Plane representations of ovoids

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Abstract

Various ways of representing an ovoid of $\mathrm{PG}(3,q)$, q even, in $\mathrm{PG}(2,q)$ are studied. These come from a kind of 'spread' of lines of $\mathrm{AG}(3,q)$, and involve five kinds of 'pencils' of ovals in the plane.

1 Introduction

An *ovoid* of $\mathrm{PG}(3,q)$ is defined to be a set of q^2+1 points, no three collinear, ($q>2$). In other words, it is a (q^2+1)-cap of $\mathrm{PG}(3,q)$. Barlotti [2] and Panella [16] showed that when q is odd all ovoids are elliptic quadrics. The even case is still not completely solved.

Any ovoid corresponds to an inversive (or Möbius) plane, by taking the structure of the points together with the non-tangent plane sections as the 'circles'. In Dembowski [3] we see that every inversive plane of even order q can be constructed from an ovoid of $\mathrm{PG}(3,q)$, whereas for q odd there might even exist inversive planes of non-prime-power orders.

Remark 1 *Henceforth we shall assume that $q=2^h$, $h\in\mathbb{Z}$, $h\geq 2$.*

There are two infinite sequences of known ovoids: these are the elliptic quadrics, which occur for every prime-power q; and also the Suzuki-Tits ovoids, which occur for $q=2^h$, h odd, $h\geq 3$. Glynn [4] has shown that any new ovoid would have a small group of central automorphisms, and that it is unlikely to contain any conics as plane sections, because then a whole sequence of perhaps non-isomorphic ovoids

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would be obtained. Many people have tried to classify the ovoids in $\mathrm{PG}(3,q)$. More recently, O'Keefe and Penttila [11], [12], and O'Keefe, Penttila and Royle [15] have succeeded for $q \leq 32$. Thus the present evidence suggests that the known ovoids are the only ones to exist in finite space.

The theory of ovoids depends to a large extent upon the theory of hyperovals and ovals of $\mathrm{PG}(2,q)$: a *hyperoval* or $(q+2)$-arc is a set of $q+2$ points of $\mathrm{PG}(2,q)$, no three collinear; an *oval* or $(q+1)$-arc is a set of $q+1$ points of $\mathrm{PG}(2,q)$, no three collinear. When q is even every oval O is contained in a unique hyperoval by the addition of a point P, that is called the *nucleus* of O. One of the main problems of $\mathrm{PG}(2,q)$ is the classification of hyperovals, which is still incomplete. See Glynn [5] for some constructions and theory.

Various properties of an ovoid are summarised as follows; see e.g. Dembowski [3].

1. The set of tangent lines (lines that intersect the ovoid in one point) are the q^3+q^2+q+1 lines of a linear complex, which is the set of absolute lines of the symplectic polarity ω associated with the ovoid.

2. The tangent lines at a point X of the ovoid are the $q+1$ lines passing through X contained in the plane X^ω, which is called the tangent plane at X.

3. Every non-tangent line intersects the ovoid in 0 or 2 points.

4. Every non-tangent plane intersects the ovoid in an oval.

Let us refer to Payne and Thas [17]. The q^3+q^2+q+1 points of $\mathrm{PG}(3,q)$ together with the totally isotropic (i.e. 'absolute' or 'fixed') lines of a symplectic polarity (i.e. the lines of a linear complex) form an incidence structure called $W(q)$. This is a 'classical' generalized quadrangle. Any ovoid of $\mathrm{PG}(3,q)$ then corresponds to an ovoid of $W(q)$: an *ovoid* of a generalized quadrangle (GQ) is a set of points, such that every line of the GQ is on a unique point of the ovoid. The dual concept in a GQ is that of *spread*. This is a set of lines, such that every point of the GQ is on a unique line of the spread. The cardinality of an ovoid or spread of $W(q)$ is q^2+1.

We need a few basic facts about $W(q)$ and related GQ's. The section numbers refer to Payne and Thas [17]. First (3.1.1), $Q(4,q)$ is the GQ of order q coming from the set of points of a non-singular quadric of $\mathrm{PG}(4,q)$, and the lines contained in it. Second (3.1.2), if O is an oval of $\mathrm{PG}(2,q)$, that in turn is embedded in a plane π of $\mathrm{PG}(3,q)$, J. Tits built a structure $T_2(O)$, that is a GQ of order q, in the following manner.

Remark 2 *The construction of* $T_2(O)$

The *points* of $T_2(O)$ are of three types:
(i) the q^3 points of $\mathrm{AG}(3,q) := \mathrm{PG}(3,q) \setminus \pi$;
(ii) the q^2+q planes Y of $\mathrm{PG}(3,q)$ for which $|Y \cap O| = 1$, i.e. Y intersects π in a line through the nucleus P of O;
(iii) a new symbol (∞).
The *lines* of $T_2(O)$ are of two types:
(a) the $q^2(q+1)$ lines of $\mathrm{PG}(3,q)$ not contained in π, intersecting O in one point;

(b) the $q+1$ points of O.
We have to say when these 'points' and 'lines' are *incident*; pairs not given below are non-incident:
(i-a) a point of type (i) is incident with the $q+1$ lines of type (a) passing through it in $\mathrm{PG}(3,q)$;
(ii-a) a point of type (ii) is incident with the q lines of type (a) contained within it;
(ii-b) a point of type (ii) is incident with the line of type (b) that it contains in $\mathrm{PG}(3,q)$;
(iii-b) the point (∞) is incident with the $q+1$ lines of type (b).

Theorem 1 *1. $W(q)$ is self-dual if and only if q is even, (3.2.1).*

2. The dual of $W(q) \cong Q(4,q)$, (q even or odd), (3.2.1).

3. $Q(4,q) \cong T_2(O) \iff O$ is an irreducible conic, (3.2.2). From these it follows that, if q is even,

4. $W(q) \cong$ the dual of $W(q) \cong T_2(O) \iff O$ is an irreducible conic, (3.2.2).

We shall use this final consequence heavily for our representations of ovoids. Let us here list various correspondences between an object of $W(q)$ and one of $T_2(O)$, where O is an irreducible conic. We take a point of $W(q)$ to a line of $T_2(O)$, and a line of $W(q)$ to a point of $T_2(O)$. Suppose that $W(q)$ has the usual representation in $\mathrm{PG}(3,q)$, q even, the lines fixed by a symplectic polarity ω.

Remark 3 *The correspondence ρ between the dual of $W(q)$ and $T_2(O)$*

1. line i of $\mathrm{PG}(3,q) \leftrightarrow$ 'point' (∞).

2. line of $W(q)$ intersecting i in a point $\leftrightarrow$ plane 'tangent' to O; that is, passing through the nucleus P of O, and not equal to the plane π of O.

3. line of $W(q)$ skew to $i \leftrightarrow$ point of $\mathrm{AG}(3,q) := \mathrm{PG}(3,q) \setminus \pi$.

4. point of the line $i \leftrightarrow$ point of the conic O.

5. point of $\mathrm{PG}(3,q)$ not on the line $i \leftrightarrow$ line not contained in π intersecting O in a point.

6. the points of a line r of $\mathrm{PG}(3,q)$ not in $W(q)$, where $r \cap i$ is a point $\leftrightarrow$ pencil of q affine lines in plane ρ secant to O, passing through a point A of O; note that $r \cap i \leftrightarrow B$, where $\rho \cap O = \{A, B\}$. Points on the conjugate line $r^\omega \leftrightarrow$ lines of the other pencil in ρ through B, and then $r^\omega \cap i \leftrightarrow A$.

7. points of a line t of $\mathrm{PG}(3,q)$ not in $W(q)$ and skew to the line $i \leftrightarrow$ regulus of $q+1$ skew lines, each line of which intersects O in a distinct point; note that the conjugate line with respect to $\omega \leftrightarrow$ the opposite regulus contained in the same hyperbolic quadric.

8. points of a plane α through $i \leftrightarrow$ lines of $\mathrm{AG}(3,q)$ through point A of O.

9. points in a plane β not through $i \leftrightarrow$ lines joining points of a fixed line n to points of O, where $n \cap O$ is a point A, and q lines in pencil of tangent plane to O at point A. Thus $\beta^\omega \leftrightarrow n$.

10. ovoid of $W(q) \leftrightarrow$ point Q (w.l.o.g. $(0,0,1,0)$), plus q 'oval-cones' of a 'spread' of $\mathrm{AG}(3,q)$; see below.

Note that there exists a unique hyperbolic quadric containing the conic O and two skew lines, not in the plane π of the conic, that intersect O in distinct points. In fact we can verify this by direct calculation as follows.

Let $O : x_1^2 = x_2x_3$ in the plane $\pi : x_0 = 0$. Then the quadric containing O and the pair of lines $\langle(0,0,1,0),(1,a,b,b)\rangle$ and $\langle(0,0,0,1),(1,c,d,d)\rangle$, where $a \neq c$, has equation

$$x_1^2 + x_2x_3 + x_0((bd+ac)x_0 + (c-a)x_1 + bx_2 + dx_3) = 0.$$

Reversing a and c gives the same quadric. In this way we obtain all $(q^4 - q^3)/2$ hyperbolic quadrics containing O and two general skew lines through points of O. The pair of reguli of the above quadric (when $b \neq 0$) contain lines of the form

$$\langle(0,s,s^2,1),(1,t,(t^2+(c-a)t+bd+ac)/b,0)\rangle,$$

where $s = (t+c)/b$, or $(t+a)/b$.

2 Ovoids and spreads of $\mathbf{T_2(O)}$

We are interested in the case that O is an irreducible conic of $\mathrm{PG}(2,q)$, and when q is even. However, some results of this section should be valid for general ovals O, and for q odd as well. Although we do not strictly need this lemma for our later results it is interesting to point out that:

Lemma 1 *An ovoid of $T_2(O)$ consists of either a set of q^2 points of type* (i), and the point (∞) of type (iii); or it has $q^2 - q$ points of type (i), and $q+1$ points of type (ii).

Proof. If the ovoid contains (∞), then it cannot contain points of type (ii), as these latter points are on lines of type (b) with (∞). We leave the remainder of the proof as an easy exercise, similar to Lemma 2.2 below. ■

Note that the former situation is related to the concept of 'ovaloid' as studied in Glynn [4]. An *ovaloid* is a set of q^2 points of $\mathrm{AG}(3,q)$ not on π, no pair of points generating a line passing through a point of O. (This condition implies that such a line cannot also pass through the nucleus P of O.) Thus an 'ovaloid' (of the Ahrens-Szekeres GQ), and an ovoid of the former type are equivalent.

The latter situation is related to the construction of a new ovoid of $\mathrm{PG}(3,q)$ from any ovoid of $\mathrm{PG}(3,q)$ containing a conic plane section. For let the ovoid pass through O. Then the remaining points of the ovoid plus the tangent planes at the points of O correspond to an ovoid of $T_2(O)$ of the latter type. Using the isomorphism between $T_2(O)$ and $W(q)$ we get a new ovoid of $\mathrm{PG}(3,q)$. In fact, the two ovoids of $\mathrm{PG}(3,q)$ are related by a quadratic transformation, and the new ovoid also contains a conic plane section; see Glynn [4]. There do exist ovoids of $T_2(O)$ of the latter type that are not parts of ovoids of $\mathrm{PG}(3,q)$, as can be seen by the representation of the Suzuki-Tits examples, which do not contain conic plane sections in their conventional $\mathrm{PG}(3,q)$ ovoid representation.

Lemma 2 *A spread of $T_2(O)$ consists of q^2 lines of type (a) and one of type (b).*

Proof. The point (∞) must be incident with a line of the spread, which in turn must be type (b). All lines of type (b) are incident with (∞) and so the spread contains exactly one line of type (b). All other lines of $T_2(O)$ are type (a), and so the spread contains q^2 of them, because a spread of $W(q)$ contains q^2+1 lines. ∎

We can assume, by using collineations of $\mathrm{PG}(3,q)$, that a spread of $T_2(O)$, where O is a conic, contains a certain point Q of O. Also, using homogeneous coordinates (x_0,x_1,x_2,x_3) over $\mathrm{GF}(q)$ for points of $\mathrm{PG}(3,q)$, we can assume that $\pi : x_0 = 0$, and that $O : x_1^2 = x_2x_3,\ x_0 = 0$. This conic has *nucleus* $P : (0,1,0,0)$. Further, we can assume that $Q = V(\infty) : (0,0,1,0)$.

Definition

An *oval-cone* $\mathrm{OC}(t)$ ($t \in \mathrm{GF}(q)$) is a set of q lines of $\mathrm{PG}(3,q)$ not in the plane π, each line passing through the point $V(t) : (0,t,t^2,1)$ of the conic O of π. Letting $\mathrm{OC}'(t) := \mathrm{OC}(t) \cup \{V(t).Q, V(t).P\}$, the resulting set of $q+2$ lines is a cone over a hyperoval: that is, it has no three lines coplanar.

Lemma 3 *A spread of $T_2(O)$ is $\{Q\} \cup \{\mathrm{OC}(t) \mid t \in \mathrm{GF}(q)\}$, where each $\mathrm{OC}(t)$ is an oval-cone with vertex $V(t)$. The q^2 lines of these oval-cones form a spread of $\mathrm{AG}(3,q)$: thus every point of $\mathrm{AG}(3,q) = \mathrm{PG}(3,q) \setminus \pi$ is on a unique line of one of the oval-cones.*

Proof. Consider the correspondence ρ from the dual of $W(q)$ to $T_2(O)$, where $W(q)$ corresponds to the symplectic polarity ω of $\mathrm{PG}(3,q)$. We refer to Remark 3. We can consider ρ to map the points of $\mathrm{PG}(3,q)$, not on a line i of $W(q)$, to the lines of $\mathrm{AG}(3,q)$ intersecting the conic O in a point. An ovoid of $\mathrm{PG}(3,q)$ has i as a tangent line, and so they intersect in a point Z, which is mapped by ρ to a point of O, which we may take to be $Q := V(\infty) := (0,0,1,0)$; see Lemma 2. Since each line of $W(q)$ is tangent to a unique point of the ovoid, we see that a spread of q^2 lines of $\mathrm{AG}(3,q)$ is obtained. It remains to show that the spread is partitioned into q oval-cones, with vertices at each point (not Q) of O. Consider a plane of $\mathrm{PG}(3,q)$ through the line i, not being tangent to the ovoid. It intersects the ovoid in an oval, with i as a tangent. Now Remark 3(8) implies that the q points (not Z) of the ovoid in this plane are mapped by ρ to q lines through a point, w.l.o.g. $V(t)$, of O. Further, by 3(2), no two of these lines can lie in a plane through the nucleus P of O, because such a plane comes from a line of $W(q)$ intersecting i in a point. This means that the "nucleus line" of the oval-cone is the line $V(t).P$. Next, suppose that two lines of the oval-cone and the line $V(t).Q$ were coplanar. Using 3(6) there would be two points (not Z) on a line not in $W(q)$ through Z. This is impossible since we have an ovoid. Hence the line $V(t).Q$ is part of the oval-cone. Finally, for the same reason

3(6), any three points (not Z) on the ovoid and on the plane through i are mapped by ρ to non-coplanar lines through $V(t)$. ■

The main conclusion of this paper follows from the fact that we can choose any plane of $\mathrm{AG}(3,q)$ and intersect it with the q oval-cones to obtain a *pencil* of q q-arcs partitioning the points of the affine plane. We can rebuild the spread of $T_2(O)$ given an association between the q-arcs and the points of O. We shall find the conditions that the arcs determine a spread. There are essentially five different ways to choose the plane with respect to the conic O, and the pair of points P and Q in π. Thus we obtain five different, but similar, planar representations of an ovoid. We label these A, B, C, D and E. Note that Penttila also discovered representation A independently at about the same time as the author. He used an algebraic method that gave a direct transformation between the ovoid and the pencil of ovals in the plane; see [18].

3 Five kinds of planar representation of an ovoid of PG$(3, q)$

Let α be a plane of $\mathrm{PG}(3,q)$, passing through neither $V(s)$ nor $V(t)$. The pair of oval-cones $\mathrm{OC}(s)$ and $\mathrm{OC}(t)$ of $\mathrm{PG}(3,q)$, with different vertices $V(s)$ and $V(t)$ respectively in π, intersect α in the pair of q-arcs, $O(s) := \mathrm{OC}(s) \cap \alpha$, and $O(t) := \mathrm{OC}(t) \cap \alpha$. Let L be the line $\alpha \cap \pi$. Finally let $L(s,t)$ be the point $L \cap V(s).V(t)$.

Lemma 4 $\mathrm{OC}(s)$ *and* $\mathrm{OC}(t)$ *are disjoint if and only if each line (not L) through $L(s,t)$ in α is either a chord of $O(s)$ and external to $O(t)$, or external to $O(s)$ and a chord of $O(t)$.*

Proof. Firstly, note that there are $q/2$ chords from $L(s,t)$ to $\mathrm{OC}(s)$, and so $q/2$ external lines through this point in the affine plane $\pi \setminus L$. The same can be said if we replace $\mathrm{OC}(s)$ by $\mathrm{OC}(t)$. There are q planes (not π) through the line $V(s).V(t)$. These intersect the plane α in q distinct lines. Two lines from each of $\mathrm{OC}(s)$ and $\mathrm{OC}(t)$ are coplanar if and only if they intersect in a point. Thus the lemma holds. ■

We can use the fact that O has a group of collineations which is triply transitive on its points and transitive on external lines. Thus we get five different situations, corresponding to the following cases:

A: L passes through $P : (0,1,0,0)$ and $Q : (0,0,1,0)$;

B: L passes through $P : (0,1,0,0)$ and $(0,0,0,1)$;

C: L passes through $Q : (0,0,1,0)$ and $(0,0,0,1)$;

D: L intersects O in $(0,0,0,1)$ and $(0,1,1,1)$;

E: L is the external line $x_3 = x_1 + \lambda x_2$ of π, where λ is second category in $\mathrm{GF}(q)$. (Equivalently, $\mathrm{trace}(\lambda) = 1$.)

Again, using the group of $\mathrm{PG}(3,q)$ fixing O, we can assume that the plane α, and the corresponding point $L(s,t)$ in the five situations have the equations and coordinates:

A: $\alpha : x_3 = 0,\ L(s,t) : (0, 1, s+t, 0)$;

B: $\alpha : x_2 = 0,\ L(s,t) : (0, st, 0, s+t)$;

C: $\alpha : x_1 = 0,\ L(s,t) : (0, 0, st, 1)$;

D: $\alpha : x_1 = x_2,\ L(s,t) : (0, st, st, s+t+1)$;

E: $\alpha : x_3 = x_1 + \lambda x_2$, λ second category, $L(s,t) : (0, 1+\lambda st, s+t+st, 1+\lambda(s+t))$.

Next, we have to see where the lines $V(s).P$ and $V(s).Q$ intersect L, for these points can be adjoined to the affine q-arc $O(s)$ of $\alpha \setminus L$ to obtain a hyperoval. Let these points be $P(s)$ and $Q(s)$ respectively.

A: $P(s) = P : (0, 1, 0, 0),\ Q(s) = Q : (0, 0, 1, 0)$;

B: $P(s) = P : (0, 1, 0, 0),\ Q(s) : (0, s, 0, 1),\ (s \neq 0)$;

C: $P(s) : (0, 0, s^2, 1),\ Q(s) = Q : (0, 0, 1, 0),\ (s \neq 0)$;

D: $P(s) : (0, s^2, s^2, 1),\ Q(s) : (0, s, s, 1),\ (s \notin \{0, 1\})$;

E: $P(s) : (0, \lambda s^2 + 1, s^2, 1),\ Q(s) : (0, s, \lambda^{-1}(s+1), 1)$.

If L contains the vertex $V(s)$ of the cone $\mathrm{OC}(s)$ then the plane α contains either one line of $\mathrm{OC}(s)$ in the cases B and C above, or zero or two lines of $\mathrm{OC}(s)$, in the case D. (For the cases A and E this situation is not applicable.) In particular, in the case D we can assume that α contains precisely two lines of $\mathrm{OC}(0)$ through $V(0) : (0, 0, 0, 1)$ and no lines of $\mathrm{OC}(1)$ through $V(1) : (0, 1, 1, 1)$.

Finally, let us rename the q-arcs $O(s)$ in each of the five cases $A(s)$, $B(s)$, ..., respectively. Hence we can summarize the resulting representations of an ovoid in each of the cases. We use a collineation γ from α to a standard $\mathrm{PG}(2,q)$ with coordinates (x, y, z). The "line at infinity" corresponding to L is always assumed to have equation $z = 0$, and so $\mathrm{AG}(2,q)$ is assumed below to be the set of all points not on this line.

Here we give the theorems that classify the five types of pencils of ovals. The completion of a q-arc $X(s)$, $X \in \{A, B, C, D, E\}$, to a hyperoval $X'(s)$ is given. Remembering the correspondence with the original ovals, the *nucleus* point is always the first of the pair of points that are adjoined. It is $V(s).P \cap L$.

For some of the pencils, $A(s)$ is not a q-arc for one or two values in $\{0, 1\}$. Then we can assume that the remaining points of the affine plane are filled out by one or two lines, which can be defined to be $X(0)$ or $X(1)$: we give equations for these lines that can be assumed using the group of collineations of $\mathrm{AG}(2,q)$.

There may be various points missed out on L, not lying on one of the hyperovals or lines. These occur in the pencils of type D and E, and come from the intersection of the line PQ with L, or from the fact about D two paragraphs before. Later, we shall give the linear pencils of conics that correspond to each of these cases (in the elliptic quadric, or *Miquelian* case). Then the one or two points of L, not appearing to occur in the union of the hyperovals and lines of the pencil, actually appear as the points of imaginary line-pairs, which are types of degenerate conics.

Finally, in each of the five theorems below, we give the point on L, through which an affine line is either a secant to $X(s)$ and external to $X(t)$, or external to $X(s)$ and a chord of $X(t)$, for any pair of distinct hyperovals $X(s)$ and $X(t)$ in the pencil. From Lemma 4, this point is $L(s,t)$.

Theorem 2 *An ovoid of* $\mathrm{PG}(3,q)$ *is equivalent to pencil* $\{A(s) \mid s \in \mathrm{GF}(q)\}$, *where each* $A(s)$ *is a* q*-arc of* $\mathrm{AG}(2,q)$. *Here* $A'(s) = A(s) \cup \{(1,0,0),(0,1,0)\}$ *and* $L(s,t) = (1, s+t, 0)$.

Proof. The collineation $\gamma : \alpha(x_3 = 0) \to \mathrm{PG}(2,q)$ is $(x_0,x_1,x_2,x_3) \mapsto (x_1,x_2,x_0)$. ∎

Theorem 3 *An ovoid of* $\mathrm{PG}(3,q)$ *is equivalent to a pencil* $\{B(s) \mid s \in \mathrm{GF}(q)\}$, *where each* $B(s)$ *is a* q*-arc of* $\mathrm{AG}(2,q)$, $s \neq 0$, *and* $B(0)$ *is the line* $x = 0$. *Here* $B'(s) = B(s) \cup \{(1,0,0),(s,1,0)\}$, $s \neq 0$ *and* $L(s,t) = (st, s+t, 0)$, $s \neq t \neq 0$.

Proof. The collineation $\gamma : \alpha(x_2 = 0) \to \mathrm{PG}(2,q)$ is $(x_0,x_1,x_2,x_3) \mapsto (x_1,x_3,x_0)$. ∎

Theorem 4 *An ovoid of* $\mathrm{PG}(3,q)$ *is equivalent to a pencil* $\{C(s) \mid s \in \mathrm{GF}(q)\}$, *where each* $C(s)$ *is a* q*-arc of* $\mathrm{AG}(2,q)$, $s \neq 0$, *and* $C(0)$ *is the line* $x = 0$. *Here* $C'(s) = C(s) \cup \{(s^2,1,0),(1,0,0)\}$,$s \neq 0$ *and* $L(s,t) = (st,1,0)$, $s \neq t \neq 0$.

Proof. The collineation $\gamma : \alpha(x_1 = 0) \to \mathrm{PG}(2,q)$ is $(x_0,x_1,x_2,x_3) \mapsto (x_2,x_3,x_0)$. ∎

Theorem 5 *An ovoid of* $\mathrm{PG}(3,q)$ *is equivalent to a pencil* $\{D(s) \mid s \in \mathrm{GF}(q)\}$, *where each* $D(s)$ *is a* q*-arc of* $\mathrm{PG}(2,q)$, $s \notin \{0,1\}$, $D(0)$ *is the line* $x = 0$, *and* $D(1)$ *is the line* $x = z$. *Here* $D'(s) = D(s) \cup \{(s^2,1,0),(s,1,0)\}$, $s \notin \{0,1\}$, *and* $L(s,t) = (st, s+t+1, 0)$. $(1,1,0)$ *and* $(1,0,0)$ *are the remaining points on* L *not on any 'real' element of the pencil.*

Proof. $\gamma : \alpha(x_1 = x_2) \to \mathrm{PG}(2,q)$ is $(x_0,x_1,x_2,x_3) \mapsto (x_2,x_3,x_0)$. ∎

Theorem 6 *An ovoid of* $\mathrm{PG}(3,q)$ *is equivalent to a pencil* $\{E(s) \mid s \in \mathrm{GF}(q)\}$, *where each* $E(s)$ *is a* q*-arc of* $\mathrm{PG}(2,q)$. *Here* $E'(s) = E(s) \cup \{(\lambda s^2+1,1,0),(s,1,0)\}$ *and* $L(s,t) = (1+\lambda st, 1+\lambda(s+t), 0)$. $(1,0,0)$ *is the point on* L *not on any 'real' element of the pencil.*

Proof. $\gamma : \alpha(x_3 = x_1 + \lambda x_2) \to \mathrm{PG}(2,q)$ is $(x_0,x_1,x_2,x_3) \mapsto (x_1,x_3,x_0)$. ∎

4 Pencils of conics corresponding to the five cases

In this section we list the various pencils of conics corresponding to the Miquelian inversive plane of even order. Figure 1 shows the five kinds of pencils that we obtain.

We refer to the catalogue of all pencils of conics in $\mathrm{PG}(2,q)$, given in Glynn [7]. There are six kinds of pencils that have an intersection of zero or one point. It turns out that the only type of pencil that doesn't seem to correspond to an inversive

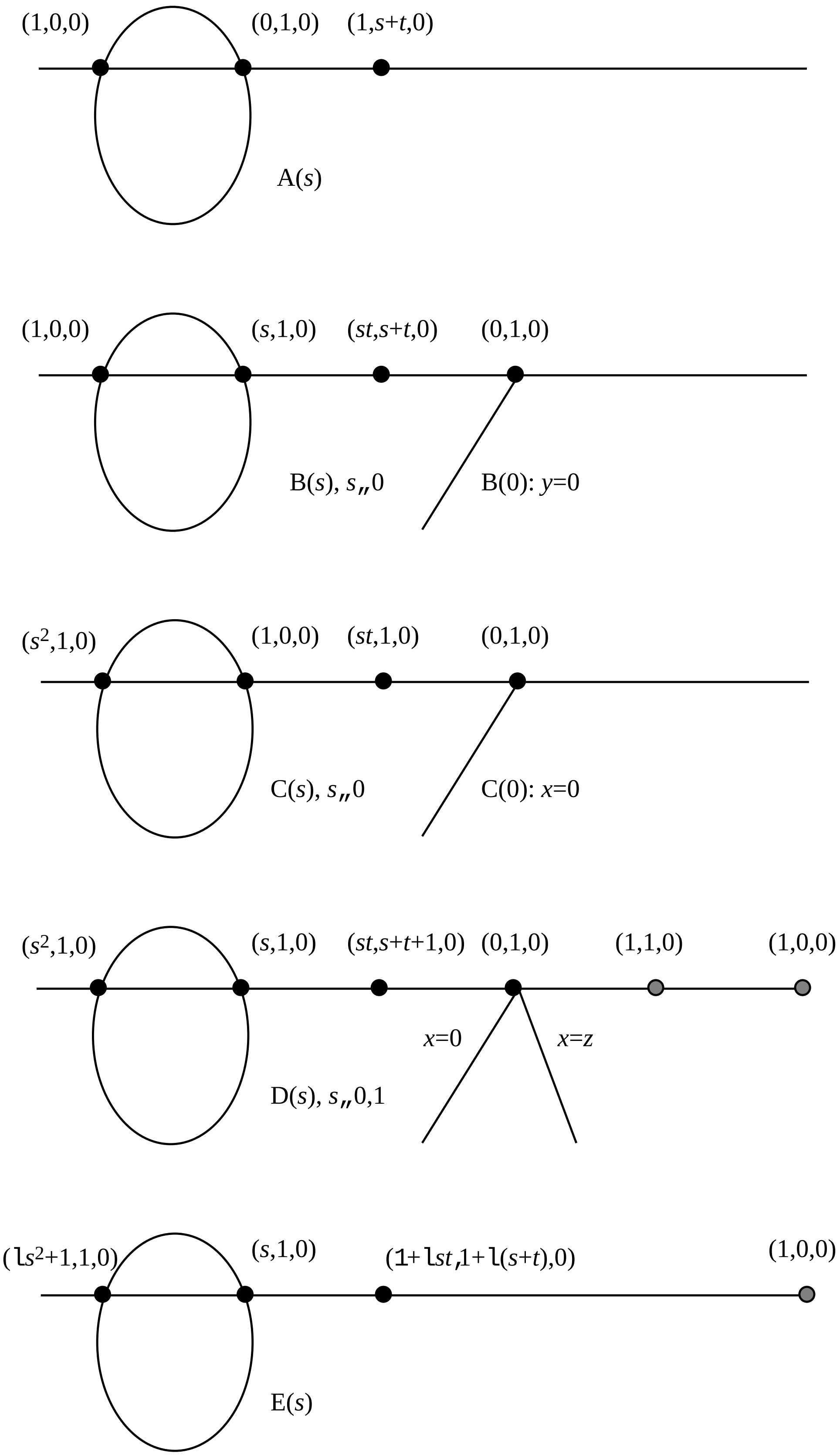

Figure 1: The five types of pencils of ovals

plane is the one with no degenerate conics; i.e. type (f). Let us proceed to the various types, giving an example in each case. In the examples below r is always a fixed element of trace 1. In other words, r is second category in $\mathrm{GF}(q)$.

To find the types of pencils in the Miquelian case it is easiest to tranform the situation from an elliptic quadric. In general we can assume, as in Glynn [4] or [6], that the ovoid is a set of points of $\mathrm{PG}(3,q)$, $\{(0,1,0,0)\} \cup \{(1, a_{sv}+sv, s, v) \mid s, v \in \mathrm{GF}(q)\}$, where $\mathfrak{A} : \mathrm{GF}(q) \times \mathrm{GF}(q) \to \mathrm{GF}(q)$ takes $(s,v) \mapsto a_{sv}$ is an "ovoid function". (It is proved in Glynn [6] that every monomial in a_{sv} is made up of even powers of s and v, when written as a reduced polynomial with individual exponents less than $q-1$.) Further we can then assume that the pencil of type A above in the plane $x_3 = 0$ is given by the set of ovals $A(s) : (1, a_{sv}, v, 0) \mid v \in \mathrm{GF}(q)\}$. This gives a set of q^2 lines forming a spread of $\mathrm{AG}(3,q)$ given by $\langle (0, s, s^2, 1), (1, v, a_{sv}, v, 0) \rangle$, where s, v vary in $\mathrm{GF}(q)$. The condition that an ovoid (or spread) is obtained is that $a_{sv} + a_{tw} \neq (s+t)(v+w)$, for all $(s,v) \neq (t,w)$ in $\mathrm{GF}(q) \times \mathrm{GF}(q)$.

In the Miquelian case an ovoid function is given by $a_{sv} := rs^2 + v^2$. First we can construct the spread of $T_2(O)$ (and of $\mathrm{AG}(3,q)$) by using the above formula. Then we intersect the spread with the correct plane and transform to standard coordinates in the (x,y,z)-plane $\mathrm{PG}(2,q)$, as given in the Theorems 2–6. Here are the pencils of conics that are obtained.

4.1 Type A

$\langle z^2, x^2 + yz \rangle$
Type (e) pencil with q irreducible conics and one repeated-line degenerate conic. The nuclei of the conics are all the same point $(1,0,0)$, and the conics pass through one point $(0,1,0)$.
$A(s) : rs^2z^2 + x^2 + yz$ and $A(s,t) : (1, s+t, 0)$. $A(s)$ has nucleus $(1,0,0)$ and passes through $(0,1,0)$ on $z = 0$.

4.2 Type B

$\langle y^2, x^2 + yz + rz^2 \rangle$
Type (b) pencil with $q-1$ irreducible conics, one repeated-line degenerate conic, and one imaginary line-pair degenerate conic. The nuclei of the conics are all the same point $(1,0,0)$, and the conics don't have any common point.
$B(s) : s^2y^2 + x^2 + yz + rz^2$ and $B(s,t) : (st, s+t, 0)$. $B(s)$ has nucleus $(1,0,0)$ and passes through $(s,1,0)$ on $z = 0$.

4.3 Type C

$\langle xz, y^2 + yz + rz^2 \rangle$
Type (d) pencil with $q-1$ irreducible conics, one line-pair degenerate conic, and one imaginary line-pair degenerate conic. The nuclei of the conics are different points on the line $z = 0$, and the conics pass through one point $(1,0,0)$.
$C(s) : xz + s^2(y^2 + yz + rz^2)$ and $C(s,t) : (st, 1, 0)$. $C(s)$ has nucleus $(s^2, 1, 0)$ and passes through $(1,0,0)$ on $z = 0$.

4.4 Type D

$\langle x^2 + xz, y^2 + yz + rz^2 \rangle$
Type (a) pencil with $q-2$ irreducible conics, one repeated-line degenerate conic, and two imaginary line-pair degenerate conics. The nuclei of the conics are different points on $z=0$, and the conics pass through different points on $z=0$.
$D(s) : x^2 + xz + s^2(y^2 + yz + rz^2)$ and $D(s,t) : (st, s+t+1, 0)$. $D(s)$ has nucleus $(s^2, 1, 0)$ and passes through $(s, 1, 0)$ on $z=0$.

4.5 Type E

$\langle \lambda x^2 + xz + yz, y^2 + yz + rz^2 \rangle$, where λ is a fixed second category element of $\mathrm{GF}(q)$. Type (c) pencil with q irreducible conics and one imaginary line-pair degenerate conic. The nuclei of the conics are different points on $z=0$, and the conics pass through different points on $z=0$.
$E(s) : \lambda x^2 + xz + yz + \lambda s^2(y^2 + yz + rz^2)$ and $E(s,t) : (1 + \lambda st, 1 + \lambda(s+t), 0)$. $E(s)$ has nucleus $(\lambda s^2 + 1, 1, 0)$ and passes through $(s, 1, 0)$ on $z=0$.

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