Subsets of association schemes corresponding to eigenvectors of the Bose-Mesner algebra

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Abstract

This paper is motivated by the following question: given a group G operating as a permutation group on a set X, which are the pairs of subsets $M, M' \subseteq X$ such that $|M \cap gM'| = c$ for a constant c and all $g \in G$? We give a characterization of these pairs in terms of eigenspaces of the corresponding association scheme, and we give further characterizing properties of these sets M. We apply our results to a generalization of a question of Cameron and Liebler in projective spaces.

1 Introduction

In [4] Cameron and Liebler proposed the problem to determine the line sets $\mathcal{B}$ of a projective space with the following property:

> Each spread has the same number of lines in common with $\mathcal{B}$.

This problem was generalized in [7], where the following question was considered:

> Let G be a group operating as a symmetrical rank 3 permutation group on the set V. Which are the pairs (M_1, M_2) of subsets of V such that $|M_1 \cap gM_2| = c$ for a constant c and all $g \in G$?

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If V is the set of lines of $\mathrm{PG}(d,q)$ and $G = \mathrm{PGL}(d+1,q)$, and if M_2 is a spread, then this reduces to the original question of Cameron and Liebler.

However, a more natural generalization of Cameron's and Liebler's question is not included in the generalization of [7], namely the question:

> Which are the sets of t-dimensional subspaces of $\mathrm{PG}(k(t+1)-1,q)$ having the same number of elements in common with every t-spread?

(Here a t-spread is a set of t-dimensional subspaces partitioning the point set.)

In this paper we extend the theory of [7] to permutation groups of higher permutation rank, thus giving a first answer to this question in Theorem 7, being a generalization of [9, Lemma 9].

To do our extension, we consider subsets of association schemes. So this paper gives a link between the theory of association schemes and Galois geometries. The main result of this paper is Theorem 5, making clear that questions of the Cameron-Liebler-Type are in fact questions about eigenspaces of association schemes.

This paper is mostly taken out of the Ph. D. Thesis [8], where in some parts more details are given.

2 Association schemes

We start with some basic results on association schemes. For a complete introduction see e.g. [3, Ch. 2], [1, Ch. 2], or [5, Ch. 17].

We start with the definition of an association scheme.

Definition 1
Let X be a finite set. An *association scheme with d classes* is a pair $(X,\mathcal{R})$, where $\mathcal{R} = (\sim_0,\ldots,\sim_d)$ is a set of binary relations on X (i.e. subsets of $X \times X$) with the following properties:

(a) For $x,y \in X$ there is exactly one i with $x \sim_i y$.

(b) $x \sim_0 y$ holds if and only if $x = y$.

(c) If $x \sim_i y$, then also $y \sim_i x$.

(d) There are numbers $p_{ij}^k \in \mathbb{R}$ with the following property: for $x,y \in X$ with $x \sim_k y$ there are exactly p_{ij}^k elements $z \in X$ with $x \sim_i z$ and $z \sim_j y$.

The number $n_i := p_{ii}^0$ is called *i-valency*.

Remarks

1. Sometimes in the literature axiom (c) is omitted and an association scheme fulfilling (c) is called *symmetrical*.

2. The case $d = 2$ corresponds to strongly regular graphs. In this case, our results reduce to results in [7].

Definition 2
A *symmetrical rank k permutation group* is a pair (G, P), where G is a group operating transitively on the set P such that the stabilizer of an element $p \in P$ has exactly k orbits, and such that for all $p_1, p_2 \in P$ the pairs (p_1, p_2) and (p_2, p_1) lie in the same orbit under G.

Lemma 1
Let (G, X) be a symmetrical rank $d+1$ permutation group. Let $\sim_0, \sim_1, \ldots, \sim_d$ be the orbits of pairs of elements of X under G, where $\sim_0 = \{(x,x) | x \in X\}$. Then $(X, \{\sim_0, \ldots, \sim_d\})$ is an association scheme with d classes.

Definition 3
Let $(X, \mathcal{R})$ be an association scheme with d classes. Let $x_1, \ldots, x_N$ be an enumeration of the elements of X.

(a) The *adjacency matrices* of $(X, \mathcal{R})$ are the matrices $A_i \in \mathbb{R}^{N \times N}$ $(i \in \{0, \ldots, d\})$ with

$$(A_i)_{st} = \begin{cases} 1 & \text{if } x_s \sim_i x_t, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The *Bose-Mesner algebra* $\mathcal{A}$ of $(X, \mathcal{R})$ is the $\mathbb{R}$-algebra generated by the adjacency matrices (see [2]), i.e.

$$\mathcal{A} = \{f(A_0, \ldots, A_d) \mid f \in \mathbb{R}[x_0, \ldots, x_d]\}.$$

(c) The *characteristic vector* of a set $M \subseteq X$ is the vector $v \in \mathbb{R}^N$ with:

$$v_i = \begin{cases} 1 & \text{if } x_i \in M, \\ 0 & \text{otherwise.} \end{cases}$$

(d) The characteristic vector of X, i.e. the all-one-vector, is denoted by $\mathbf{1}$. The unit matrix is denoted by I. The all-one-matrix is denoted by J.

Theorem 1
[see [3, 2.2]]

(a) *The Bose-Mesner algebra $\mathcal{A}$ is a $(d+1)$-dimensional commutative algebra of symmetrical matrices.*

(b) *The space $\mathbb{R}^N$ is the direct sum of $d+1$ maximal common eigenspaces of the matrices of $\mathcal{A}$. One of these eigenspaces has dimension 1 and is spanned by $\mathbf{1}$.*

Remark

If we normalize C_i such that $C_i v_i = v_i$, these matrices are the *minimal idempotents* of the Bose-Mesner algebra (see [3, 2.6]).

From now on let $V_0, \ldots, V_d$ be the eigenspaces of the matrices from $\mathcal{A}$, where $V_0 = \langle \mathbf{1} \rangle$.

3 Properties of subsets corresponding to eigenspaces of the Bose-Mesner algebra

In this section we shall see that subsets of association schemes whose characteristic vectors decompose into few eigenvectors of the Bose-Mesner algebra have some characterizations, one of them being a generalization of the Cameron-Liebler-problem (see Theorem 5).

The first two theorems can be stated more generally for graphs.

Theorem 2
Let $(X, \mathcal{R})$ be an association scheme with d classes. Let M be a subset of X, and let $r \in \{1, \dots, d\}$. Then

$$\frac{|M|}{|X|}(n_r|M| + \alpha(|X| - |M|)) \leq \Big|\{(x, y) \in M \times M \mid x \sim_r y\}\Big| \leq \frac{|M|}{|X|}(n_r|M| + \beta(|X| - |M|)),$$

where n_r is the eigenvalue of A_r to the eigenvector $\mathbf{1}$, while α (resp. β) is the smallest (resp. biggest) other eigenvalue of A_r.

Equality holds if and only if the characteristic vector of M is contained in the span of $\mathbf{1}$ and the eigenspace of A_r to the eigenvalue α (for the left hand side) resp. β (for the right hand side).

Proof. Let $v = (v_1, \dots, v_N)$ be the characteristic vector of M. Then $v^T v = |M|$. We write v as the sum of eigenvectors: $v = w_0 + \dots + w_d$ with $w_i \in V_i$. All elements of V_i $(i \geq 1)$ are orthogonal to the vector $\mathbf{1}$, hence the sum of their entries is zero. As the sum of the entries of v is $|M|$, the sum of the entries of w_0 is equal to $|M|$ i.e. $w_0 = |M|/|X| \cdot \mathbf{1}$ and so $w_0^T w_0 = |M|^2/|X|$. As the eigenspaces are orthogonal, $|M| = v^T v = w_0^T w_0 + \dots + w_d^T w_d$. Hence

$$|w_1|^2 + |w_2|^2 + \dots + |w_d|^2 = |v|^2 - |w_0|^2 = |M| - |M|^2/|X| = |M|(|X| - |M|)/|X|. \quad (*)$$

On the other hand

$$v^T A_r v = \sum_i \sum_j v_i (A_r)_{ij} v_j.$$

Here the only summands not vanishing are those where all three factors are one, i.e. where $x_i \in M$, $x_i \sim_r x_j$ and $x_j \in M$. Hence

$$\begin{aligned}\Big|\{(x, y) \in M \times M \mid x \sim_r y\}\Big| &= v^T A_r v = w_0^T A_r w_0 + \dots + w_d^T A_r w_d \\ &= n_r |M|^2/|X| + \alpha_1 |w_1|^2 + \dots + \alpha_d |w_d|^2,\end{aligned}$$

where α_i is the eigenvalue of A_r to the eigenspace V_i. As $|w_i|^2 \geq 0$, this expression has its minimal (resp. maximal) value, if the whole sum in $(*)$ consists of the vector with the smallest (resp. biggest) eigenvalue. From this the assertion follows. ∎

Remark

The same statement holds for linear combinations of adjacencies.

Theorem 3
Let $(X, \mathcal{R})$ be an association scheme with d classes. Let M be a subset of X, and let $i \in \{1, \dots, d\}$. Then the following statements are equivalent.

(a) *There are numbers $c_1, c_2 \in \mathbb{R}$ such that each element of M is i-adjacent to exactly c_1 elements of M, and each element of $X \setminus M$ is i-adjacent to exactly c_2 elements of M.*

(b) *The characteristic vector v of M is contained in the span of the unit vector $\mathbf{1}$ and an eigenspace of the adjacency matrix A_i.*

In this case $c_1 - c_2$ is the corresponding eigenvalue.

Proof. Assertion (a) holds if and only if $A_i v = c_1 v + c_2(\mathbf{1} - v)$.
Suppose that (a) holds. Then for every constant α the equation

$$A_i(v - \alpha\mathbf{1}) = (c_1 - c_2)v + (c_2 - \alpha n_i)\mathbf{1}$$

holds. If $c_1 - c_2 < n_i$, we set $\alpha := c_2/(n_i - c_1 + c_2)$. This yields the equality

$$A_i(v - \alpha\mathbf{1}) = (c_1 - c_2)(v - \alpha\mathbf{1}).$$

Hence $v - \alpha\mathbf{1}$ is an eigenvector of A_i to the eigenvalue $c_1 - c_2$, from which (b) follows. If on the other hand $c_1 - c_2 \geq n_i$, then $c_1 = n_i$ and $c_2 = 0$ (for $0 \leq c_1, c_2 \leq n_i$), which implies $A_i v = (c_1 - c_2)v$, from which (b) follows.

Now suppose that (b) holds. Then $v = v_0 + \alpha\mathbf{1}$, where v_0 is an eigenvector of A_i (to the eigenvalue c) and $\alpha \in \mathbb{R}$. Hence

$$A_i v = c v_0 + \alpha n_i \mathbf{1} = cv + (n_i - c)\alpha\mathbf{1} = (c + \alpha(n_i - c))v + (n_i - c)\alpha(\mathbf{1} - v),$$

from which (a) follows. ∎

Remark

An analogous statement holds for linear combinations of adjacencies. This leads to the following theorem characterizing the span of eigenspaces.

Theorem 4
Let $(X, \mathcal{R})$ be an association scheme with d classes. Let $M \neq \emptyset$ be a subset of X with characteristic vector v. We define the matrix $B \in \mathbb{R}^{X \times \{0, \dots, d\}}$ by $B_{xi} := |\{y \in M \mid y \sim_i x\}|$. Let $k \in \{1, \dots, d\}$. Then the following statements are equivalent:

(a) *There are k eigenspaces $V_{r_1}, \dots, V_{r_k}$ of the Bose-Mesner algebra of $(X, \mathcal{R})$ such that $v \in \langle \mathbf{1}, V_{r_1}, \dots, V_{r_k} \rangle$.*

(b) *The matrix B has rank $\leq k + 1$.*

Proof. Let $v = v_0 + \cdots + v_d$ with $v_i \in V_i$. Because of $M \neq \emptyset$, we have $v_0 \neq 0$. We renumber the coefficients such that $v_0, \ldots, v_t \neq 0$ and $v_{t+1} = \cdots = v_d = 0$. Let p_{ij} be the eigenvalue of the matrix A_i corresponding to the eigenspace V_j. Let $P = (p_{ij})$.

The i-th column of B is equal to $A_i v$. Hence (b) is equivalent to the statement that for each $k+2$ values $0 \leq s_0 < \cdots < s_{k+1} \leq d$ there are coefficients c_i not all equal to zero such that $\sum_i c_i(A_{s_i} v) = 0$. This means:

$$0 = \sum_{i=0}^{k+1} c_i A_{s_i} \left(\sum_{j=0}^{t} v_j \right) = \sum_{i=0}^{k+1} c_i \sum_{j=0}^{t} p_{s_i j} v_j = \sum_{j=0}^{t} \left(\sum_{i=0}^{k+1} c_i p_{s_i j} \right) v_j .$$

As the v_j are linearly independent, the expression in parentheses vanishes. This holds for every choice of the s_i, which means that the submatrix of P formed by the columns $0, 1, \ldots, t$ has rank at most $k+1$. As the matrix P is regular (otherwise $A_0, \ldots, A_d$ would be linearly dependent), this means that $t \leq k$, i.e. (a) holds. The other direction follows analogously. ■

For the rest of the paper we suppose that we have the situation of Lemma 1, i.e. (G, X) is a rank $d+1$ permutation group, and $(X, \mathcal{R})$ is the corresponding association scheme with d classes. The matrices A_i and the algebra $\mathcal{A}$ are defined as usual.

Each element $g \in G$ can be regarded as a permutation matrix from $\mathbb{R}^{N \times N}$: if $gx_i = x_j$, then the corresponding permutation matrix maps the i-th unit vector (i.e. the characteristic vector of $\{x_i\}$) to the j-th unit vector.

Let $\mathcal{G} = \langle \{g \mid g \in G\} \rangle_{\mathbb{R}}$ be the span of these permutation matrices as $\mathbb{R}$-vector space (or as subalgebra of $\mathbb{R}^{N \times N}$).

As above let $V_0, \ldots, V_d$ be the common eigenspaces of the matrices from $\mathcal{A}$.

The following lemmata can be concluded from the fact that the permutation representation of (G, X) is the direct sum of $d+1$ distinct irreducible representations (see e.g. [1, II.1]). However we give more basic proofs.

Lemma 2
[compare [1, Thm. II.1.3]] The algebra $\mathcal{A}$ consists of exactly the matrices commuting with all permutation matrices of G (or, equivalently, with all elements of $\mathcal{G}$).

Proof. As permutation matrices are orthogonal, a matrix A commutes with a permutation matrix $g \in G$ if and only if $g^T A g = A$, i.e. if

$$(ge_i)^T A (ge_j) = e_i^T g^T A g e_j = e_i^T A e_j$$

for all unit vectors e_i, e_j. If $gx_i = x_{i'}$, $gx_j = x_{j'}$, this means that $e_{i'}^T A e_{j'} = e_i A e_j$, i.e. the (i', j')-entry of A is equal to the (i, j)-entry of A. In other words: if (x_i, x_j) and $(x_{i'}, x_{j'})$ are in a common orbit under G, then the entries in A on the positions (i, j) and (i', j') are equal. The matrices A with this property are by definition the linear combinations of the matrices $A_0, \ldots, A_d$, i.e. the matrices from $\mathcal{A}$. ■

Lemma 3
The elements of $\mathcal{G}$ map the eigenspaces V_s onto themselves.

Proof. Is is sufficient to prove the assertion for the permutation matrices from G. Let $w_s \in V_s$ and $g \in G$. Let $gw_s = v_0 + \cdots + v_d$ with $v_i \in V_i$. Let A_t be one of the adjacency matrices. Then

$$A_t g w_s = A_t v_0 + \cdots + A_t v_d = \alpha_0 v_0 + \cdots + \alpha_d v_d,$$

where α_i is the eigenvalue of A_t to the eigenspace V_i. On the other hand by Lemma 2 we have

$$A_t g w_s = g A_t w_s = g \alpha_s w_s = \alpha_s g w_s = \alpha_s v_0 + \cdots + \alpha_s v_d.$$

Hence

$$(\alpha_0 - \alpha_s) v_0 + (\alpha_1 - \alpha_s) v_1 + \cdots + (\alpha_d - \alpha_s) v_d = 0.$$

This holds for all adjacency matrices A_t. As for each $i \neq s$ there is an adjacency matrix A_t whose eigenvalues for V_i and V_s are different, we get $v_i = 0$ for all $i \neq s$. Hence $gw_s \in V_s$. ∎

Lemma 4
Let $v = v_0 + \cdots + v_d \in \mathbb{R}^N$, where $v_i \in V_i$.

(a) *If $v_i \neq 0$, then for each $w_i \in V_i$ there is an $M \in \mathcal{G}$ with $Mv_i = w_i$.*

(b) *If $v_i \neq 0$ and $j \neq i$, then there is an $M \in \mathcal{G}$ with $Mv_j = 0$ and $Mv_i \neq 0$.*

(c) *If $v_i \neq 0$ and $w_i \in V_i$, then there is an $M \in \mathcal{G}$ with $Mv = w_i$.*

Proof. (a) Let $W := \{Mv_i \mid M \in \mathcal{G}\}$. We want to show that $W = V_i$. By Lemma 3, $W \subseteq V_i$. Suppose that W is a true subspace of V_i. Let $W' := W^\perp \cap V_i$. Then W, W' are complementary subspaces of V_i which are mapped into themselves by $\mathcal{G}$. (For W' this holds because $\mathcal{G}$ is spanned by orthogonal (permutation) matrices.) Let A be the matrix inducing the identity on W and mapping W' and the spaces V_j with $j \neq i$ to zero. This matrix commutes with all elements of G. By Lemma 2, A is an element of $\mathcal{A}$. This produces a contradiction, because V_i is not an eigenspace of A. Hence $W = V_i$.

(b) If $v_j = 0$, the assertion is clear. Let now $v_j \neq 0$. Suppose that for all $M \in \mathcal{G}$ with $Mv_j = 0$ we have $Mv_i = 0$. Let A be the matrix mapping all V_s ($s \neq j$) to zero, while the operation of A on V_j is defined by $AMv_j := Mv_i$ for all $M \in \mathcal{G}$. By (a) this yields values for all elements of V_j. The map is well-defined: if $Mv_j = M'v_j$, then $(M - M')v_j = 0$, hence $(M - M')v_i = 0$, and so $Mv_i = M'v_i$. The matrix A commutes with all elements of $\mathcal{G}$. (For $M \in \mathcal{G}$, $v_j \in V_j$ we have $AMv_j = Mv_i = MAv_j$.) By Lemma 2, $A \in \mathcal{A}$. This is a contradiction, because $Av_j = v_i$ such that A does not map V_j into itself.

(c) Apply first (b) for all $j \neq i$ and then (a). ∎

Theorem 5
Let (G, X) be a rank $d+1$ permutation group. Let M, M' be subsets of X with characteristic vectors v, w. Let $v = v_0 + \cdots + v_d$ and $w = w_0 + \cdots + w_d$ the decompositions of v, w into eigenvectors of the Bose-Mesner algebra $\mathcal{A}$ (i.e. v_i, w_i are elements of the eigenspace V_i, where $V_0 = \langle \mathbf{1} \rangle$). Then the following statements are equivalent:

(a) *There is a constant* $c \in \mathbb{R}$ *such that* $|M \cap gM'| = c$ *for all* $g \in G$.

(b) *For each* $i \in \{1, \ldots, d\}$ *one of the vectors* v_i, w_i *is equal to zero.*

Proof. As the group G operates transitively on X, the stabilizer of an element $x \in X$ has exactly $|G|/|X|$ elements, and the same number of elements maps x onto an arbitrary $x' \in X$. Hence the number of triples $(g, x, x') \in G \times M \times M'$ with $x = gx'$ is equal to $|M| \cdot |M'| \cdot |G|/|X|$. This is the number of pairs $(g, x) \in G \times M$ with $x \in gM'$. Hence the average number of elements of $M \cap gM'$ is equal to $|M||M'|/|X|$. Thus (a) can hold only with the value $c = |M||M'|/|X|$.

Each element of $V_i\,(i \geq 1)$ is orthogonal to the all-one-vector $\mathbf{1} \in V_0$, so the sum of its entries is 0. As the sum of entries of v is $|M|$, also the sum of entries of v_0 is $|M|$, i.e. $v_0 = |M|/|X| \cdot \mathbf{1}$. Analogously, for each $g \in G$, we have $gw_0 = |gM'|/|X| \cdot \mathbf{1} = |M'|/|X| \cdot \mathbf{1}$, and so $v_0^T gw_0 = |M||M'|/|X| = c$.

The number of elements of $M \cap gM'$ is equal to

$$v^T gw = v_0^T gw_0 + \cdots + v_d^T gw_d = c + v_1^T gw_1 + \cdots + v_d^T gw_d.$$

(Here we use Lemma 3.) Hence (a) holds if and only if

$$v_1^T gw_1 + \cdots + v_d^T gw_d = 0 \quad \text{for all} \quad g \in G.$$

This obviously is true if (b) holds.

Now suppose that (a) holds. Then

$$(v_1 + \cdots + v_d)^T g(w_1 + \cdots + w_d) = 0 \quad \text{for all} \quad g \in G.$$

This equality holds for all $g \in \mathcal{G}$, too. Suppose that for some $i \in \{1, \ldots, d\}$ we have $w_i \neq 0$. By Lemma 4(c), for each $u_i \in V_i$ there is an element $g \in \mathcal{G}$ with $g(w_1 + \cdots + w_d) = u_i$. For this g we get

$$0 = (v_1 + \cdots + v_d)^T g(w_1 + \cdots + w_d) = v_i^T u_i.$$

Hence v_i is orthogonal to V_i, and so $v_i = 0$. This yields (b). ∎

4 Application to projective spaces

We give now the application of Theorem 5 to our original problem on spreads in projective spaces, namely the generalization of [9, Lemma 9].

Let $\mathcal{P} = \mathrm{PG}(k(t+1), q)$ be a projective space, and let $\mathcal{L}_i$ be the set of i-dimensional subspaces of $\mathcal{P}$ for all i.

We need the characterization of the eigenspaces of the Bose-Mesner-Algebra corresponding to projective spaces, given in [6]. From this we need the following result [6, Thm. 2.7]:

Theorem 6
For $r \in \{0, \ldots, \min(t+1, d-n)\}$ *let* V_r *be the vector space of functions* $f : \mathcal{L}_t \to \mathbb{R}$ *that can be written in the form*

$$f(L_t) = \sum_{L_{r-1} \subseteq L_t} g(L_{r-1}),$$

where $g : \mathcal{L}_{r-1} \to \mathbb{R}$ *is a function for which holds:*

$$\sum_{L_{r-1} \supseteq L_{r-2}} g(L_{r-1}) = 0 \quad \textit{for all } L_{r-2} \in \mathcal{L}_{r-2}.$$

(In particular, V_0 *is the space of constant functions.)*

Then the V_r *form a complete system of eigenspaces of the association scheme formed by the* t*-dimensional subspaces of* $\mathcal{P}$.

Theorem 7

(a) *Let* $f : \mathcal{L}_0 \to \mathbb{R}$ *be a function such that for the function*

$$g : \mathcal{L}_t \to \mathbb{R}, \qquad L_t \mapsto \sum_{P \in L_t} f(P)$$

there are two constants $c_1, c_2 \in \mathbb{R}$ *such that* $g(L_t) \in \{c_1, c_2\}$ *for all* $p \in P$. *Then the set*

$$M := \{L_t \in \mathcal{L}_t \mid g(L_t) = c_1\}$$

is a subset of $\mathcal{L}_t$, *having the same number of elements in common with each* t*-spread of* $\mathcal{P}$

(b) *Let* $M \subseteq \mathcal{L}_t$ *be a set of* t*-dimensional subspaces of* $\mathcal{P}$, *having the same number of elements in common with each regular* t*-spread. If* $k \geq 3$ *or if* $k = 2, t \leq 2$, *then* M *can be expressed as in (a).*

Proof. By Theorem 6, the sets constructed in (a) are exactly the subsets of $\mathcal{L}_t$ whose characteristic vector is contained in $\langle \mathbf{1}, V_1 \rangle$. As a spread covers each point exactly once, its characteristic vector lies in $\langle V_0, V_2, V_3, \ldots, V_{t+1} \rangle$. From Theorem 5 we get immediately (a).

For the proof of (b) we have to show that the characteristic vector of a regular spread, when decomposed into eigenvectors, contains a non-zero part of each V_i $(i \neq 1)$. We show this for every spread.

Let M be a t-spread of $\mathcal{P}$. We must show that the characteristic vector of M is not contained in the span of $\langle \mathbf{1} \rangle$ and $t - 1$ other eigenspaces. By Theorem 4 we have to show that the matrix $B \in \mathcal{R}^{\mathcal{L}_t \times \{0,\ldots,t+1\}}$ defined by

$$B_{L_t i} := |\{L'_t \in M \mid \dim(L'_t \cap L_t) = t - i\}|$$

has at least rank $t + 1$. Therefore we must find $t + 1$ linear independent rows of B. It suffices to show that for each $s = 0, \ldots, t$ there is a row or B whose first $t - s - 1$ entries vanish, while the $(t - s)$-th entry is non-zero.

This can be done combinatorially by observing that for an element $L \in M$ and an s-dimensional subspace $T \subseteq L$ the number of elements of L_t intersecting L in T is bigger than the number of elements of L_t intersecting L in T and intersecting some other element of M in at least a line. (More details are given in [8].) ■

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