Monomial flocks of monomial cones in even characteristic

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Abstract

We investigate the generalization of the concept of a flock of a quadratic cone to more general types of cones. Several new flocks are found and a geometric construction of one of these is given.

1 Introduction

Let $\mathcal{F} = \mathrm{GF}(q)$ with $q = 2^e$. Define a *monomial cone* in $\mathrm{PG}(3, q)$ to be the set of points,

$$\Sigma_\beta = \{(x, y, z, w) \mid y^\beta = xz^{\beta-1}\},$$

together with the vertex $(0, 0, 0, 1)$, where $(\beta, q-1) = (\beta - 1, q-1) = 1$. A *flock* of Σ_β is a set of q planes of $\mathrm{PG}(3, q)$ not passing through the vertex which do not intersect each other at a point of Σ_β. Without loss of generality, the planes of a flock may be represented by

$$a_t x + b_t y + c_t z + w = 0, \qquad t \in \mathcal{F}.$$

If, in such a representation, each of the functions $a_t, b_t,$ and c_t is a monomial function of t, we shall call the flock a *monomial flock*. Finally, we define

$${\mathcal{D}_\beta}^0 = \{y \in \mathcal{F} \mid y = x^\beta + x, \text{ for some } x \in \mathcal{F}\}$$

and its complement, ${\mathcal{D}_\beta}^1 = \mathcal{F} \backslash {\mathcal{D}_\beta}^0$.

Received by the editors August 1997.
Communicated by Jef Thas.
1991 *Mathematics Subject Classification.* 51E20.
Key words and phrases. flocks, cones, clans, herds.

The simplest example of a flock is the *linear* flock, which is a flock consisting of q planes passing through a fixed line L skew to Σ_β. Examples of *non-linear* flocks are also known. The case of the *quadratic cone*, Σ_2, has been extensively examined (see [10]).

One of the most interesting features of flocks is that they are related to a large variety of other structures. Flocks of quadratic cones give rise to certain generalized quadrangles of order (q^2, q) and can be used to define translation planes. Recently, in the even case, a third connection was found [3]. A flock of a quadratic cone gives rise to a set of $q+1$ ovals in a projective plane $\mathrm{PG}(2,q)$, q even, called a *herd of ovals*. This link, found previously by Payne [7], led to the discovery of the Payne ovals ([7]) and to the discovery of the Subiaco ovals ([3]).

However, not all ovals are related to flocks of quadratic cones (via herds), and so, we were led to investigate flocks of the cones Σ_β, for which $t \mapsto t^\beta$ is an automorphism of $\mathcal{F}$ of maximal order. These flocks are called *α-flocks*. The study of these flocks led to a proof of the fact that the Cherowitzo ovals form an infinite family, and provided a uniform method for proving that all known monomial ovals are ovals (see [4]).

The connection between ovals and flocks is still not completely determined, so we are naturally led to look at a larger collection of cones in this note. Our restriction to monomial flocks is spurred by the recent classification of monomial flocks of quadratic cones ([8]).

In the next section, we collect some general results concerning monomial flocks of monomial cones. Section 3 provides all the known examples. In section 4, the geometric structure of one of these new flocks is examined and a geometric construction is given for a flock of this type. Finally, we end with a number of open problems.

2 General results

The algebraic condition that a set of q planes in the above representation form a flock of Σ_β is,

$$\frac{(a_t + a_s)^{\frac{1}{\beta-1}}(c_t + c_s)}{(b_t + b_s)^{\frac{\beta}{\beta-1}}} \in \mathcal{D}_\beta{}^1, \qquad \forall t \neq s.$$

It is easily seen that this condition forces each of $a_t, b_t,$ and c_t to be a permutation of $\mathcal{F}$ and that we may, by reparameterization, choose any one of these functions to be any permutation we like. Such a normalization proved useful in earlier work ([4],[3]), but has not been fruitful in this more general context, so we shall not make any such normalization.

Specializing to monomial flocks, we may take $a_t = At^a, b_t = Bt^b,$ and $c_t = Ct^c$, where A, B, C are nonzero constants. The algebraic condition then becomes,

$$\kappa\frac{(t^a + s^a)^{\frac{1}{\beta-1}}(t^c + s^c)}{(t^b + s^b)^{\frac{\beta}{\beta-1}}} \in \mathcal{D}_\beta{}^1, \qquad \forall t \neq s, \text{ where } \kappa = \frac{A^{\frac{1}{\beta-1}}C}{B^{\frac{\beta}{\beta-1}}}. \tag{1}$$

On setting $s = 0$ in (1) we have that

$$\kappa t^{\frac{a-c-\beta(b-c)}{\beta-1}} \in \mathcal{D}_\beta{}^1, \qquad \forall t \neq 0. \tag{2}$$

From this we can conclude that $\kappa \in {\mathcal{D}_\beta}^1$ (set $t = 1$), and so, the necessary condition in (2) will be satisfied if $\beta \equiv \frac{a-c}{b-c} \pmod{q-1}$. Monomial flocks with $\beta \equiv \frac{a-c}{b-c} \pmod{q-1}$ will be referred to as *flocks of type I*, and all others as *flocks of type II*.

Remark 1 There are situations in which a monomial flock must be of type I. For instance, it can be shown to be so when $t \mapsto t^\beta$ is an automorphism of the field (of maximal order) or if $q-1$ is a prime. Computer results show that if $q < 2^8$ or $q = 2^9$ then the only monomial flocks are of type I. However, flocks of type II do exist when $q = 2^8$.

Using (1), we can replace a flock by an equivalent one in which the constants of the general form have been absorbed into the single constant κ. The planes of this *normalized* monomial flock are given by:

$$t^a x + t^b y + \kappa t^c z + w = 0, \tag{3}$$

with $\kappa \in {\mathcal{D}_\beta}^1$ for each $t \in \mathcal{F}$. In keeping with previous usage, we will represent each plane of the monomial flock by an upper triangular 2×2 matrix of the form

$$\begin{pmatrix} t^a & t^b \\ & \kappa t^c \end{pmatrix}.$$

A set of q matrices of this form is called a β - *clan* if the corresponding planes form a flock of a β-cone. We remark that we are only using the matrices as a notational device and we will not be exploring the algebraic ramifications of this notation.

Note that, with $t = 0$ in (3), we always have the plane $w = 0$ in a monomial flock. We will view this plane as the *carrier plane* of the cone. The points of the *carrier* of the cone, i.e., the points of intersection of the cone and $w = 0$, are:

$$\{(x^\beta, x, 1, 0) \mid x \in \mathcal{F}\} \cup \{(1, 0, 0, 0)\}.$$

Consider a *generator* of the cone, i.e., a line joining the vertex $(0, 0, 0, 1)$ with a point of the carrier of the cone. This line meets each plane of (3) at the point $(x^\beta, x, 1, t^a x^\beta + t^b x + \kappa t^c)$ for a fixed $x \in \mathcal{F}$ or $(1, 0, 0, t^a)$. No two planes of (3) meet at a point of the cone, if and only if, for each $x \in \mathcal{F}$ the functions $t^a x^\beta + t^b x + \kappa t^c$, and t^a are permutations (in t). We refer to this set of $q + 1$ polynomials as the *herd* corresponding to the set of planes given by (3). Thus, we have:

Theorem 1 $\begin{pmatrix} t^a & t^b \\ & \kappa t^c \end{pmatrix}$ *is a β-clan if and only if each polynomial in its herd is a permutation polynomial.* ■

Remark 2 In [3] the term "herd of ovals" was introduced, for 2-cones, to denote the geometric point sets given by the functions that we have here called a "herd" with a different parameterization. From the more general viewpoint that we have taken, it is clear that it is the set of functions rather than the point sets that they determine which are of primary interest. This shift in emphasis on what a herd is, is embodied in our definition.

Remark 3 This theorem appears in [3] for 2-clans, and again, in [4] for α-clans. It should be clear from the proof that, even in its current form, this is not the most general statement that can be made. Indeed, there is nothing about the geometric nature of the carrier that is needed in this proof, so the statement really applies to cones of the most general kind.

In keeping with the above remark, we shall make a modest extension of our definitions. Given a monomial cone Σ_β, if we add a point to the carrier of the cone and then construct the cone with the same vertex over the expanded carrier, we shall refer to the new cone as an *extended cone*. A flock of the extended cone is defined in the same way as a flock of Σ_β, only with reference to the extended cone, i.e., a set of q planes of $\mathrm{PG}(3,q)$ not passing through the vertex which do not intersect each other at a point of the extended cone.

Corollary 1 *The point* $(0,1,0,0)$ *may be added to the carrier of any monomial cone and any monomial flock of the original cone will remain a flock of the extended cone.*

Proof. The point $(0,1,0,0)$ in $w=0$ is on no other plane of the flock, else t^b would not be a permutation. The additional generator of the extended cone will add the permutation polynomial t^b to the herd. By Theorem 1 the original flock is a flock of the extended cone. ∎

Remark 4 In the case of a 2-cone, this point is the nucleus of the conic carrier of the cone and the extended cone is a regular hyperoval cone. Thas ([9]) proved this corollary in the case that the cone is a cone over an arbitrary oval.

Any collineation of $\mathrm{PG}(3,q)$ will map a flock of a cone to a flock of the image cone. In particular, if we apply a collineation which fixes the vertex and stabilizes the carrier plane ($w=0$), and furthermore, induces in the carrier plane a monomial preserving collineation, then we should expect to see a fairly simple relationship between the flocks of equivalent β-cones. More precisely, we have:

Theorem 2 *The following are equivalent:*

(1) $\begin{pmatrix} t^a & t^b \\ & \kappa t^c \end{pmatrix}$ *is a* β*-clan.*

(2) $\begin{pmatrix} t^b & t^a \\ & \kappa t^c \end{pmatrix}$ *is a* $\frac{1}{\beta}$*-clan.*

(3) $\begin{pmatrix} t^a & t^c \\ & \kappa^{\frac{1-\beta}{\beta}} t^b \end{pmatrix}$ *is a* $(q-\beta)$*-clan.*

(4) $\begin{pmatrix} t^b & t^c \\ & \kappa^{\beta-1} t^a \end{pmatrix}$ *is a* $(q-\frac{1}{\beta})$*-clan.*

(5) $\begin{pmatrix} t^c & t^a \\ & \kappa^{\frac{1-\beta}{\beta}} t^b \end{pmatrix}$ *is a* $\frac{1}{q-\beta}$*-clan.*

(6) $\begin{pmatrix} t^c & t^b \\ & \kappa^{\beta-1}t^a \end{pmatrix}$ *is a* $\frac{\beta}{\beta-1}$*-clan.*

Proof. The group of six homographies of $\mathrm{PG}(2,q)$ whose associated matrices are given by

$$\left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

preserve monomials in the plane ([2]). Using any one of them as M in

$$\begin{pmatrix} & & & 0 \\ & M & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

will give a matrix whose associated homography fixes $(0,0,0,1)$ and stabilizes the plane $w = 0$, while preserving any monomial in that plane. We shall carry out the computation for one of these equivalences, the rest are similar.

Assume that $\begin{pmatrix} t^a & t^b \\ & \kappa t^c \end{pmatrix}$ is a β-clan, and apply the homography, ρ, whose matrix (for points) is,

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

to the points of the β-cone's carrier together with $(0,1,0,0)$ (see Corollary 1) to obtain,

$$(x^\beta, x, 1, 0)^\rho = (x, 1, x^\beta, 0) = \begin{cases} (s^{q-\frac{1}{\beta}}, s, 1, 0) & \text{if } x \neq 0 \quad (s \neq 0), \\ (0,1,0,0) & \text{if } x = 0. \end{cases}$$
$$(0,1,0,0)^\rho = (1,0,0,0)$$
$$(1,0,0,0)^\rho = (0,0,1,0).$$

By removing the point $(0,1,0,0)$, we see that the "image" of the β-cone is a $(q - \frac{1}{\beta})$-cone. Now we apply ρ to the planes of the flock to obtain,

$$[t^a, t^b, \kappa t^c, 1]^\rho = [t^b, \kappa t^c, t^a, 1].$$

When we normalize this flock it has the form, $[t^b, t^c, \kappa^{\beta-1}t^a, 1]$. Thus we have shown that (1) $\Rightarrow$ (4), and ρ^{-1} provides the reverse implication. ∎

Monomial flocks are in a sense simpler than other kinds of flocks since they admit a cyclic group of automorphisms. More precisely, we have:

Theorem 3 *Let λ be a primitive element of $\mathcal{F}$, then the cyclic group of order $q-1$ given by,*

$$\left\langle \begin{pmatrix} \lambda^a & 0 & 0 & 0 \\ 0 & \lambda^b & 0 & 0 \\ 0 & 0 & \lambda^c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

fixes the plane $w = 0$ and acts sharply transitively on the remaining planes of the flock, $t^a x + t^b y + \kappa t^c z + w = 0$. Furthermore, if this is a flock of a β-cone, the β-cone is also stabilized, if and only if, the flock is of type I.

Proof. Since we are dealing with a flock, t^a is a permutation polynomial, and so, $(a, q-1) = 1$. Thus, the group has order $q-1$. Clearly, the group fixes the plane $w = 0$. The action of the generator of the group on the planes of the flock is given by $[t^a, t^b, \kappa t^c, 1] \to [(\lambda^{-1}t)^a, (\lambda^{-1}t)^b, \kappa(\lambda^{-1}t)^c, 1]$. Thus, the action is sharply transitive on the remaining planes of the flock.

The generator fixes the vertex of the β-cone and its action on the points of the carrier of the cone is given by $(x^\beta, x, 1, 0) \to (\lambda^a x^\beta, \lambda^b x, \lambda^c, 0) \equiv (\lambda^{a-c+\beta(c-b)} s^\beta, s, 1, 0)$. Since the point $(1, 0, 0, 0)$ is fixed, we see that the β-cone is stabilized if and only if $\beta \equiv \frac{a-c}{b-c} \pmod{q-1}$, i.e., the flock is of type I. ■

3 Some families of β-clans

A complete computer search was made for monomial flocks of monomial cones over the fields $\mathrm{GF}(q), q = 2^e$, for $e = 3, 4, \ldots, 8$, and a partial search for $e = 9$. Theorem 2 was used to significantly reduce the amount of search time. Besides the α-flocks, which are reported in [4], all the remaining β-clans that were found will be discussed in this section. We will, when appropriate, only display one of the six equivalent clans given by Theorem 2.

Theorem 4 *Let $\mathcal{L} = \mathrm{GF}(2^r)$ be a proper subfield of $\mathcal{F} = \mathrm{GF}(q)$ and $\mathcal{L}^* = \mathcal{L} \setminus \{0\}$, then*

$$\begin{pmatrix} t^{2^a} & t \\ & \kappa t^{2^b} \end{pmatrix}$$

with $0, a$ and b distinct, is a β-clan for

$$\beta \equiv \frac{j(q-1) + (2^r-1)(2^b-2^a)}{(2^b-1)(2^r-1)} \pmod{q-1} \text{ where } 0 \le j < 2^r - 1,$$

whenever $(\beta, q-1) = (\beta-1, q-1) = 1$ and $\kappa\mathcal{L}^* \subseteq \mathcal{D}_\beta{}^1$.

Proof. In this case (1) becomes,

$$\kappa \frac{(t^{2^a} + s^{2^a})^{\frac{1}{\beta-1}} (t^{2^b} + s^{2^b})}{(t+s)^{\frac{\beta}{\beta-1}}} = \kappa(t+s)^{\frac{2^a - 2^b + \beta(2^b-1)}{\beta-1}}.$$

With β as given in the statement of the theorem, this exponent reduces to $\frac{j(q-1)}{(\beta-1)(2^r-1)}$. Since $(\beta-1, q-1) = 1$, we have that

$$\kappa\left((t+s)^{\frac{1}{\beta-1}}\right)^{\frac{j(q-1)}{2^r-1}} \in \kappa\mathcal{L}^* \quad \text{for all } t \ne s.$$

Thus, provided $(\beta, q-1) = (\beta-1, q-1) = 1$, and $\kappa\mathcal{L}^* \subseteq {\mathcal{D}_\beta}^1$, this is a β-clan. ∎

Remark 5 If $\mathcal{L} = \mathrm{GF}(2)$ then these β-clans are of type I, and the last condition reduces to $\kappa \in {\mathcal{D}_\beta}^1$. For other subfields we obtain type II flocks, and, indeed, the only type II flocks found in the examined planes were of this form. Specifically, over the field GF(256), with primitive element λ satisfying $\lambda^8 = \lambda^4 + \lambda^3 + \lambda^2 + 1$, we have that

$$\begin{pmatrix} t^2 & t \\ & \lambda^{29} t^4 \end{pmatrix} \text{ is a 29-clan,}$$

and

$$\begin{pmatrix} t^4 & t \\ & \lambda^{37} t^{32} \end{pmatrix} \text{ is a 53-clan.}$$

In both of these cases, $\mathcal{L} = \mathrm{GF}(4)$.

Remark 6 The β-clans of this theorem provide a set of matrices that is closed under matrix addition. This property does not seem to be shared with any other known types of non-linear clans in even characteristic. Indeed, Johnson [6] has shown that, for quadratic cones in even characteristic, this property implies that the flock is linear.

The other β-clans that have been found seem to be rare in comparison with those given in Theorem 4.

Proposition 1

$$\begin{pmatrix} t & t^5 \\ & t^7 \end{pmatrix} \textit{ is a 3-clan}$$

whenever $q = 2^e, e \equiv 1$ *or* $5 \pmod 6$.

Proof. Using Theorem 1, we examine the polynomials t and $x^3 t + x t^5 + t^7$ for each $x \in \mathcal{F}$. The latter are Dickson permutation polynomials under the given conditions ([5], pg. 57). ∎

Remark 7 The above proposition shows that $1 \in {\mathcal{D}_3}^1$ if $e \equiv 1$ or $5 \pmod 6$. However, it can be shown that $1 \in {\mathcal{D}_3}^1$ if and only if $e \not\equiv 0 \pmod 3$.

Lemma 1 *If* $\beta = 2^i + 1$ *then*

$$\begin{pmatrix} t^a & t^b \\ & \kappa t^c \end{pmatrix} \textit{ is a } \beta\textit{-clan if and only if } \begin{pmatrix} t^{(\beta-1)c} & t^b \\ & \kappa t^{\frac{a}{\beta-1}} \end{pmatrix} \textit{ is a } \beta\textit{-clan.}$$

Proof. Since $\beta - 1 = 2^i$ corresponds to a field automorphism, (1) becomes,

$$\kappa \frac{(t^a + s^a)^{\frac{1}{2^i}} (t^c + s^c)}{(t^b + s^b)^{\frac{\beta}{\beta-1}}} = \kappa \frac{(t^{2^i c} + s^{2^i c})^{\frac{1}{2^i}} (t^{\frac{a}{2^i}} + s^{\frac{a}{2^i}})}{(t^b + s^b)^{\frac{\beta}{\beta-1}}} \in {\mathcal{D}_\beta}^1.$$

∎

Remark 8 With $\beta = 2$ $(i = 0)$ this reduces to a well known equivalence of 2-clans.

Proposition 2

$$\begin{pmatrix} t^{14} & t^5 \\ & t^{\frac{1}{2}} \end{pmatrix} \text{ is a 3-clan}$$

whenever $q = 2^e, e \equiv 1$ *or* $5 \pmod 6$.

Proof. Apply lemma 1 to the 3-clan of proposition 1. ∎

The only remaining β-clans to be found in the planes that were examined are the 5-clans given by,

$$\begin{pmatrix} t^{-22} & t^{14} \\ & t^{23} \end{pmatrix}, \tag{5}$$

and its companion obtained from lemma 1, over the fields GF(32) and GF(128) (and not over GF(512).) While there is little doubt that these are the initial members of infinite families, this representation may not be the one that generalizes.

4 A geometric construction of a flock

Some of the flocks of the β-clans given in Theorem 4 have a remarkably simple geometric structure. We shall describe one such structure, and provide a geometric construction for this type of flock.

Definition 1 The lines of intersection of $w = 0$ with the $q - 1$ other planes of the flock will be called *base lines.* The intersection of two distinct base lines is called a *base point.*

Lemma 2 *Let* $b_t,\ t \in \mathcal{F}\backslash\{0\}$, *denote the base line of the plane* $t^{2^a}x+ty+\kappa t^{2^b}z+w = 0$. *Then* b_{t+s} *passes through the intersection of* b_t *and* b_s *for* $t \neq s$. *Furthermore, the line of intersection of the planes with base lines* b_t *and* b_s *lies in the plane determined by the vertex* Z *and the base line* b_{t+s}.

Proof. Let the (not necessarily distinct) base lines b_t and b_s have a common point P. Let the planes, for which these are the base lines, meet in the line r. Clearly, P is on r. The projection of r from the point $Z = (0, 0, 0, 1)$ into the plane $w = 0$ is the line $(t^{2^a}+s^{2^a})x+(t+s)y+\kappa(t^{2^b}+s^{2^b})z = 0$. The point P, which is in the plane $w = 0$ is certainly a point of this line. Since $t^{2^i} + s^{2^i} = (t + s)^{2^i}$ in fields of characteristic two, this projected line has the equation $(t + s)^{2^a}x + (t + s)y + \kappa(t + s)^{2^b}z = 0$ and so, is the base line b_{t+s}. ∎

Consequently, all of the $\binom{q}{2}$ lines of intersection of planes in the flock lie in the planes determined by the vertex and each of the base lines.

Proposition 3 *With* $q = 2^e$, *the configuration of base points and lines, formed by the planes* $t^{2^a}x + ty + \kappa t^{2^b}z + w = 0$, $t \neq 0$, *is the dual of an embedded* $\mathrm{PG}(e - 1, 2)$, *if the base lines are distinct and meet three at a point.*

Proof. Since there are $q-1$ distinct base lines, meeting three at a point, the number of base points on a base line is $\frac{q-2}{2} = 2^{e-1} - 1$, and the total number of configuration points is $\frac{1}{3}\binom{q-1}{2} = \frac{(2^e-1)(2^{e-1}-1)}{(2^2-1)(2^1-1)} = \begin{bmatrix} e \\ 2 \end{bmatrix}_2$. The point-line dual of this configuration of base points and lines thus has the same parameters as the point-line design of a $\mathrm{PG}(e-1, 2)$.

To prove that these designs are isomorphic, we must show that the Pasch Axiom (Veblen's Axiom) holds in the dual configuration, or, as we shall do, show that the dual of the Pasch Axiom holds in the configuration. The dual of the Pasch axiom in this setting is: *Given a base point not on any side of a triangle formed by three base lines, if the given point is joined to two of the vertices of the triangle by base lines it is also joined to the third vertex by a base line.* Let the three sides of the triangle be given by base lines b_r, b_s and b_t. Without loss of generality and due to the fact that there are only three base lines through any base point, we may assume that the given base point, P, is the intersection of the base lines b_{r+s} and b_{r+t}, and thus, the third vertex of the triangle, Q, is the intersection of the lines b_s and b_t. By Lemma 2, the third base line through Q is b_{s+t}, while the third base line through P is $b_{(r+s)+(r+t)} = b_{s+t}$, thus proving the validity of the dual of the Pasch Axiom. ■

Remark 9 There are some, but not all, β-clans which meet the conditions of the above proposition. For instance, all the known 3-clans satisfy these conditons.

Lemma 3 *Under the assumptions of Propositon* 3*, given four base lines, no three of which are concurrent, if one of the diagonal lines of the quadrilateral they form is a base line, then the three diagonal lines, which are concurrent, are all base lines.*

Proof. Let the four base lines be b_t, b_s, b_r and b_u. Label the points of intersection as follows: $A = b_t \cap b_s, B = b_r \cap b_u, C = b_s \cap b_r, D = b_s \cap b_u, E = b_t \cap b_u$ and $F = b_t \cap b_r$. Without loss of generality, we may assume that the diagonal line AB is a base line. The $\triangle ABC$ has sides that are all base lines, and E is joined to A and B by base lines. Thus, by the dual of the Pasch Axiom, the diagonal line EC is a base line. Now, $\triangle CDE$ also has sides that are base lines, and F is joined to C and E by base lines, hence we can again conclude that the diagonal line FD is a base line. Since these lines are in a Desarguesian plane of even order, the three diagonal lines of this quadrilateral are concurrent (Fano's configuration). ■

Proposition 4 *Under the assumptions of Propositon 3, every point of* $\mathrm{PG}(3, q)$ *lies on* $0, 1, 2$ *or* 4 *planes of the flock.*

Proof. The vertex of the cone lies on no plane of the flock. A non-vertex point of the cone lies on exactly one plane of the flock. A point on a base line which is not a base point lies on exactly two planes of the flock. Base points lie on four planes of the flock. The proposition will be proved by showing that any point not in the carrier plane, which lies on three planes of the flock, must lie on exactly four such planes.

Let P be such a point, lying on three distinct planes with base lines b_t, b_s and b_r. By lemma 2, since P lies on the line of intersection of the planes with base lines b_t and b_s, it is in the plane π determined by Z and b_{t+s}. If $b_r = b_{t+s}$, then the plane determining b_r would be π, a contradiction since flock planes do not contain Z. The plane with base line b_r therefore intersects π in a line passing through P (and not containing Z). This line meets b_{t+s} at a point Q, which is also on b_r, and so, is a base point. There is a third base line through Q, say b_u. The line of intersection of the planes with bases lines b_r and b_u must lie in π by lemma 2, and therefore, must be the line PQ. Hence, the point P lies on at least four planes of the flock.

Consider the four base lines b_t, b_s, b_r and b_u. Note that $b_{t+s} = b_{r+u}$ by the definition of b_u. They form a quadrilateral in $w = 0$. By lemma 3, the two diagonal lines, other than b_{t+s}, are also base lines. Thus, they are $b_{t+r} = b_{s+u}$, and $b_{t+u} = b_{s+r}$, and R, the common point of the three diagonal lines, is a base point. Consider the plane π' determined by Z and one of these diagonal lines, say b_{t+r}. Then π' intersects π in the line ZR. The line of intersection of the planes with base lines b_t and b_r lies in π' (lemma 2), and P lies on that line. Thus, P lies on the line ZR. Now suppose that there is another plane of the flock, with base line b_x, through P. Using the same argument, with b_x replacing b_r, the plane determined by Z and b_{t+x} meets π in the line ZP, and so, b_{t+x} must contain the point R. Since there are only three base lines through R, we must have that $x = r, u$ or s. Therefore, there are exactly four planes of the flock through P. ∎

We can provide a geometric construction of the these flocks.

Construction 1 *The following construction provides a flock of the type considered above:*

1. *Embed* $\mathrm{PG}(e-1,2)$ $(e \geq 3)$ *in a plane* $\nu_1 = \mathrm{PG}(2, 2^e)$.

2. *Take the dual of this embedding as the set of* $q-1$ *base lines.*

3. *Let* Z *be a point of* $\mathrm{PG}(3, 2^e)$ *not in* ν_1, ℓ *one of the base lines,* π_1 *the plane determined by* Z *and* ℓ, *and* P *a point of* ℓ *where* 3 *base lines meet.*

4. *Let* μ *be any line of* π_1 *through* P, *not* ℓ *nor passing through* Z.

5. *Construct* $\frac{q}{2} - 2$ *other lines in* π_1, *each associated to a base point of* ℓ, *other than* P, *in the following way: Label the base points of* ℓ *with* $Q_i, 1 \leq i \leq \frac{q}{2} - 1$, *where* $P = Q_1$. *Fix a base line through* P, *other than* ℓ, *call it* t. *For each base point* Q_i *on* ℓ, *other than* P, *select a base line in* ν_1 *through* Q_i *other than* ℓ, *call it* s. *Let* $t \cap s = R$. *The third base line through* R *intersects* ℓ *at a point* $Q_{i'}$, *where* $1 \neq i' \neq i$. *Join* $Q_{i'}$ *to* Z *in* π_1, *and call the intersection of this line with* μ, R'. *Finally, join* Q_i *to* R'. *This is the line associated with* Q_i.

6. *With* μ *being taken as the line associated with* $P = Q_1$, *for each base point* Q_i *on* ℓ, *form the planes* $\nu_{i,1}$ *and* $\nu_{i,2}$ *determined by the line of* π_1 *associated with* Q_i *and each of the base lines through* Q_i *other than* ℓ, *with the plane determined by* μ *and* t *being labelled* $\nu_{1,1}$.

7. *Let* π_2 *be the plane determined by* Z *and the base line through* P, *other than* t *or* ℓ. *Let* $\pi_2 \cap \nu_{1,1} = r$ *and construct the plane determined by* r *and* ℓ, *call it* ν_2.

8. *The $q-2$ planes, $\nu_{i,j}$, $1 \leq i \leq \frac{q}{2}-1$, $j = 1,2$ constructed in step 6, and the planes ν_1 and ν_2, form the flock.*

There are a few comments to be made concerning this construction.

1. The embedding of step 1 can be found explicitly in Brown [1]. It is also shown there that there is a cyclic group acting sharply transitively on the points of the embedding.

2. In step 4, there are $q-1$ choices for the line μ, and each leads to a flock. These flocks are closely related by a parameter which is a non-zero field element. They all normalize to the same flock.

3. In step 5, there appears to be a choice of which base line to take through each of the points on ℓ. However, these choices are irrelevant, as is, which base line through P is choosen as t. Consider the two base lines through each of P and Q_i, other than ℓ. These four lines determine a quadrilateral, and, by the proof of lemma 3, the diagonal lines of this quadrilateral are seen to be base lines that meet on ℓ. Thus, no matter which of the four vertices of the quadrangle is picked to be R (corresponding to making different choices of the base lines through P and Q_i), the third base line through it will always meet ℓ at the same point, $Q_{i'}$.

Theorem 5 *Construction 1 provides a flock for a cone with vertex Z whose carrier is* **any** *set of points in ν_1 not in the union of the points in the base lines.*

Proof. Clearly, the construction provides a set of q planes which do not pass through the point Z. The theorem will be proved if it can be shown that every line of intersection determined by two of these planes lies in a plane determined by Z and one of the base lines. Trivially, this is true of all intersections with ν_1 (these are the base lines) and, by construction, it is true for the intersection of $\nu_{i,1}$ and $\nu_{i,2}$, for all i (these lines lie in π_1.)

For the remainder of the proof, we shall fix some notation. For a plane $\nu_{i,j}$ with $i \neq 1$ and $j = 1$ or 2 (see comment 3 following the construction), the point R, is the intersection of the base line of this plane with t, the base line through P. We will choose the labelling so that the third base line through R is the base line of the plane $\nu_{i',j}$. In π_1, $R' = \mu \cap ZQ_{i'}$, and, by the construction, is a point of $\nu_{i,j}$. Thus, the line RR' is the line of intersection of $\nu_{i,j}$ and $\nu_{1,1}$. For a second plane, $\nu_{k,m}$, with $k \neq 1, i$ and $m \in \{1,2\}$, the points corresponding to R and R' will be labelled T and T', respectively. Thus, $TT' = \nu_{k,m} \cap \nu_{1,1}$. We will denote by $\nu_{i,j'}$ the plane constructed through Q_i other than $\nu_{i,j}, \nu_1$ and ν_2.

We start by showing that the intersection of any other plane with $\nu_{1,1}$ has the required property.

Consider the plane $\nu_{i,j}$. The plane determined by RR' and the base line $RQ_{i'}$ contains the line $R'Q_{i'}$, which, in turn, contains the point Z. Therefore, RR' is contained in a plane determined by Z and a base line. Now, the intersection of $\nu_{1,1}$ with ν_2 is the line r, which is contained in the plane π_2, of the required type. Hence, all the other planes of the construction meet $\nu_{1,1}$ in lines that lie in planes determined by Z and a base line.

Had the base line of $\nu_{1,2}$ been used in the role of t in the construction, the constructed planes would be the same (comment 3). Repeating the above argument in this case shows that $\nu_{1,2}$ has the same property.

We now consider the the intersection of $\nu_{i,j}$ with $\nu_{k,m}$, where $i \neq k$, $i, k > 1$, and $j, m = 1, 2$. There are two cases to consider, depending on whether or not $k = i'$.

Let $k = i'$. With $S = Q_i R' \cap Q_{i'} T'$, the line of intersection of $\nu_{i,j}$ and $\nu_{i',m}$ is SR. In π_1, the quadrangle $Q_i Q_{i'} R' T'$ has diagonal line ZSP. Therefore, the plane determined by Z and SR contains the base line $t = PR$.

Now, assume $k \neq i'$. Let V be the intersection of the base lines of $\nu_{i,j}$ and $\nu_{k,m}$ and let $S = RR' \cap TT' \in \nu_{1,1}$. The line of intersection of $\nu_{i,j}$ and $\nu_{k,m}$ is VS. Projecting $\nu_{1,1}$ from Z to ν_1, we see that S projects to the point L, which is the intersection of the base lines of $\nu_{i',j}$ and $\nu_{k',m}$. Thus, the plane determined by Z and VS contains the line VL. The sides of $\triangle RTV$ are base lines, and L is joined to R and T by base lines, so, by the dual of the Pasch Axiom, VL is a base line.

Finally, we must show that the lines of intersection in ν_2 have the required property. Recall, from the construction of ν_2 that $r = \pi_2 \cap \nu_{1,1}$, where π_2 is the plane determined by Z and the base line of $\nu_{1,2}$. Consider the plane $\nu_{i,j}$. Let $S = r \cap RR'$. The line SQ_i is the intersection of $\nu_{i,j}$ and ν_2. Projecting $\nu_{1,1}$ from Z to ν_1, we see that S projects to the point L, which is the intersection of the base lines of $\nu_{i',j}$ and $\nu_{1,2}$. Thus, the plane determined by Z and SQ_i contains the line LQ_i. The sides of $\triangle RPQ_i$ are base lines, and L is joined to R and P by base lines, so, by the dual of the Pasch Axiom, LQ_i is a base line. ∎

5 Open problems

This work should be considered as an initial foray into largely uncharted waters. It raises more questions than it answers. Some of the open problems are:

1. What are the actual families to which the flocks given in (5) belong?

2. Are there other type II monomial flocks besides those given by Theorem 4?

3. An initial impetus for examining monomial cones was to find new occurances of o-polynomials naturally associated with herds. No such examples were found, which leads to the following question. If t^a appears as the coefficient of x in a type I monomial flock, whose y coefficient is $t^{\frac{1}{\beta}}$, and is an o-polynomial, must t^β also be an o-polynomial?

4. No known cone whose carrier is an oval or hyperoval has a flock of the type given by Construction 1. Is it possible that the base line configuration used in that construction is a blocking set for ovals in the plane?

5. The base line configuration for all the known monomial α-flocks are subsets of the duals of translation hyperovals (in particular, $q - 1$ lines, no three of which are concurrent). The base line configurations of all the β- flocks given in Section 3 are the same as those given by the flocks of Theorem 4. What are the other types of base line configurations for monomial flocks? Can these configurations be classified?

6. What can be said about the flocks of cones when the conditions $(\beta, q-1) = (\beta-1, q-1) = 1$ are relaxed? More generally, for flocks of arbitrary cones?

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