Semipartial geometries and generalized quadrangles of order (r, r^2)

Matthew R. Brown

Abstract

We show how a semipartial geometry can be constructed from a particular type of subquadrangle of order r of a generalized quadrangle of order (r, r^2). We also determine conditions under which two such semipartial geometries are isomorphic. As a result, a new semipartial geometry will be constructed from a generalized quadrangle constructed by Kantor.

1 Introduction

We begin with the definition of a finite generalized quadrangle. (For more details on generalized quadrangles see [15]). A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) Each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \mathrm{~I~} m \mathrm{~I~} Y \mathrm{~I~} \ell$.

Received by the editors September 1997.
Communicated by Frank De Clerck.
1991 *Mathematics Subject Classification.* 51E12, 51D20 (05B25).
Key words and phrases. Generalized quadrangle, semipartial geometry, cover, flock.

The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* (s,t). If $s=t$, then $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s,t), then it follows that $|\mathcal{P}|=(s+1)(st+1)$ and $|\mathcal{B}|=(t+1)(st+1)$ ([15, 1.2.1]).

If $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ is a GQ of order (s,t) then the incidence structure $\mathcal{S}^\wedge=(\mathcal{B},\mathcal{P},\mathrm{I})$ is a GQ of order (t,s) known as the *dual* of $\mathcal{S}$. In any GQ $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$, if two distinct points X,Y are collinear, we write $X\sim Y$ and denote the line incident with X and Y by $\langle X,Y\rangle$. If X and Y are not collinear then we write $X\not\sim Y$. If X is a point of $\mathcal{S}$, then we define $X^\perp=\{Y\in\mathcal{P}: Y\sim X\}\cup\{X\}$. If $\{X_1,\ldots,X_n\}$ is a set of n points of $\mathcal{S}$, pairwise non-collinear, then we define $\{X_1,\ldots,X_n\}^\perp=\cap_{i=1}^n X_i^\perp$. A set of three, pairwise non-collinear, points of a GQ is called a *triad*.

Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ be a GQ of order (s,t) and let $\mathcal{S}'=(\mathcal{P}',\mathcal{B}',\mathrm{I}')$ be a GQ of order (s',t'). The GQ $\mathcal{S}'$ is a *subquadrangle* (subGQ) of $\mathcal{S}$ if $\mathcal{P}'\subseteq\mathcal{P}$, $\mathcal{B}'\subseteq\mathcal{B}$ and I' is the restriction of I to $(\mathcal{P}'\times\mathcal{B}')\cup(\mathcal{B}'\times\mathcal{P}')$. If $\mathcal{S}'$ is a subquadrangle of $\mathcal{S}$, then we write $\mathcal{S}'\subseteq\mathcal{S}$. If $\mathcal{S}\neq\mathcal{S}'$ then we say that $\mathcal{S}'$ is a *proper subquadrangle* of $\mathcal{S}$ and write $\mathcal{S}'\subset\mathcal{S}$. If $\mathcal{S}'\subset\mathcal{S}$, then it follows that $\mathcal{P}'\neq\mathcal{P}$ and $\mathcal{B}'\neq\mathcal{B}$.

As an example, the GQ $Q(5,q)$, of order (q,q^2), arises as the geometry of points and lines of a non-singular elliptic quadric in $\mathrm{PG}(5,q)$, which has a canonical form given by the equation $f(x_0,x_1)+x_2x_5+x_3x_4=0$ where f is an irreducible quadratic binary form. The GQ $Q(4,q)$, of order q, arises similarly as the geometry of points and lines of a non-singular (parabolic) quadric in $\mathrm{PG}(4,q)$, which has a canonical form given by the equation $-{x_0}^2+x_1x_4+x_2x_3=0$. We note that $Q(5,q)$ contains subquadrangles isomorphic to $Q(4,q)$ (see [15, 3.1.1 and 3.5 (a)]).

An *ovoid* of a GQ $\mathcal{S}$ of order (s,t) is a set θ of points such that each line of $\mathcal{S}$ is incident with precisely one point of θ. It follows that θ has $st+1$ points. Dually, a *spread* of a GQ $\mathcal{S}$ of order (s,t) is a set $\mathtt{S}$ of $st+1$ lines of $\mathcal{S}$ such that each point of $\mathcal{S}$ is incident with exactly one line of $\mathtt{S}$.

Lemma 1.1 ([16], [11], see [15], 2.2.1) *Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ be a GQ of order (r,r^2) with a subquadrangle $\mathcal{S}'=(\mathcal{P}',\mathcal{B}',\mathrm{I}')$ of order r and let $P\in\mathcal{P}\setminus\mathcal{P}'$. Then the set of points of $\mathcal{S}'$ which are collinear with P form an ovoid of $\mathcal{S}'$.*

An ovoid defined as in Lemma 1.1 is said to be *subtended* by P, or just *subtended*. The ovoids of $\mathcal{S}'$ subtended by the points in $\mathcal{P}\setminus\mathcal{P}'$ are said to be the ovoids *subtended by* $\mathcal{S}$ or just the *subtended* ovoids.

A *rosette* based at a point X of a GQ $\mathcal{S}$ of order (s,t) is a set $\mathcal{R}$ of ovoids with pairwise intersection $\{X\}$ and such that $\{\theta\setminus\{X\}:\theta\in\mathcal{R}\}$ is a partition of the points of $\mathcal{S}$ not collinear with X. The point X is called the *base point* of $\mathcal{R}$. It follows that a rosette $\mathcal{R}$ has s ovoids.

If $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ is a GQ of order (r,r^2) with a subquadrangle $\mathcal{S}'=(\mathcal{P}',\mathcal{B}',\mathrm{I}')$ of order r, then every line of $\mathcal{S}$ is either a line of $\mathcal{S}'$ or is incident with exactly one point of $\mathcal{S}'$ (by [15, 2.1] and a count). A line of $\mathcal{S}$ meeting $\mathcal{S}'$ in exactly one point is called a *tangent*. Given a tangent line ℓ to $\mathcal{S}'$, the set of r ovoids subtended by points of ℓ not in $\mathcal{S}'$ form a rosette of $\mathcal{S}'$. To see this, first observe that if $X,Y\ \mathrm{I}\ \ell$ and $X,Y\in\mathcal{P}\setminus\mathcal{P}'$, then the ovoids θ_X and θ_Y subtended by X and Y both contain the point $\ell\cap\mathcal{S}'$. Also if θ_X and θ_Y contain a further point Z of $\mathcal{S}'$, then X,Y,Z form a triangle, contradicting GQ axiom (iii). We say that this rosette is the rosette *subtended* by the line ℓ or that the rosette is *subtended*.

A (finite) *semipartial geometry* (SPG) is an incidence structure $\mathcal{T} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines* respectively, and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) Each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) If X is a point and ℓ is a line not incident with X, then the number of pairs $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \mathrm{\ I\ } m \mathrm{\ I\ } Y \mathrm{\ I\ } \ell$ is either a constant α ($\alpha > 0$), or 0;

(iv) For any pair of non-collinear points (X, Y) there are μ ($\mu > 0$) points Z such that Z is collinear with both X and Y.

The integers s, t, α, μ are the *parameters* of $\mathcal{T}$. For more information on SPGs see [4].

In Section 2 we will show that if $\mathcal{S}$ is a GQ of order (r, r^2) and $\mathcal{S}' \subset \mathcal{S}$ is a subGQ of order r such that each subtended ovoid of $\mathcal{S}'$ is subtended by precisely two points, then the subtended ovoid/rosette structure is an SPG with parameters $s = r - 1$, $t = r^2$, $\alpha = 2$ and $\mu = 2r(r-1)$. It will also be shown that the above condition is equivalent to the existence of an involution of $\mathcal{S}$ that fixes $\mathcal{S}'$ pointwise.

In Section 3 we consider the isomorphism problem of SPGs constructed as above. In particular we consider $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_{\mathcal{S}}$ two SPGs constructed from GQs $\mathcal{W}' \subset \mathcal{W}$ and $\mathcal{S}' \subset \mathcal{S}$, respectively. We show that $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_{\mathcal{S}}$ are isomorphic if and only if there is an isomorphism from $\mathcal{W}'$ to $\mathcal{S}'$, taking $\mathcal{T}_{\mathcal{W}}$ to $\mathcal{T}_{\mathcal{S}}$.

In Section 4 we outline the construction of a GQ, $\mathcal{S}(\mathcal{C})$, of order (q^2, q) from a q-clan $\mathcal{C}$ and state some results on GQs constructed in this manner. In Section 5, for q odd and non-prime, we consider a q-clan GQ, $\mathcal{S}(\mathcal{C}_\sigma)$ constructed by Kantor in [10]. We construct a new SPG from a subGQ of $\mathcal{S}(\mathcal{C}_\sigma)$ isomorphic to $Q(4, q)$.

2 SPGs from GQs of order (r, r^2)

Consider the SPG (constructed by Metz, see [5] and by Hirschfeld and Thas [7]) with parameters $s = q - 1$, $t = q^2$, $\alpha = 2$, $\mu = 2q(q-1)$, where q is a prime power. The construction due to Metz, is as follows: let $\mathcal{Q} = Q(4, q)$, $\mathcal{P}$ be the set of three dimensional, non-singular elliptic quadrics contained in $\mathcal{Q}$. Let a *bundle* of $\mathcal{Q}$ be a set of q elements of $\mathcal{P}$ that meet pairwise in a common point. Let $\mathcal{B}$ be the set of bundles of $\mathcal{Q}$. Define incidence $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ to be symmetrized containment. Since each bundle is a set of q elliptic quadric ovoids of $\mathcal{Q}$ sharing a common tangent plane at the point where the elliptic quadrics intersect, and two elements of $\mathcal{P}$ that are tangent are incident with exactly one common bundle, it follows that the structure $\mathcal{T} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a SPG with the above parameters.

Now consider the GQ $\mathcal{S} = Q(5, q)$. $\mathcal{S}$ contains $\mathcal{Q}$ as a subGQ and the subtended ovoids are exactly the elements of $\mathcal{P}$. Moreover, the subtended rosettes are the bundles and $\mathcal{T}$ is the incidence structure obtained by taking subtended ovoids as points and subtended rosettes as lines. This relation between $Q(5,q)$, $Q(4,q)$ and the SPG $\mathcal{T}$ seems to depend on the combinatorics of the situation, rather than the specific geometry. So, it is natural to try and extend this result to a more general scenario. The following work leads to such a generalisation.

To begin we recall a result of Bose and Shrikhande interpreted in the GQ context.

Lemma 2.1 ([1], see [15], 1.2.4) *If $\mathcal{S}$ is a* GQ *of order (r,r^2) and $\{X,Y,Z\}$ is a triad of $\mathcal{S}$, then $|\{X,Y,Z\}^\perp| = r+1$.*

Corollary 2.2 *Let $\mathcal{S}$ be a* GQ *of order (r,r^2) and $\mathcal{S}'$ a sub*GQ *of order r. A subtended ovoid of $\mathcal{S}'$ is subtended by at most two points of $\mathcal{S}$. Further, if an ovoid θ is subtended by two points X, X', then the size of the intersection of θ with any other subtended ovoid θ_Y, $Y \neq X, X'$, is determined: if $Y \sim X$ or $Y \sim X'$, then $|\theta \cap \theta_Y| = 1$ and if $Y \not\sim X, X'$, then $|\theta \cap \theta_Y| = r+1$.*

Proof. Suppose that an ovoid θ is subtended by three points X, Y, Z. These three points are necessarily pairwise non-collinear and so form a triad of $\mathcal{S}$. Since $|\{X,Y,Z\}^\perp| \geq r^2+1$ we have a contradiction of Lemma 2.1. Thus any ovoid may be subtended by at most two points.

Now suppose that θ is subtended by exactly two points X and X'. Let θ_Y be the ovoid subtended by the point Y, $Y \neq X, X'$. Suppose that $Y \sim X$ or $Y \sim X'$; without loss of generality we may suppose that $Y \sim X$. Thus θ_X and θ_Y are contained in the rosette subtended by the line $\langle X, Y\rangle$ and so $\theta_X \cap \theta_Y = \{P\}$ for some point P and $|\theta \cap \theta_Y| = |\{X,X',Y\}^\perp| = 1$. Suppose now that $Y \not\sim X, X'$, then $\{X,X',Y\}$ is a triad of $\mathcal{S}$ and so $|\theta \cap \theta_Y| = |\{X,X',Y\}^\perp| = r+1$. ∎

If a GQ $\mathcal{S}$ of order (r,r^2) has a subGQ $\mathcal{S}' \subset \mathcal{S}$ of order r such that each subtended ovoid of $\mathcal{S}'$ is subtended by exactly two points of $\mathcal{S}$, then we say that $\mathcal{S}'$ is *doubly subtended* in $\mathcal{S}$. In some sense a doubly subtended subquadrangle is an 'extreme' subquadrangle, so it is not surprising that we get some nice geometry from it. At this stage we introduce a slight abuse of notation. If X is a point of $\mathcal{S}$, then we denote this by $X \in \mathcal{S}$. If X is a point of $\mathcal{S}$ but not a point of $\mathcal{S}'$, then we denote this by $X \in \mathcal{S} \setminus \mathcal{S}'$.

We might now ask how many subtended rosettes can two subtended ovoids have in common. Let θ_X and θ_Y be two subtended ovoids; subtended by X and Y, respectively, where $\theta_X \neq \theta_Y$.

There are three cases to consider: (i) neither θ_X nor θ_Y is subtended by a second point, (ii) θ_X is subtended by another point X' and θ_Y is not subtended by a second point, (iii) θ_X and θ_Y are each subtended by a second point, say X' and Y' respectively. For case (i) θ_X and θ_Y are contained in exactly one common subtended rosette if $X \sim Y$ and in none otherwise. In case (ii) Y may be collinear with at most one of X, X' since otherwise $Y \in \{X,X'\}^\perp \subset S'$ which contradicts $Y \in \mathcal{S} \setminus \mathcal{S}'$. Thus for case (ii) θ_X and θ_Y have one common rosette if $Y \sim X$ or $Y \sim X'$ (not both) and none otherwise. For case (iii), to each unordered, incident pair taken from $\{X, X', Y, Y'\}$, there corresponds a subtended rosette containing θ_X and θ_Y. So there may be none, one or two subtended rosettes containing θ_X and θ_Y. Note that for the case where there are two such subtended rosettes, that they need not be distinct. In this case two distinct lines subtend the same rosette.

We have already seen that a GQ $\mathcal{S}$ of order (r, r^2) with a subGQ $\mathcal{S}'$ of order r that has all subtended ovoids of $\mathcal{S}$ being subtended twice is a special and also extremal case of a subGQ of $\mathcal{S}$. We now give a result which shows the relationship between double subtending and the existence of a particular type of involution of the GQ $\mathcal{S}$. The idea for the construction of the involution comes from Thas [17]. For the following we will denote the ovoid subtended by a point X by θ_X.

Lemma 2.3 *Let* $\mathcal{S}$ *be a* GQ *of order* (r, r^2) *and* $\mathcal{S}'$ *a sub*GQ *of order* r*. Then* $\mathcal{S}'$ *is doubly subtended in* $\mathcal{S}$ *if and only if there exists a non-identity involution of* $\mathcal{S}$ *that fixes* $\mathcal{S}'$ *pointwise.*

Proof. First, let τ be an involution of $\mathcal{S}$ that fixes $\mathcal{S}'$ pointwise. We first show that τ fixes no point of $\mathcal{S} \setminus \mathcal{S}'$. Suppose that $X \in \mathcal{S} \setminus \mathcal{S}'$ and $\tau(X) = X$. Consider a line ℓ such that $X \in \ell$ and let $\ell \cap \mathcal{S}' = P$, say. Now $\tau(P) = P$ and so $\tau(\ell) = \ell$. Now let $R \in \ell$, $R \neq P$. Then $\theta_R = \theta_{\tau(R)}$ but $\tau(R) \in \ell$ and so $R = \tau(R)$. Thus ℓ is fixed pointwise by τ. Now consider a point $Y \in \mathcal{S} \setminus \mathcal{S}'$, $Y \not\sim X$. If $\theta_Y = \theta_X$ then since an ovoid may only be subtended twice and $\tau(Y)$ subtends θ_Y we must have that $\tau(Y) = Y$. If $\theta_Y \neq \theta_X$, then there exists an incident point/line pair (ℓ', R') such that $X \in \ell'$, $R' \in \mathcal{S} \setminus \mathcal{S}'$ and $Y \sim R'$. However, R' is fixed by τ as it is collinear with X and so every point collinear with R' is fixed, so τ also fixes Y. Thus τ fixes every point of $\mathcal{S}$ and so is the identity, which is a contradiction. Hence τ fixes no point in $\mathcal{S} \setminus \mathcal{S}'$.

Thus we can now say that for any point $X \in \mathcal{S} \setminus \mathcal{S}'$ we have $\tau(X) = X' \neq X$ and so θ_X is subtended by the distinct points X, X'.

Now, suppose that $\mathcal{S}'$ is doubly subtended in $\mathcal{S}$. If $X \in \mathcal{S} \setminus \mathcal{S}'$ then let X' be the second point subtending θ_X. Now define τ to be the following map:

$$\begin{aligned} \tau : \quad & X \mapsto X' \quad X \in S \setminus \mathcal{S}' \\ & X \mapsto X \quad X \in \mathcal{S}'. \end{aligned}$$

Consider points $P, Q \in \mathcal{S}$, with $P \sim Q$. If $P, Q \in \mathcal{S}'$, then $P = \tau(P) \sim \tau(Q) = Q$. If $P \in \mathcal{S}'$, $Q \in \mathcal{S} \setminus \mathcal{S}'$, then $P \in \theta_Q = \theta_{Q'} = \theta_{\tau(Q)}$ and so $P \sim \tau(Q)$, that is $\tau(P) \sim \tau(Q)$.

Now if $P, Q \in \mathcal{S} \setminus \mathcal{S}'$ then $|\theta_P \cap \theta_Q| = |\theta_{\tau(P)} \cap \theta_{\tau(Q)}| = 1$; so by Corollary 2.2 we have that $\tau(Q) \sim P$ or $\tau(Q) \sim \tau(P)$. Since $Q \sim P$ it must be that $\tau(Q) \not\sim P$ and so $\tau(P) \sim \tau(Q)$. Thus τ is an automorphism of $\mathcal{S}$ and clearly an involution. ■

Corollary 2.4 *If* $\mathcal{S}$ *is a* GQ *of order* (r, r^2) *that has a doubly subtended sub*GQ $\mathcal{S}'$ *of order* r*, then for each incident point, subtended ovoid pair* (X, θ)*, for* $X \in \mathcal{S}'$*, there exists an unique subtended rosette* $\mathcal{R}$*, containing* θ *and with base point* X*.*

Proof. If θ is subtended by the points Y and Y', then the only subtended rosettes containing θ and with base point X, are those subtended by the lines $\langle X, Y\rangle$ and $\langle X, Y'\rangle$. However $\langle X, Y\rangle$ is the image of $\langle X, Y'\rangle$ under the involution constructed in Lemma 2.3, and vice-versa. Since the involution in Lemma 2.3 fixes the subquadrangle $\mathcal{S}'$ pointwise (and linewise), the rosette subtended by $\langle X, Y\rangle$ and the rosette subtended by $\langle X, Y'\rangle$ are the same. ■

Now we show that if a subGQ is doubly subtended, then we get an SPG from its subtended ovoid/rosette structure.

Theorem 2.5 *Let $\mathcal{S}$ be a* GQ *of order (r, r^2) containing a sub*GQ *$\mathcal{S}'$ of order r, such that $\mathcal{S}'$ is doubly subtended in $\mathcal{S}$. Consider the incidence structure $\mathcal{T}$:*

Points : Subtended ovoids of $\mathcal{S}'$.
Lines : Subtended rosettes of $\mathcal{S}'$.
Incidence : Symmetrized containment.

Then $\mathcal{T}$ is a SPG *with parameters $s = r-1$, $t = r^2$, $\alpha = 2$ and $\mu = 2r(r-1)$.*

Proof. A rosette contains r ovoids, thus $s = r-1$. By Corollary 2.4 there are r^2+1 subtended rosettes containing a subtended ovoid θ_X, that is, $t = r^2$.

Now consider a subtended rosette $\mathcal{R}$ with basepoint P and not containing the ovoid θ_X. Recall that ovoids of $\mathcal{R}$ partition the points of $\mathcal{S}'$ that are not collinear with P. Suppose that $P \in \theta_X$. Then $\theta_X \subset \mathcal{S}' \setminus P^{\perp}$. Let n_1 and n_{r+1} be the number of ovoids of $\mathcal{R}$ that meet θ_X in 1 and $r+1$ points, respectively (see Corollary 2.2). Then we have the following equations:

$$\begin{aligned} n_{r+1} \cdot r + n_1 \cdot 0 &= r^2 \\ n_{r+1} + n_1 &= r \end{aligned}$$

Solving simultaneously gives $n_{r+1} = r$ and $n_1 = 0$, that is, θ_X meets each ovoid in $\mathcal{R}$ in $r+1$ points.

Suppose now that $P \notin \theta_X$. Then θ_X has $r^2 - r$ points non-collinear with P and so we have the following equations:

$$\begin{aligned} n_{r+1} \cdot (r+1) + n_1 &= r^2 - r \\ n_{r+1} + n_1 &= r \end{aligned}$$

Solving simultaneously we have $n_{r+1} = r-2$ and $n_1 = 2$.

In terms of $\mathcal{T}$ the above means that if we have a non-incident point/line pair (A, ℓ) in $\mathcal{T}$ there are 0 or 2 point/line pairs (B, m) such that $A \mathrm{\ I\ } m \mathrm{\ I\ } B \mathrm{\ I\ } \ell$.

Now consider two subtended ovoids of $\mathcal{S}'$, say θ_X and θ_Y, such that $|\theta_X \cap \theta_Y| = r+1$. By Corollary 2.4, for each $Q \in \theta_X \setminus \theta_Y$ there exists exactly one subtended rosette $\mathcal{R}$, with base point Q and containing θ_X. By the above, we see that there are two subtended rosettes containing θ_Y and an ovoid in $\mathcal{R}$, and so two subtended ovoids that are contained in a subtended rosette with θ_X and contained in a distinct subtended rosette with θ_Y. This is true for each point in $\theta_X \setminus \theta_Y$ and so there are $2(r^2 - r)$ subtended ovoids that are contained in a rosette with both θ_X and θ_Y. In $\mathcal{T}$, this means that given two non-collinear points A and B there are $2r(r-1)$ points collinear to both A and B. ∎

Corollary 2.6 *Let $\mathcal{S}$ be a* GQ *of order (r, r^2) containing a sub*GQ *$\mathcal{S}'$ of order r such that there exists a non-identity involution of $\mathcal{S}$ that fixes $\mathcal{S}'$ pointwise. Then there is an associated* SPG *with parameters $s = r-1$, $t = r^2$, $\alpha = 2$ and $\mu = 2r(r-1)$.*

3 The isomorphism problem for SPGs

In this section we will establish when two SPGs constructed as in Section 2 are isomorphic. We first introduce algebraic 2-fold covers of SPGs.

3.1 Algebraic 2-fold covers of SPGs

Suppose that $\mathcal{S}$ is a GQ of order (r, r^2) and $\mathcal{S}'$ is a subquadrangle of order r that is doubly subtended in $\mathcal{S}$. Let $\mathcal{T}$ be the SPG constructed from $\mathcal{S}$ and $\mathcal{S}'$ as in Theorem 2.5. Let Γ be the point graph of $\mathcal{T}$ (the graph whose vertices are the points of $\mathcal{T}$ and with adjacency given by collinearity in $\mathcal{T}$). Let c be a function such that

$$c : \ \{(P, Q) \ : \ P, Q \text{ adjacent vertices of } \Gamma\} \to \mathbb{Z}_2$$

and $c(P, Q) = c(Q, P)$ for any pair of adjacent vertices P, Q. Now, define the graph $\overline{\Gamma}$ as follows. Let the set of vertices of $\overline{\Gamma}$ be $\{(P, \alpha) : P \in \Gamma, \alpha \in \mathbb{Z}_2\}$ and define two vertices (P, α) and (Q, β) of $\overline{\Gamma}$ to be adjacent if P and Q are adjacent in Γ and $c(P, Q) = \alpha + \beta$. Any graph that is isomorphic to $\overline{\Gamma}$ is an *algebraic* 2*-fold cover* of Γ. The vertices $(P, 0)$ and $(P, 1)$ of $\overline{\Gamma}$ are said to *cover* the vertex P of Γ, while P is said to be *covered* by $(P, 0)$ and $(P, 1)$. For more details on covers and algebraic covers see [2]. Note that in the work that follows on algebraic 2-fold covers, since c maps into $\mathbb{Z}_2$, arithmetic will be modulo 2.

Now, suppose that c has the additional property that if P, Q, R are three collinear points of $\mathcal{T}$ then $\delta c(P, Q, R) = c(P, Q) + c(P, R) + c(Q, R) = 0$. Let ℓ be the line of $\mathcal{T}$ incident with the points $\{X_1, X_2, \cdots, X_{s+1}\}$. Each X_i, $i = 1, \ldots, s+1$, is covered by two vertices of $\overline{\Gamma}$ and this set of $2(s+1)$ vertices of $\overline{\Gamma}$ form two disjoint complete graphs of size $s+1$: $\{(X_1, 0), (X_2, c(X_1, X_2)), \cdots, (X_{s+1}, c(X_1, X_{s+1}))\}$ and $\{(X_1, 1), (X_2, c(X_1, X_2)+1), \cdots, (X_{s+1}, c(X_1, X_{s+1})+1)\}$ (note that $s+1 = r$). Each of these sets is said to cover the clique of Γ corresponding to the points of ℓ, or to simplify matters, to cover ℓ. Now consider the geometry $\overline{\mathcal{T}}$ that has pointset the vertex set of $\overline{\Gamma}$ and lineset the set of covers of lines of $\mathcal{T}$ and so has point graph $\overline{\Gamma}$. Any geometry that is isomorphic to $\overline{\mathcal{T}}$ is an *algebraic* 2*-fold cover* of $\mathcal{T}$ and is said to be *defined* by the function c.

3.2 The GQ condition

Let $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{T}$ be as in Section 3.1 and let $\overline{\mathcal{T}}$ be the algebraic 2-fold cover of $\mathcal{T}$ defined by the function c. Then $\overline{\mathcal{T}}$ is said to satisfy the *GQ condition* if for each set $\{P, Q, R\}$ of pairwise collinear points of $\mathcal{T}$

$$\delta c(P, Q, R) = c(P, Q) + c(P, R) + c(Q, R) = 0 \Longleftrightarrow P, Q, R \text{ are collinear} \qquad (1)$$

Theorem 3.1 *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a* GQ *of order* (r, r^2) *and* $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ *a doubly subtended sub*GQ *of order* r*. Let* $\mathcal{T}$ *be the* SPG *constructed from* $\mathcal{S}$ *and* $\mathcal{S}'$*, as in Theorem 2.5 and let* $\mathcal{T}^*$ *be the geometry* $(\mathcal{P} \setminus \mathcal{P}', \mathcal{B} \setminus \mathcal{B}', \mathrm{I}^*)$*, where* I^* *is the appropriate restriction of* I*. Let* Θ *be the set of subtended ovoids of* $\mathcal{S}'$ *and represent*

$\mathcal{P} \setminus \mathcal{P}'$ as the set $\{(\theta,0),(\theta,1) : \theta \in \Theta\}$. Let c be the function defined by

$$c(\theta_i,\theta_j) = \begin{cases} 0 & \text{if } |\theta_i \cap \theta_j| = 1 \text{ and } (\theta_i,0) \text{ and } (\theta_j,0) \text{ are collinear.} \\ 1 & \text{if } |\theta_i \cap \theta_j| = 1 \text{ and } (\theta_i,0) \text{ and } (\theta_j,0) \text{ are not collinear.} \end{cases}$$

Then $\mathcal{T}^*$ is an algebraic 2-fold cover of $\mathcal{T}$ defined by c. Furthermore, c satisfies the GQ *condition* (1).

Proof. Let θ_1 and θ_2 be two collinear points of $\mathcal{T}$ and so $|\theta_1 \cap \theta_2| = 1$. Clearly, $c(\theta_1,\theta_2) = c(\theta_2,\theta_1)$. Since θ_1 and θ_2 are collinear points of $\mathcal{T}$, they are two subtended ovoids of $\mathcal{S}'$ contained in a common subtended rosette, that is, subtended by two lines of $\mathcal{S}$. The point $(\theta_1,0)$ of $\mathcal{T}^*$ is incident with one of these lines and $(\theta_1,1)$ is incident with the other, and similarly for $(\theta_2,0)$ and $(\theta_2,1)$. Thus (θ_1,α) is collinear with (θ_2,β) if and only if $c(\theta_1,\theta_2) = \alpha+\beta$. Thus c defines an algebraic 2-fold cover of the point graph of $\mathcal{T}$.

To show that $\mathcal{T}^*$ is an algebraic 2-fold cover of the geometry $\mathcal{T}$, defined by c, we need to show that $\delta c(\theta_1,\theta_2,\theta_3) = 0$ whenever θ_1, θ_2 and θ_3 are distinct collinear points of $\mathcal{T}$. Let θ_1, θ_2 and θ_3 be three distinct collinear points of $\mathcal{T}$, then they are contained in a common subtended rosette $\mathcal{R}$ of $\mathcal{S}'$. Now $(\theta_1,0)$ is collinear with $(\theta_2,c(\theta_1,\theta_2))$ and with $(\theta_3,c(\theta_1,\theta_3))$. Since $\langle(\theta_1,0),(\theta_2,c(\theta_1,\theta_2))\rangle$ and $\langle(\theta_1,0),(\theta_3,c(\theta_1,\theta_3))\rangle$ both subtend the rosette $\mathcal{R}$, it follows that $(\theta_2,c(\theta_1,\theta_2))$ and $(\theta_3,c(\theta_1,\theta_3))$ are collinear and so $\delta c(\theta_1,\theta_2,\theta_3) = 0$. Thus c defines a cover of the geometry $\mathcal{T}$.

If θ_1, θ_2 and θ_3 are pairwise collinear but not incident with a common line of $\mathcal{T}$, then it follows that they are not contained in a common subtended rosette of $\mathcal{S}'$. Thus $(\theta_1,0)$, $(\theta_2,c(\theta_1,\theta_2))$ and $(\theta_3,c(\theta_1,\theta_3))$ are not incident with a common line of $\mathcal{T}^*$ and so $(\theta_2,c(\theta_1,\theta_2))$ and $(\theta_3,c(\theta_1,\theta_3))$ are not collinear since this would be a triangle in $\mathcal{S}$. Hence, $\delta c(\theta_1,\theta_2,\theta_3) = 1$ and c satisfies the GQ condition. ■

Given the notation of Theorem 3.1 and $\mathcal{T}^* \cong \overline{\mathcal{T}}$ (since $\mathcal{T}$ is an algebraic 2-fold cover of $\mathcal{T}$), consider the following description of $\mathcal{S}$.

Points	(i)	Points of $\mathcal{S}'$.
	(ii)	Points of $\overline{\mathcal{T}}$.
Lines	(a)	Lines of $\mathcal{S}'$.
	(b)	$\ell \cup P$ where ℓ is a line of $\overline{\mathcal{T}}$ and P the base point of the subtended rosette covered by ℓ. (2)
Incidence	(i),(a)	As in $\mathcal{S}'$.
	(i),(b)	A point P of type (i) is incident with a line $\ell \cup Q$ of type (b) if and only if $P = Q$.
	(ii),(a)	None.
	(ii),(b)	A point P of type (ii) is incident with a line $\ell \cup Q$ of type (b) if and only if P is incident with ℓ in $\overline{\mathcal{T}}$

Now suppose that in the above incidence structure instead of using the algebraic 2-fold cover of $\mathcal{T}$ from Theorem 3.1 we use an arbitrary algebraic 2-fold cover of $\mathcal{T}$. The following theorem specifies the conditions under which this new incidence structure is a GQ.

Theorem 3.2 *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a* GQ *of order* (r, r^2) *and* $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ *a doubly subtended sub*GQ *of order* r*. Let* $\mathcal{T}$ *be the* SPG *constructed from* $\mathcal{S}$ *and* $\mathcal{S}'$*, as in Theorem 2.5 and let* $\overline{\mathcal{T}}$ *be an algebraic* 2*-fold cover of* $\mathcal{T}$ *defined by a function* c*. Let* $\mathcal{W}$ *be the incidence structure defined by* (2)*. incident Then* $\mathcal{W}$ *is a* GQ *of order* (r, r^2) *if and only if* c *satisfies the* GQ *condition* (1)*. In this case* $\mathcal{W}$ *contains* $\mathcal{S}'$ *as a subquadrangle and* $\mathcal{S}'$ *is doubly subtended by* $\mathcal{W}$*. The* SPG *constructed from* $\mathcal{W}$ *and* $\mathcal{S}'$ *as in Theorem 2.5 is* $\mathcal{T}$*.*

Proof. Any line of $\mathcal{S}'$ is incident with $r+1$ points of $\mathcal{S}'$ and so $r+1$ points of $\mathcal{W}$. A line $\ell \cup P$ of type (b) is incident with P and with the r points of $\overline{\mathcal{T}}$ incident with ℓ. Thus each line of $\mathcal{W}$ is incident with $r+1$ points.

Let Q be a point of type (i), then Q is incident with $r+1$ lines of $\mathcal{S}'$. There are $(r^2-r)/2$ subtended rosettes that have Q as a base point and so there are r^2-r lines of $\mathcal{W}$ of type (b) that are incident with Q. Thus Q is incident with r^2+1 lines of $\mathcal{S}$. By Corollary 2.4 each subtended ovoid of $\mathcal{S}'$ is contained in r^2+1 subtended rosettes and so each type (ii) point of $\mathcal{W}$ is incident with r^2+1 lines of $\mathcal{W}$.

We check the third GQ axiom for each non-incident point/line pair, (P, ℓ) of $\mathcal{W}$. If P is of type (i) and ℓ is of type (a), then since $\mathcal{S}'$ is a GQ the property holds. Let P be of type (i) and $\ell \cup Q$ of type (b). If P and Q are collinear then there is no ovoid of $\mathcal{S}'$ containing both P and Q. Thus, Q is the unique point of $\ell \cup Q$ that is collinear with P. If P is not collinear with Q then P is contained in a unique ovoid in the rosette subtended by ℓ. There is a unique subtended rosette containing this ovoid and with basepoint P.

Let P be of type (ii) and ℓ of type (a). The ovoid θ, corresponding to P meets ℓ in exactly one point X. There is a unique subtended rosette containing θ and with basepoint X and thus a unique line of type (b) containing P and X.

Let P be of type (ii) and let $\ell \cup Q$ of type (b). Let θ be the ovoid of $\mathcal{S}'$ corresponding to P and $R = \{\theta_1, \ldots, \theta_r\}$ the subtended rosette of $\mathcal{S}'$ corresponding to ℓ. Without loss of generality suppose that $P = (\theta, 0)$. There are two possibilities for ℓ, either

$\ell = \{(\theta_1, 0), (\theta_2, c(\theta_1, \theta_2)), \ldots, (\theta_r, c(\theta_1, \theta_r))\}$ or

$\ell = \{(\theta_1, 1), (\theta_2, c(\theta_1, \theta_2) + 1), \ldots, (\theta_r, c(\theta_r, \theta_1) + 1)\}$. Suppose that $\theta \in R$ and that without loss of generality $\theta = \theta_1$. Then since $(\theta, 0)$ is not incident with ℓ we have that

$\ell = \{(\theta_1, 1), (\theta_2, c(\theta_1, \theta_2) + 1), \ldots, (\theta_r, c(\theta_1, \theta_r) + 1)\}$ and $(\theta, 0)$ is collinear with none of the points on ℓ. Thus Q is the unique point on $\ell \cup Q$ that is collinear with P. Now suppose that $\theta \notin R$ and that without loss of generality

$\ell = \{(\theta_1, 0), (\theta_2, c(\theta_1, \theta_2)), \ldots, (\theta_r, c(\theta_r, \theta_1))\}$. If $Q \in \theta$ then θ meets each of the θ_i in $r+1$ points and is contained in a unique subtended rosette with Q as the basepoint, which gives a unique line incident with P and a point of $\ell \cup Q$. If $Q \notin \theta$, then there are two ovoids of R that meet θ in precisely one point. Without loss of generality let these ovoids be θ_1 and θ_2. Now $(\theta_1, 0)$ is collinear to $(\theta_2, c(\theta_1, \theta_2))$ (on ℓ) and $(\theta_1, 1)$ is collinear to $(\theta_2, c(\theta_1, \theta_2) + 1)$, while $(\theta, 0)$ is collinear to exactly one point of the form $(\theta_1, -)$ and one of the form $(\theta_2, -)$. So $(\theta, 0)$ is collinear to exactly one point on $\ell \cup Q$ if and only if either $(\theta, 0)$ is collinear to $(\theta_1, 0)$ and $(\theta_2, c(\theta_1, \theta_2) + 1)$ or $(\theta, 0)$ is collinear to $(\theta_1, 1)$ and $(\theta_2, c(\theta_1, \theta_2))$. This occurs if and only if $c(\theta,\theta_2)=c(\theta,\theta_1)+c(\theta_1,\theta_2)+1$. That is, if and only if

$$\begin{aligned}\delta c(\theta,\theta_1,\theta_2) &= c(\theta,\theta_1)+(c(\theta,\theta_1)+c(\theta_1,\theta_2)+1)+c(\theta_1,\theta_2)\\ &= 1.\end{aligned}$$

This is precisely the GQ condition 1. Thus $\mathcal{W}$ is a GQ of order (r,r^2) if and only if c satisfies the GQ condition.

Now suppose that $\mathcal{W}$ is a GQ of order (r,r^2). If X is a point of type (ii) of $\mathcal{S}$, then let θ_X be the ovoid of $\mathcal{S}'$ that is covered by X. The set of lines of $\mathcal{W}$ incident with X meets $\mathcal{S}'$ in the set of basepoints of subtended rosettes containing θ_X. So, in $\mathcal{W}$, X subtends the ovoid θ_X in $\mathcal{S}'$. It then follows that a line $\ell\cup P$ of $\mathcal{W}$ subtends the rosette that is covered by the line ℓ of $\overline{\mathcal{T}}$. Thus $\mathcal{S}'$ is doubly subtended in $\mathcal{W}$ and the subtended ovoid/rosette structure is $\mathcal{T}$. ■

3.3 Isomorphisms of SPGs

In this section we determine when two SPGs, constructed from the double subtending process, are isomorphic. We also calculate the group of such an SPG.

Theorem 3.3 *Let $\mathcal{W}$ and $\mathcal{S}$ be two* GQ*s of order (r,r^2) and let $\mathcal{W}'$ and $\mathcal{S}'$ be sub*GQ*s of $\mathcal{W}$ and $\mathcal{S}$ respectively, of order r. Let $\mathcal{W}'$ and $\mathcal{S}'$ be doubly subtended in $\mathcal{W}$ and $\mathcal{S}$ and let the* SPG*s constructed as in Theorem 2.5 be $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_{\mathcal{S}}$. The* SPG*s $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_{\mathcal{S}}$ are isomorphic if and only if there exists an isomorphism from $\mathcal{W}'$ to $\mathcal{S}'$ that induces an isomorphism from $\mathcal{T}_{\mathcal{W}}$ to $\mathcal{T}_{\mathcal{S}}$.*

Proof. First, let $c_{\mathcal{S}}$ define an algebraic 2-fold cover of $\mathcal{T}_{\mathcal{S}}$, as in Theorem 3.1, and let $i:\mathcal{T}_{\mathcal{W}}\to\mathcal{T}_{\mathcal{S}}$ be an isomorphism. If θ and θ' are two points of $\mathcal{T}_{\mathcal{W}}$, then we may easily show that the function $c_{\mathcal{W}}$ acting by $c_{\mathcal{W}}(\theta,\theta')=c_{\mathcal{S}}(i(\theta),i(\theta'))$ defines an algebraic 2-fold cover of $\mathcal{T}_{\mathcal{W}}$, that satisfies the GQ condition. Let $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}$ be the algebraic 2-fold cover of $\mathcal{T}_{\mathcal{W}}$ defined by $c_{\mathcal{W}}$, $\mathcal{T}_{\mathcal{S}}^{c_{\mathcal{S}}}$ the algebraic 2-fold cover of $\mathcal{T}_{\mathcal{S}}$ defined by $c_{\mathcal{S}}$ and $\overline{\mathcal{S}}$ the GQ of order (r,r^2) constructed from $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}$ and $\mathcal{W}'$ as in Theorem 3.2. Now, let $\bar{\imath}$ be the map from the point set of $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}$ to the point set of $\mathcal{T}_{\mathcal{S}}^{c_{\mathcal{S}}}$, which acts by $(\theta,\alpha)\mapsto(i(\theta),\alpha))$, for θ a point of $\mathcal{T}_{\mathcal{W}}$ and $\alpha\in\mathbb{Z}_2$. If the lines of $\mathcal{T}_{\mathcal{W}}$ are considered as sets of points of $\mathcal{T}_{\mathcal{W}}$, then $\bar{\imath}$ induces an isomorphism from $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}$ to $\mathcal{T}_{\mathcal{S}}^{c_{\mathcal{S}}}$, which we also denote by $\bar{\imath}$. We show that $\bar{\imath}$ may be extended to an isomorphism from $\overline{\mathcal{S}}$ to $\mathcal{S}$.

Let ℓ and m be two lines of $\overline{\mathcal{S}}$ that are tangent to $\mathcal{W}'$. A line of $\overline{\mathcal{S}}$ that is tangent to $\mathcal{W}'$ is said to be a *transversal* to ℓ and m if it is concurrent to both ℓ and m.

Let P be the unique point of $\mathcal{W}'$ that is incident with ℓ. Let Q be the unique point of $\mathcal{W}'$ that is incident with m. Now if P and Q are not collinear, then by the third GQ axiom one point of the set $\ell\setminus\{P\}$ is collinear with Q and each of the remaining $r-1$ points of $\ell\setminus\{P\}$ is collinear with a unique point of $m\setminus\{Q\}$. Thus there are $r-1$ transversals to ℓ and m in this case. If P is collinear to Q (with the line incident with P and Q necessarily a line of $\mathcal{W}'$), then each point of $\ell\setminus\{P\}$ is collinear with a unique point of $m\setminus\{Q\}$. Thus there are r transversals to ℓ and m.

Now if ℓ and m are not skew, but meet in a point of $\mathcal{W}'$, that is $P=Q$, then the third GQ axiom implies that there are no transversals to ℓ and m.

Note that the above shows that the geometry $\mathcal{T}_{\mathcal{W}}^{c\mathcal{W}}$ satisfies the axiom $(*)$ of De Clerck and Van Maldeghem [3].

is

Let ℓ' and m' be the lines of $\mathcal{T}_{\mathcal{W}}^{c\mathcal{W}}$ such that $\ell = \ell' \cup \{P\}$ and $m = m' \cup \{Q\}$. Let a line of $\mathcal{T}_{\mathcal{W}}^{c\mathcal{W}}$ be a *transversal* to ℓ' and m' if it is concurrent to both ℓ' and m'. Consider the incidence structure which has as points the sets $\mathcal{L}$ of lines of $\mathcal{T}_{\mathcal{W}}^{c\mathcal{W}}$ of size $q^2 - q$ such that any pair of lines of $\mathcal{L}$ has no transversals $\mathcal{T}_{\mathcal{W}}^{c\mathcal{W}}$ a pair of lines of $\ell, m \in \mathcal{L}$ and m, and and points $\mathcal{L}_1$ and $\mathcal{L}_2$ collinear if ℓ_1 and ℓ_2 have exactly r transversals, for all $\ell_1 \in \mathcal{L}_1$ and $\ell_2 \in \mathcal{L}_2$ (lines are the maximal sets of pairwise collinear points). The above calculations show that this incidence structure is $\mathcal{W}'$.

Thus $\bar{\imath}$ may be extended to an isomorphism from $\overline{\mathcal{S}}$ to $\mathcal{S}$. The restriction of $\bar{\imath}$ to $\mathcal{W}'$ is an isomorphism from $\mathcal{W}'$ to $\mathcal{S}'$ that induces i from $\mathcal{T}_{\mathcal{W}}$ to $\mathcal{T}_{\mathcal{S}}$.

Now suppose that there exists an isomorphism from $\mathcal{W}'$ to $\mathcal{S}'$ that takes $\mathcal{T}_{\mathcal{W}}$ to $\mathcal{T}_{\mathcal{S}}$. Since such an isomorphism maps ovoids to ovoids, rosettes to rosettes and preserves inclusion of an ovoid in a rosette, it induces an isomorphism from $\mathcal{T}_{\mathcal{W}}$ to $\mathcal{T}_{\mathcal{S}}$. ∎

As a corollary of Theorem 3.3 we state the automorphism group of an SPG arising from the double subtending process.

Corollary 3.4 *Let $\mathcal{S}$ be a* GQ *of order (r, r^2) and $\mathcal{S}'$ a sub*GQ *of order r. Let $\mathcal{S}'$ be doubly subtended in $\mathcal{S}$, with* SPG *$\mathcal{T}$ constructed as in Theorem 2.5. The automorphism group of $\mathcal{T}$ is the stabiliser of $\mathcal{T}$ in the automorphism group of $\mathcal{S}'$.*

Proof. From the proof of Theorem 3.3, if $\mathcal{T} = \mathcal{T}_{\mathcal{W}} = \mathcal{T}_{\mathcal{S}}$ and i is an automorphism of $\mathcal{T}$, then there is an automorphism of $\mathcal{S}'$ that induces i. Also, any automorphism of $\mathcal{S}'$ that fixes $\mathcal{T}$ induces an automorphism of $\mathcal{T}$. Since any point of $\mathcal{S}'$ may be expressed as the intersection of two ovoids that are points of $\mathcal{T}$, any automorphism of $\mathcal{S}'$ that induces the identity on $\mathcal{T}$ must be the identity. So, if we consider the group of $\mathcal{S}'$ that fixes $\mathcal{T}$ acting on $\mathcal{T}$ by the automorphism it induces, the action is faithful and so it is the group of $\mathcal{T}$. ∎

4 The q-clan GQs

In this section we give a summary of results on the q-clan construction of GQs of order (q^2, q). First consider the group coset construction of a GQ as introduced by Kantor [8]. Let G be a finite group of order s^2t, $s \geq 2$, $t \geq 2$ and two families of subgroups $\mathcal{F} = \{S_0, \ldots, S_t\}$, $\mathcal{F}^\star = \{S_0^\star, \ldots, S_t^\star\}$ such that $|S_i| = s$, $|S_i^\star| = st$ and $S_i \subseteq S_i^\star$. If $\mathcal{F}$ and $\mathcal{F}^\star$ satisfy:

$$\begin{array}{lll} K1 & S_iS_j \cap S_k = 1 & \text{for } k \neq i, j \\ K2 & S_i^\star \cap S_j = 1 & \text{for } i \neq j \end{array}$$

Then $\mathcal{F}$ is a 4-*gonal family* for G. The following point-line geometry $\mathcal{S}(G, \mathcal{F})$ is a GQ of order (s, t).

$$\begin{array}{rll} \text{Points}: & \text{(i)} & \text{Elements of } G \\ & \text{(ii)} & \text{Right cosets, } S_i^\star g,\ i = 0, \ldots, t,\ g \in G \\ & \text{(iii)} & (\infty) \\ \text{Lines}: & \text{(a)} & \text{Right cosets } S_i g,\ i = 0, \ldots, t,\ g \in G \\ & \text{(b)} & \text{Symbols } [S_i],\ i = 0, \ldots, t. \end{array}$$

A point g of type (i) is incident with each line $S_i g$. A point $S_i^\star$ of type (ii) is incident with $[S_i]$ and with each line $S_i h \subset S_i^\star g$. The point (∞) is incident with each line $[S_i]$ of type (b). These are all of the incidence relations. The GQ $\mathcal{S}(G, \mathcal{F})$ is an elation GQ with elation group G. If $g \in G$, then the elation corresponding to g is induced by right multiplication by g.

Now we consider q-clans and give the construction of a 4-gonal family from a q-clan, the development of which is due to Payne [12, 13] and Kantor [10]. For q a prime power a q-*clan* is a set $\mathcal{C} = \{A_t : t \in \mathrm{GF}(q)\}$ of q 2×2 matrices over $\mathrm{GF}(q)$ such that for distinct $s, t \in \mathrm{GF}(q)$, $(a, b)(A_s - A_t)(a, b)^T = 0$ has only the trivial solution $a = b = 0$. We can normalise a q-clan such that for q odd $A_t = \begin{pmatrix} x_t & y_t/2 \\ y_t/2 & z_t \end{pmatrix}$ and for q even $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$.

Now consider the particular group $G = \{(\alpha, c, \beta) : \alpha, \beta \in \mathrm{GF}(q)^2, c \in \mathrm{GF}(q)\}$ and define a binary operation on G by

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta(\alpha')^T, \beta + \beta').$$

This binary operation makes G into a group where $(\alpha, c, \beta)^{-1} = (-\alpha, \alpha\beta^T - c, -\beta)$. Let $\mathcal{C}$ be a normalised q-clan and $K_t = A_t + A_t^T$ for $t \in \mathrm{GF}(q)$. Define the following subgroups of G:

$$\begin{aligned} A(\infty) &= \{(\overline{0}, 0, \beta) : \beta \in \mathrm{GF}(q)^2\} \\ A^\star(\infty) &= \{(\overline{0}, c, \beta) : c \in \mathrm{GF}(q), \alpha \in \mathrm{GF}(q)^2\} \\ A(t) &= \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) : \alpha \in \mathrm{GF}(q)^2\} \quad \text{for } t \in \mathrm{GF}(q) \\ A^\star(t) &= \{(\alpha, c, \alpha K_t) : c \in \mathrm{GF}(q), \alpha \in \mathrm{GF}(q)^2\} \quad \text{for } t \in \mathrm{GF}(q). \end{aligned}$$

Then $\mathcal{F}(\mathcal{C}) = \{A(t) : t \in \mathrm{GF}(q) \cup \{\infty\}\}$ is a 4-gonal family for G which gives a GQ of order (q^2, q), $\mathcal{S}(G, \mathcal{F}(\mathcal{C}))$. We will denote $\mathcal{S}(G, \mathcal{F}(\mathcal{C}))$ by $\mathcal{S}(\mathcal{C})$.

The following three theorems are results on the automorphism group of a q-clan GQ which will prove useful in Section 5.

Theorem 4.1 ([14], III.(1)) *Suppose G_1, G_2 are groups and $\mathcal{F}$ a 4-gonal family of G_1. If $\Theta : G_1 \to G_2$ is a group isomorphism or a group anti-isomorphism, then $\mathcal{S}(G_1, \mathcal{F})$ and $\mathcal{S}(G_2, \Theta(\mathcal{F}))$ are isomorphic* GQ*s.*

In particular, if we have a group automorphism of G then it induces an automorphism of $\mathcal{S}(G, \mathcal{F})$.

Theorem 4.2 ([14], IV.1.) *Let $\mathcal{C}$ be a q-clan, q odd, with $A_0 = K_0 = 0$ and A symmetric for all $A \in \mathcal{C}$. Let $\mathcal{S}(\mathcal{C}) = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be the associated* GQ *of order (q^2, q). Let Θ be a collineation of $\mathcal{S}(\mathcal{C})$ which fixes (∞), $[A(\infty)]$, and $(0, 0, 0)$. Then there is*

a permutation $t \mapsto \overline{t}$ of the elements of $\mathrm{GF}(q)$,
a nonzero element λ of $\mathrm{GF}(q)$,
an automorphism σ of $\mathrm{GF}(q)$ *and ,*
a 2×2 matrix $D \in GL(2, q)$

for which the following holds

$$A_{\overline{t}} = \lambda D^T A_t^\sigma D + A_{\overline{0}} \text{ for all } t \in \mathrm{GF}(q).$$

Moreover, Θ induces an automorphism of G, given by

$$\Theta : \ (\alpha, c, \beta) \mapsto (\alpha^\sigma \lambda^{-1} D^{-T}, \lambda^{-1} c^\sigma + \lambda^{-2} \alpha^\sigma D^{-T} A_{\overline{0}} D^{-1} (\alpha^\sigma)^T, \beta^\sigma D + \alpha^\sigma \lambda^{-1} D^{-T} K_{\overline{0}}).$$

Conversely, given D, σ, λ and $t \mapsto \overline{t}$ as just described, the $\Theta = \Theta(\pi, \lambda, \sigma, D)$ as above is a collineation of $\mathcal{S}(\mathcal{C})$.

The automorphisms in Theorem 4.2 all fix the line $[A(\infty)]$, in the following theorem we consider automorphisms not fixing $[A(\infty)]$.

Theorem 4.3 ([14], III.5.) *If the set $\mathcal{C} = \{A_t : \ t \in \mathrm{GF}(q)\}$ is a q-clan with $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $\mathcal{C}' = \{A_t^{-1} : \ 0 \neq t \in \mathrm{GF}(q)\} \cup \{A_0\}$ is a q-clan with $\mathcal{S}(\mathcal{C}) \cong \mathcal{S}(\mathcal{C}')$. In fact, the switch from $\mathcal{C}$ to $\mathcal{C}'$ just amounts to interchanging the roles of $A(\infty)$ and $A(0)$ in the coordinatisation of $\mathcal{S}$.*

Note that for q odd, the isomorphism in Theorem 4.3 is given by the group automorphism $\Theta : \ G \to G : \ (\alpha, c, \beta) \mapsto (\beta, 2c, 2\alpha)$. Thus the subgroup $\mathcal{Q}_1 = (\mathrm{GF}(q) \times 0) \times \mathrm{GF}(q) \times (\mathrm{GF}(q) \times 0)$ is fixed under the isomorphism.

5 A GQ of Kantor, an ovoid of Kantor and a new SPG

In [10] Kantor constructed the q-clan $\mathcal{C}_\sigma$, which has associated GQ $\mathcal{S}(\mathcal{C}_\sigma)$ of order (q^2, q). In this section we investigate the connection between the GQ $\mathcal{S}(\mathcal{C}_\sigma)$ and the ovoid θ_σ constructed by Kantor in [9]. The dual GQ $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$ possesses a subquadrangle isomorphic to $Q(4, q)$ and we show that each subtended ovoid of the $Q(4, q)$ subquadrangle is isomorphic to θ_σ. We also show that the $Q(4, q)$ subquadrangle is doubly subtended in $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$. Since the subtended ovoids are isomorphic to θ_σ, Theorem 3.3 shows that the SPG constructed as in Theorem 2.5 is distinct from the Metz/Hirschfeld and Thas SPG constructed in the classical case. Hence we have a new SPG. Note that the work in this section relies heavily on the work of Payne and Rogers in [14].

Consider the q-clan

$$\mathcal{C}_\sigma = \left\{ A_t = \begin{pmatrix} t & 0 \\ 0 & -mt^\sigma \end{pmatrix} : t \in \mathrm{GF}(q) \right\}$$

where q is an odd prime power, m a fixed non-square of $\mathrm{GF}(q)$ and $\sigma \in \mathrm{Aut}(GF(q))$, as constructed in [10]. Recall from Section 4 that if G is the group with elements $\{(\alpha, c, \beta) : \alpha, \beta \in \mathrm{GF}(q)^2, c \in \mathrm{GF}(q)\}$ and operation

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta(\alpha')^T, \beta + \beta'),$$

then the family of subgroups of G (of order q^2)

$$\begin{array}{rcll} A(\infty) & = & \{(\overline{0}, 0, \beta) : \beta \in \mathrm{GF}(q)^2\} & \\ A(t) & = & \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) : \alpha \in \mathrm{GF}(q)^2\} & \text{for } t \in \mathrm{GF}(q), \text{ where } K_t = A_t + A_t^T \\ & = & \{(\alpha, \alpha A_t \alpha^T, 2\alpha A_t) : \alpha \in \mathrm{GF}(q)^2\} & \text{for } t \in \mathrm{GF}(q) \end{array}$$

is a 4-gonal family for G which we denote by $\mathcal{F}(\mathcal{C}_\sigma)$. The family of subgroups

$$\begin{array}{rcll} A^\star(\infty) & = & \{(\overline{0}, c, \beta) : c \in \mathrm{GF}(q), \alpha \in \mathrm{GF}(q)^2\} & \\ A^\star(t) & = & \{(\alpha, c, 2\alpha A_t) : c \in \mathrm{GF}(q), \alpha \in \mathrm{GF}(q)^2\} & \text{for } t \in \mathrm{GF}(q) \end{array}$$

is denoted by $\mathcal{F}^\star(\mathcal{C}_\sigma)$. The GQ of order (q^2, q) constructed from the 4-gonal family $\mathcal{F}(\mathcal{C}_\sigma)$ is denoted by $\mathcal{S}(\mathcal{C}_\sigma)$. Note that Kantor observes that $\mathcal{S}(\mathcal{C}_\sigma) \cong \mathcal{S}(\mathcal{C}_{\sigma^{-1}})$.

In [10] Kantor observes that if $\mathcal{Q}_1 = (\mathrm{GF}(q) \times 0) \times \mathrm{GF}(q) \times (\mathrm{GF}(q) \times 0)$ and $\mathcal{Q}_2 = (0 \times \mathrm{GF}(q)) \times \mathrm{GF}(q) \times (0 \times \mathrm{GF}(q))$ are subgroups of G, then for $i = 1$ or 2, $\mathcal{F}_i = \{A_i(t) = A(t) \cap \mathcal{Q}_i : t \in \mathrm{GF}(q) \cup \{\infty\}\}$ is a 4-gonal family for $\mathcal{Q}_i$ giving rise to an $\mathrm{Sp}(4, q)$ subquadrangle, that is, a subquadrangle of $\mathcal{S}(\mathcal{C}_\sigma)$ isomorphic to $W(q)$. We saw in Lemma 1.1 that if $\mathcal{W}$ is a GQ of order (r, r^2) with $\mathcal{W}'$ a subquadrangle of order r, then each point of $\mathcal{W}$, external to $\mathcal{W}'$, subtends an ovoid of $\mathcal{W}'$. Dually, $\mathcal{W}^\wedge$ is a GQ of order (r^2, r) with $(\mathcal{W}')^\wedge$ a subquadrangle of order r and each line external to $(\mathcal{W}')^\wedge$ subtends a spread of $(\mathcal{W}')^\wedge$. If $\mathcal{W}'$ is doubly subtended in $\mathcal{W}$, then we say that $(\mathcal{W}')^\wedge$ is doubly subtended in $\mathcal{W}^\wedge$. So, we are interested in the subtended spreads of the GQ determined by $\mathcal{F}_1$ and $\mathcal{F}_2$. It can be shown using Theorem 4.2 ([14, IV.1.]) that there exists an isomorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ that maps the subquadrangle determined by $\mathcal{F}_1$ to the subquadrangle determined by $\mathcal{F}_2$. Since this is the case, we will consider only the subquadrangle determined by $\mathcal{F}_1$, which will be referred to as $W(q)$ from here on.

Firstly, we show that the subtended spreads of $W(q)$ are pairwise isomorphic. We do this by showing that the group of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$ is transitive on the lines of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$. The lines of $\mathcal{S}(\mathcal{C}_\sigma)$ that are external to $W(q)$ are $A(\infty)g$, for $g \in G$, such that $A(\infty)g \cap \mathcal{Q}_1$ is empty, and $A(t)g$, for $t \in \mathrm{GF}(q)$ and $g \in G$, such that $A(t)g \cap \mathcal{Q}_1$ is empty. The following lemma deals with the external lines of the form $A(\infty)g$.

Lemma 5.1 *The stabiliser of $W(q)$ in $\mathcal{S}(\mathcal{C}_\sigma)$ is transitive on lines external to $W(q)$ that have the form $A(\infty)g$.*

Proof. The coset $A(\infty)g$ is the set $\{((g_1, g_2), g_3 + u_1 g_1 + u_2 g_2, (g_4 + u_1, g_5 + u_2)) : u_1, u_2 \in \mathrm{GF}(q)\}$ and so the line $A(\infty)g$ is external to $W(q)$ if and only if $g_2 \neq 0$. We show that g can be mapped to an element of G of the form $g' = ((g'_1, g'_2), -, (-, -))$ by an automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$.

Let $k = g'_2/g_2$ and $g_z = ((z, 0), 0, (0, 0))$ where $z = g'_1/k - g_1$. The elation of $\mathcal{S}(\mathcal{C}_\sigma)$ corresponding to g_z acts on g by right multiplication

$$g_z : \ g \mapsto ((\frac{g'_1}{k}, \frac{g'_2}{k}), -, (-, -)).$$

By Theorem 4.1 an automorphism of G that fixes $\mathcal{F}_\sigma$ induces an automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$. Let σ_k be the automorphism of G acting by $(\alpha, c, \beta) \mapsto (k\alpha, k^2 c, k\beta)$. The automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ induced by σ_k acts on $g \cdot g_z$ by

$$\sigma_k : \ g \cdot g_z \mapsto ((g'_1, g'_2), -, (-, -)).$$

Clearly both the elation of $\mathcal{S}(\mathcal{C}_\sigma)$ induced by g_z and the automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ induced by σ_k fix $W(q)$.

So now we may assume that g has the form $g = ((g'_1, g'_2), g_3, (g_4, g_5))$. Let $x = g'_3 - g_3 + (g_4 - g'_4) \cdot g'_1 + (g_5 - g'_5) \cdot g'_2$ and $g_x = ((0, 0), x, (0, 0))$. The elation of $\mathcal{S}(\mathcal{C}_\sigma)$ induced by g_x maps $A(\infty)g$ to $A(\infty)g'$ and fixes $W(q)$. ■

Lemma 5.2 *The stabiliser of* $W(q)$ *in* $\mathcal{S}(\mathcal{C}_\sigma)$ *is transitive on lines external to* $W(q)$.

Proof. First we show that the stabiliser of $W(q)$ in $\mathcal{S}(\mathcal{C}_\sigma)$ is transitive on the set $\{[A(t)] : t \in \mathrm{GF}(q)\}$. Let $s \in \mathrm{GF}(q)$ and let π be a permutation of $\mathrm{GF}(q)$, $\pi : \ t \mapsto \overline{t} = t + s$. Then

$$\begin{aligned} A_{\overline{t}} = A_{t+s} &= \begin{pmatrix} t+s & 0 \\ 0 & -m(t+s)^\sigma \end{pmatrix} \\ &= \begin{pmatrix} t & 0 \\ 0 & -mt^\sigma \end{pmatrix} + \begin{pmatrix} s & 0 \\ 0 & -ms^\sigma \end{pmatrix} \\ &= A_t + A_s = A_t + A_{\overline{0}} \end{aligned}$$

So by Theorem 4.2 there is an automorphism Θ of $\mathcal{S}(\mathcal{C}_\sigma)$ with $\lambda = 1$, σ the identity, $D = I$ (see Theorem 4.2 for notation) and associated permutation π. By Theorem 4.2 we have that

$$\begin{aligned} \Theta \ &: \ [A(t)] \mapsto [A(t+s)] \\ &: \ (\alpha, c, \beta) \mapsto (\alpha, c + \alpha A_s \alpha^T, \beta + 2\alpha A_s) \end{aligned}$$

Now Θ fixes $W(q)$ and if we let s vary over $\mathrm{GF}(q)$, then we have the desired transitivity on $\{[A(t)] : t \in \mathrm{GF}(q)\}$.

From this it can be shown that any line external to $W(q)$, of the form $A(t)g$ may be mapped to an external line of the form $A(0)g'$, for some g'. We now wish to find an automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$ and swaps $[A(\infty)]$ and $[A(0)]$ (and hence maps a line of the form $A(0)g$ to one of the form $A(\infty)g'$ and vice versa). Consider the q-clan $\mathcal{C}'_\sigma = \{A_t^{-1} : t \in \mathrm{GF}(q) \setminus \{0\}\} \cup \{A_0\}$. Now if $S = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ then the automorphism of G given by $(\alpha, c, \beta) \mapsto (\alpha S^{-1}, c, \beta S)$ induces an isomorphism from $\mathcal{S}(\mathcal{C}'_\sigma)$ to $\mathcal{S}(\mathcal{C}_\sigma)$ which maps $([A(t)], [A(0)], [A(\infty)])$ to $([A(t^{-1})], [A(0)], [A(\infty)])$ for $t \in \mathrm{GF}(q) \setminus \{0\}$. Composing this isomorphism with that in Theorem 4.3 yields an automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ mapping $[A(\infty)] \leftrightarrow [A(0)]$ and fixing $W(q)$.

Thus any line of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$, that has the form $A(t)g$ may be mapped to a line of the form $A(\infty)g'$ by an automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ fixing $W(q)$. Lemma 5.1 then implies that the stabiliser of $W(q)$ in $\mathcal{S}(\mathcal{C}_\sigma)$ is transitive on the lines of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$. ■

We now show that each spread of $W(q)$ subtended by a line of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$, is dual to an ovoid of $Q(4,q)$ that is isomorphic to the ovoid θ_σ constructed by Kantor in [9]. That is, under the duality from $W(q)$ to $Q(4,q)$ any subtended spread of $W(q)$ is mapped to an ovoid of $Q(4,q)$ isomorphic to θ_σ. Given Lemma 5.2, if one subtended spread of $W(q)$ is dual to an ovoid isomorphic to θ_σ, then all subtended spreads must be.

Let $Q(4,q)$ be defined by the equation $x_0x_4 + x_1x_3 + x_2^2 = 0$ then θ_σ is given by

$$\theta_\sigma = \{(0,0,0,0,1)\} \cup \left\{(1, y, z, -my^\sigma, -z^2 + my^{\sigma+1}) \text{ for } y, z \in \mathrm{GF}(q)\right\}.$$

Here m is the same fixed non-square and σ the same automorphism of $\mathrm{GF}(q)$ used in the definition of $\mathcal{C}_\sigma$. The ovoid θ_σ may also be written as the intersection of $Q(4,q)$ with the variety defined by the equation $mx_1^\sigma + x_0^{\sigma-1}x_3 = 0$.

The GQ $W(q)$ may be represented as the set of absolute points and lines of a symplectic polarity in $\mathrm{PG}(3,q)$. The canonical form of $W(q)$ in $\mathrm{PG}(3,q)$ is given by the polarity which has associated bilinear form $x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0$. In the following lemma we give an explicit isomorphism between the $W(q)$ as a subquadrangle of $\mathcal{S}(\mathcal{C}_\sigma)$ and the canonical representation in $\mathrm{PG}(3,q)$.

Lemma 5.3 *Let $\mathcal{F}_1 = \{A_1(t) = A(t) \cap \mathcal{Q}_1 : t \in \mathrm{GF}(q) \cup \{\infty\}\}$ be the 4-gonal family for $W(q)$ as above. Let $W(q)'$ be the GQ arising in $\mathrm{PG}(3,q)$ as the set of absolute points and absolute lines of the symplectic polarity with associated bilinear form $x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0$. Then the map ρ is an isomorphism from $W(q)$ to $W(q)'$ where ρ acts as follows:*

$$\begin{array}{rcl}
(\infty) & \mapsto & (0,1,0,0) \\
[A_1(t)] & \mapsto & \langle (0,0,2t,1), (0,1,0,0) \rangle \\
[A_1(\infty)] & \mapsto & \langle (0,1,0,0), (0,0,1,0) \rangle \\
((g_1,0), g_2, (g_3,0)) & \mapsto & (1, 2g_2 - g_1g_3, g_3, g_1) \\
A_1(t)((g_1,0), g_2, (g_3,0)) & \mapsto & \langle (1, 2g_2 - g_1g_3, g_3, g_1), (0, 2g_1t - g_3, 2t, 1) \rangle \\
A_1(\infty)((g_1,0), g_2, (g_3,0)) & \mapsto & \langle (1, 2g_2 - g_1g_3, g_3, g_1), (0, g_1, 1, 0) \rangle \\
A_1^\star(t)((g_1,0), g_2, (g_3,0)) & \mapsto & (0, 2tg_1 - g_3, 2t, 1) \\
A_1^\star(\infty)((g_1,0), g_2, (g_3,0)) & \mapsto & (0, 2g_1, 1, 0)
\end{array}$$

where $t, g_1, g_2, g_3 \in \mathrm{GF}(q)$.

Theorem 5.4 *Let $W(q)$ be the subquadrangle of $\mathcal{S}(\mathcal{C}_\sigma)$ given by $\mathcal{F}_1$. Then, each spread of $W(q)$ subtended by a line of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$ is dual to an ovoid of $Q(4,q)$ that is isomorphic to θ_σ.*

Proof. Given Lemma 5.2, we may take our favourite fixed line of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$ and find its subtended spread $\mathtt{S}$. Let $\ell = A(\infty)g$ be the external line where $g = ((0,1), 0, (0,0))$. Then the points on $A(\infty)g$ are

$$\{((0,1), u_2, (u_1, u_2)) \;:\; u_1, u_2 \in \mathrm{GF}(q)\} \cup \{A^\star(\infty)g\}$$

where $A^\star(\infty)g = \{((0,1), u_3, (u_4, u_5)) : u_3, u_4, u_5 \in \mathrm{GF}(q)\}$.

For each point on $A(\infty)g$ there is a unique line of $W(q)$ incident with it, which is a line of the subtended spread. For the point $A^\star(\infty)$ the corresponding line is $[A(\infty)]$. Now let $h(u_1, u_2) = ((0,1), u_2, (u_1, u_2))$ for $u_1, u_2 \in \mathrm{GF}(q)$, then for each pair (u_1, u_2) we need to find $t \in \mathrm{GF}(q)$ such that $A(t)h(u_1, u_2)$ is a line of $W(q)$. Now

$$\begin{aligned} A(t)h(u_1,u_2) \;&=\; \Big\{\big((v_1, v_2+1), v_1^2 t - m v_2^2 t^\sigma + u_2 - 2 m v_2 t^\sigma, \\ &\quad (2v_1 t + u_1, -2 m v_2 t^\sigma + u_2)\big) \;:\; v_1, v_2 \in \mathrm{GF}(q)\Big\} \end{aligned}$$

and is a line of $W(q)$ if it contains any point of $W(q)$ (in which case it contains $q+1$ points of $W(q)$). This occurs if, as a coset of $A(t)$ in G, $A(t)h(u_1, u_2)$ contains an element of $\mathcal{Q}_1$ (in which case it contains q elements of $\mathcal{Q}_1$). So $v_2 = -1$ and $-2mv_2 t^\sigma = -u_2$ which implies $t = -\left(\dfrac{u_2}{2m}\right)^{\sigma^{-1}}$.

Thus the spread of $W(q)$ subtended by $\mathcal{S}(\mathcal{C}_\sigma)$ has the form:

$$\mathtt{S} = \begin{cases} [A(\infty)] \\[2ex] A\left(-\left(\dfrac{u_2}{2m}\right)^{\sigma^{-1}}\right) h(u_1,u_2), \quad u_1, u_2 \in \mathrm{GF}(q) \end{cases}$$

To express the spread in the group coset representation of $W(q)$ without reference to $\mathcal{S}(\mathcal{C}_\sigma)$, we need a representative of the coset $A\left(-\left(\dfrac{u_2}{2m}\right)^{\sigma^{-1}}\right) h(u_1,u_2)$ that is in $\mathcal{Q}_1$, say $((0,0), u_2 + 2mt^\sigma, (u_1, 0)) = ((0,0), u_2/2, (u_1,0))$. Thus the spread is

$$\mathtt{S} = \begin{cases} [A(\infty)] \\[2ex] A\left(-\left(\dfrac{u_2}{2m}\right)^{\sigma^{-1}}\right) ((0,0), \dfrac{u_2}{2}, (u_1,0)), \quad u_1, u_2 \in \mathrm{GF}(q). \end{cases}$$

By using the isomorphism in Lemma 5.3, $\mathtt{S}$ has the following form in $W(q)'$:

$$\mathtt{S} = \begin{cases} \{x_0 = 0; x_3 = 0\} = \langle (0,1,0,0), (0,0,1,0)\rangle \\ \langle (1, u_2, u_1, 0), (0, -u_1, 2t, 1)\rangle, \; t = -\left(\dfrac{u_2}{2m}\right)^{\sigma^{-1}} \quad \text{for } u_1, u_2 \in \mathrm{GF}(q). \end{cases}$$

We use Plücker coordinates and the Klein correspondence (see [6, Chapter 15]) to give a duality from $W(q)'$ to $Q(4,q)$. Thus, in $Q(4,q)$ the spread $\mathcal{S}$ becomes an ovoid θ, say, which has the form:

$$\theta = \begin{cases} (0,0,0,0,1) \\ (1, 2t, -u_1, u_2, -2u_2 t - u_1^2), \quad t = -\left(\dfrac{u_2}{2m}\right)^{\sigma^{-1}} \quad \text{for } u_1, u_2 \in \mathrm{GF}(q). \end{cases}$$

Now since

$$\begin{aligned} m(2t)^\sigma + 1^{\sigma-1}u_2 &= m\left(2\left(\frac{-u_2}{2m}\right)^{\sigma^{-1}}\right)^\sigma + u_2 \\ &= m\left(\frac{-u_2}{m}\right) + u_2 \\ &= 0, \end{aligned}$$

it follows that every point of θ satisfies the equation $mx_1^\sigma + x_0^{\sigma-1}x_3 = 0$ and so θ is the ovoid θ_σ. ■

Theorem 5.5 *There is an involution of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$.*

Proof. Consider the automorphism $\Theta_D = \Theta(\pi, \lambda, \rho, D)$ of $\mathcal{S}(\mathcal{C}_\sigma)$, using the notation of Theorem 4.2, where π is the identity permutation on GF(q), $\lambda = 1$, ρ the identity automorphism of GF(q) and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL(2, q)$. Then

$$\Theta_D : (\alpha, c, \beta) \mapsto ((\alpha_1, -\alpha_2), c, (\beta_1, -\beta_2)) \text{ where } \alpha = (\alpha_1, \alpha_2) \text{ and } \beta = (\beta_1, \beta_2).$$

Thus Θ_D is an involution of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$. ■

Corollary 5.6 *The* SPG *constructed by the* GQ $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$ *and the sub*GQ *isomorphic to $Q(4, q)$, as in Theorem 2.5, is not isomorphic to the known* SPG *with parameters $s = q - 1$, $t = q^2$, $\alpha = 2$ and $\mu = 2q(q - 1)$.*

Proof. The GQ $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$ of order (q, q^2) has a subquadrangle of order q isomorphic to $Q(4, q)$ such that there is a collineation of $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$ that fixes this subquadrangle pointwise. Thus, by Corollary 2.6 we have an SPG and since each subtended ovoid of the subquadrangle is non-classical, by Theorem 3.3 it is not isomorphic to the known SPG of that order. ■

The author gratefully acknowledges the assistance and guidance of Dr. Christine O'Keefe and thanks her for detailed proof reading of the paper. The author also acknowledges the support of an A.P.R.A. scholarship in this research.

References

[1] R. C. Bose and S. S. Shrikhande. Geometric and pseudo-geometric graphs $(q^2 + 1, q + 1, 1)$. *J. Geometry*, 2/1:75–94, 1973.

[2] P. J. Cameron. Covers of graphs and EGQs. *Discrete Math.*, 97:83–92, 1991.

[3] F. De Clerck and H. Van Maldeghem. On linear representations of (α, β)-geometries. *European J. Combin.*, 15:3–11, 1994.

[4] F. De Clerck and H. Van Maldeghem. Some classes of rank 2 geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, chapter 10, pages 433–475. Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 1995.

[5] I. Debroey and J. A. Thas. On semipartial geometries. *J. Combin. Theory Ser. A*, 25:242–250, 1978.

[6] J. W. P. Hirschfeld. *Finite Projective Spaces of Three Dimensions.* Clarendon Press, Oxford, 1985.

[7] J. W. P. Hirschfeld and J. A. Thas. Sets of type $(1, n, q+1)$ in $\mathrm{PG}(d, q)$. *Proc. London Math. Soc. (3)*, 41:254–278, 1980.

[8] W. M. Kantor. Generalized quadrangles associated with $G_2(q)$. *J. Combin. Theory Ser. A*, 29:212–219, 1980.

[9] W. M. Kantor. Ovoids and translation planes. *Canad. J. Math.*, 34:1195–1203, 1980.

[10] W. M. Kantor. Some generalized quadrangles with parameters q^2, q. *Math. Z.*, 192:45–50, 1986.

[11] S. E. Payne. A restriction on the parameters of a subquadrangle. *Bull. Amer. Math. Soc.*, 79:747–748, 1973.

[12] S. E. Payne. Generalized quadrangles as group coset geometries. *Congr. Numer.*, 29:115–128, 1980.

[13] S. E. Payne. A garden of generalized quadrangles. *Algebras Groups Geom.*, 2(3):323–354, 1985.

[14] S. E. Payne and L. A. Rogers. Local group actions on generalized quadrangles. *Simon Stevin*, 64:249–284, 1990.

[15] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles.* Pitman, London, 1984.

[16] J. A. Thas. A remark concerning the restriction on the parameters of a 4-gonal subconfiguration. *Simon Stevin*, 48:65–68, 1974–75.

[17] J. A. Thas. 4-gonal configurations with parameters $r = q^2 + 1$ and $k = q + 1$, part II. *Geom. Dedicata*, 8, no. 4:51–59, 1975.

Matthew R. Brown
Department of Pure Mathematics
University of Adelaide
SA 5005, Australia
email: mbrown@maths.adelaide.edu.au