6-3-6^{(27)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
137	$6-2-6^{(26)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
138	$6-3-6^{(27)}$	$6-2-6^{(24)}$	$6-3-6$
139	$6-3-6^{(28)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
140	$6-2-6^{(26)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
141	$6-3-6^{(28)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
142	$6-2-6^{(26)}$	$6-3-6^{(29)}$	$6-2-6^{(31)}$
143	$6-2-6^{(25)}$	$6-2-6^{(25)}$	$6-2-6^{(31)}$
144	$6-3-6^{(29)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
145	$6-3-6^{(28)}$	$6-2-6^{(25)}$	$6-3-6$
146	$6-3-6^{(27)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
147	$6-3-6^{(29)}$	$6-2-6^{(25)}$	$6-3-6$
148	$6-2-6^{(26)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
149	$6-2-6^{(26)}$	$6-2-6^{(25)}$	$6-2-6^{(31)}$
150	$6-3-6^{(27)}$	$6-2-6^{(25)}$	$6-3-6$
151	$6-3-6^{(28)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
152	$6-3-6^{(28)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
153	$6-2-6^{(26)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
154	$6-2-6^{(26)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
155	$6-3-6^{(27)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
156	$6-3-6^{(29)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
157	$6-2-6^{(26)}$	$6-3-6^{(28)}$	$6-3-6$
158	$6-3-6^{(27)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
159	$6-3-6^{(28)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
160	$6-2-6^{(26)}$	$6-3-6^{(28)}$	$6-2-6^{(30)}$
161	$6-3-6^{(27)}$	$6-3-6^{(28)}$	$6-3-6$
162	$6-2-6^{(26)}$	$6-3-6^{(27)}$	$6-2-6^{(31)}$

Table 4: The truncations of the thin geometries

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
163	$6-2-6^{(26)}$	$6-3-6^{(27)}$	$6-2-6^{(31)}$
164	$6-3-6^{(27)}$	$6-3-6^{(27)}$	$6-2-6^{(30)}$
165	$6-2-6^{(26)}$	$6-3-6^{(27)}$	$6-2-6^{(31)}$
166	$6-2-6^{(26)}$	$6-3-6^{(27)}$	$6-2-6^{(30)}$
167	$6-2-6^{(24)}$	$6-2-6^{(24)}$	$6-3-6$
168	$6-2-6^{(25)}$	$6-2-6^{(25)}$	$6-3-6$
169	$6-2-6^{(26)}$	$6-2-6^{(26)}$	$6-3-6$
170	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-3-6$
171	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-2-6^{(30)}$
172	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
173	$6-3-6$	$6-2-6^{(31)}$	$6-2-6^{(30)}$
174	$6-3-6$	$6-2-6^{(31)}$	$6-3-6$
175	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-3-6$
176	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
177	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
178	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-3-6$
179	$6-3-6$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
180	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-3-6$
181	$6-3-6$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
182	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-2-6^{(30)}$
183	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-2-6^{(30)}$

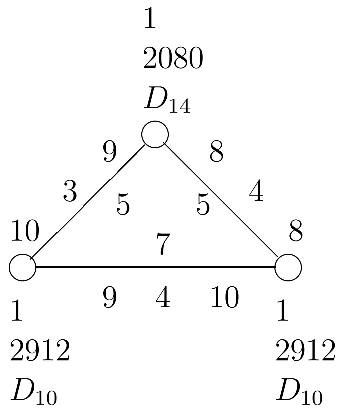
4 Some observations

We give in this section some observations arising from the tables given in the preceding section.

First, let us remark that we get only 15 distinct diagrams for 183 non-isomorphic geometries. It might be a good idea to add some information on our diagrams. We now decide to add three parameters on each edge of the diagram, that are the three parameters of the rank 2 truncation obtained by the two vertices of an edge.

Definition 4.1 *The* extended diagram *of a firm, residually connected, flag-transitive geometry is a diagram (as defined in section 2), whose structure is completed by adding between every pair of vertices three parameters that are the diameters and the gonality of the corresponding rank 2 truncation.*

For example, geometry number 9 has an extended diagram as follows.



B = 1

Looking at the truncations of table 4, we obtain 75 distinct extended diagrams.

It is of course not sufficient to look at the truncations in order to say whether two geometries are isomorphic or not. If two geometries have distinct rank 2 truncations, they are non-isomorphic. But the converse is not necessarily true. There are several examples appearing here. The geometries having three rank 2 truncations in common are (33,39), (53,54), (55,56), (59,69), (62,64,66), (84,85), (105,106), (109,116), (115,120), (129,135), (132,136), (137,140), (141,151,152), (148,153), (155,158), (162,163,165).

If we observe table 4, we see that a lot of truncations do not satisfy the $(IP)_2$ condition. A new condition to impose on our geometries is the following.

Condition 4.2 Γ *satisfies the* strong intersection property of rank 2 $(SIP)_2$ *if* Γ *is* $(IP)_2$ *and for every edge of the diagram of* Γ*, the corresponding truncation satisfies the* $(IP)_2$ *condition.*

Remark that when a geometry has a rank 2 residue which is a generalized digon, the rank 2 truncation obtained by the two vertices that are not joined by an edge does not satisfy $(IP)_2$ or is a generalized digon. This is why we restrict ourselves to truncations corresponding to edges of the diagram. Out of the 183 thin geometries of Sz(8), only 50 are $(SIP)_2$. This new condition seems attractive because the intersection property (IP) implies it for every geometry of finite rank.

Theorem 4.3 *Let* Γ *be a geometry of rank* $n < \infty$*. If* Γ *satisfies* (IP)*, then* Γ *satisfies* $(SIP)_2$*.*

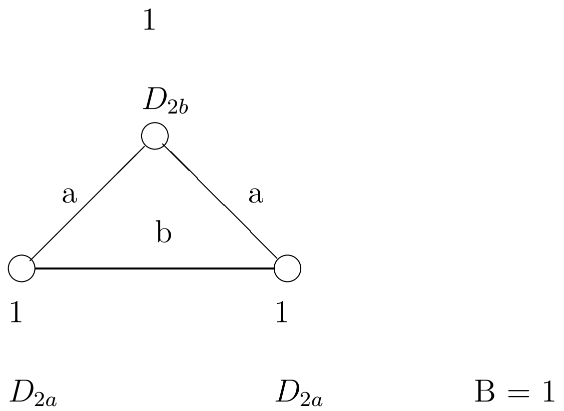
Proof. Let I be the set of types of Γ. Suppose Γ is not $(SIP)_2$. To show that Γ cannot be (IP), we suppose it is (IP) and we derive a contradiction. Because Γ is not $(SIP)_2$, there exist two types 0 and 1 in I such that the residue of a flag of type $I \setminus \{0,1\}$ has gonality at least 3 and the truncation $\Gamma^{\{0,1\}}$ has a circuit of length 4. Hence there are two 1-elements, say e and e', such that $\mid \sigma_0(e) \cap \sigma_0(e') \mid \geq 2$. Because Γ satisfies (IP), it must have a flag F such that $e * F * e'$ and $\sigma_0(F) = \sigma_0(e) \cap \sigma_0(e')$. Thus $\sigma_0(F)$ has at least two 0-elements. It implies that F is a flag of type $t(F) \subseteq I \setminus \{0,1\}$.
If $t(F) = I \setminus \{0,1\}$, then the geometry Γ_F is not $(IP)_2$ and thus Γ cannot be (IP), a contradiction.
Otherwise, the geometry Γ_F, whose rank is smaller than the rank of Γ, satisfies (IP) and does not satisfy $(SIP)_2$. We may then suppose $\Gamma = \Gamma_F$ and start the discussion again until we get the contradiction. ∎

In a first version of this paper, we stated theorem 4.3 for flag-transitive geometries. Antonio Pasini showed us that it was not necessary to impose the flag-transitivity in the hypotheses of this theorem.

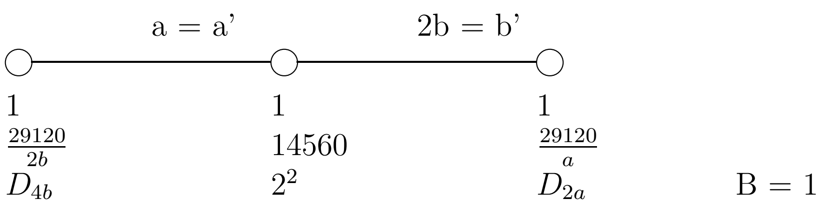
Because buildings satisfy (IP), it might be interesting to impose this new property. Also, theorem 4.3 shows that $(SIP)_2$ is a necessary condition for (IP). This is very helpful because (IP) is not easy to test with a computer whereas $(SIP)_2$ is. For example, the second rank 4 geometry mentionned in [10] for the Janko group J_1 cannot satisfy (IP) because one of its rank 2 truncations, namely the one with $PSL(2,11)$ and $S_3 \times D_{10}$ is not $(IP)_2$ and hence this geometry is not $(SIP)_2$.

5 Applying the Neumaier construction

This construction is detailed in [15]. It would be too long to recall it here. We just give a rough idea of how it works. We start from a diagram as the one given below.



Applying the Neumaier construction to a geometry corresponding to the diagram given above, we obtain another geometry whose diagram is given below, for the group Sz(8)×2. This group acts chirally or regularly on the geometry, depending on the correlation group of the starting geometry. Roughly speaking, if the starting geometry has a duality involving the two D_{2a}, then Sz(8)×2 acts regularly on the new geometry. Otherwise, it acts chirally. Remark that the starting diagram was a triangle whereas the diagram of the geometry obtained by the Neumaier construction is linear. This construction may be seen as a linearisation process of the diagram of a geometry.



In table 5 we give for each thin geometry obtained in section 3 the geometry obtained by applying this construction (when the construction is applyable) and we mention whether the group Sz(8)×2 acts chirally or regularly.

Table 5: The geometries obtained by the Neumaier construction

Nr.	a	b	c	a'	b'	Chiral or Regular
3	2	7	7	4	7	Chiral
7	2	13	13	4	13	Chiral
8	5	5	5	3	5	Chiral
9-12	5	5	7	5	14	Chiral
13	5	5	7	5	14	Regular
14-17	5	5	13	5	26	Chiral
18	5	5	13	5	26	Regular
19-31	5	7	7	7	10	Chiral
32	5	7	7	7	10	Regular
59-71	5	13	13	10	13	Chiral
72	5	13	13	10	13	Regular
73-83	7	7	7	7	14	Chiral
84-85	7	7	7	7	14	Regular
86	7	7	7	3	7	Chiral
87-125	7	7	13	7	26	Chiral
126-128	7	7	13	7	26	Regular
129-166	7	13	13	13	14	Chiral
167-169	7	13	13	13	14	Regular
170-179	13	13	13	13	26	Chiral
180-181	13	13	13	13	26	Regular
182-183	13	13	13	3	13	Chiral

Remark that, according to Francis Buekenhout, Jacques Tits arrived at the same construction independently.

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Dimitri Leemans
Université Libre de Bruxelles
Département de Mathématique
C.P.216 - Géométrie
Boulevard du Triomphe
B-1050 Bruxelles
E-Mail : dleemans@cso.ulb.ac.be