On strong basis

G.M. Deheri

Efforts have been made to characterize a strong basis. It has been established that each Schauder basis in a DF-nuclear space is a strong basis, which in turn, is applied to prove that a Schauder basis $\{x_i, f_i\}$ in a Frechet space E is strong iff $\{f_i, Jx_i\}$ is a strong basis for E^*_β. As an application of strong basis it is observed that a Schauder basis $\{x_i, f_i\}$ in a barrelled space E is a fully-$\lambda(P)$-basis iff $\{f_i, Jx_i\}$ is a fully-$\lambda(P)$-basis for E^*_β, $\lambda(P)$ being a nuclear G_∞-space.

In order that a Frechet space with a basis be nuclear each basis in it should be an absolute basis. On the other hand Mertins [10], shows that presence of a single 'strong' basis ensures the nuclearity of a Frechet space. This establishes that the impact of a strong basis is rather stronger as compared to that of absolute bases. Characterization of absolute bases and the related application aspects has been investigated by Pietsch. DeGrande-DeKimpe studied strong bases and their applications in [7]. Some of the results incorporated therein regarding the impact of strong basis, suggested us to carry out the study further.

For an appropriate understanding of the material incorporated in this short paper, one is assumed to have familiarity with (i) the rudiments of the theory of locally convex spaces (cf. [2], [8]), (ii) a general study of sequence spaces (cf. [4], [8]), (iii) nuclearity and its ramifications (cf. [11], [12]) and (iv) Schauder bases and their types (cf. [5], [6]). However, for the notion of strong basis we request the reader to have a glance at [7]. Lastly, the generalized bases, namely, fully-$\lambda(P)$-bases and the related aspects can be had from [5] and [6].

For an l.c. TVS E, $\mathbb{D}_E$ denotes the fundamental system of seminorms determining the topology of E while $\mathbb{B}_E$ is a fundamental system of closed, absolutely convex and bounded subsets of E.

Received by the editors December 1996 – In revised form March 1997.
Communicated by J. Schmets.
1991 *Mathematics Subject Classification* : 46A12, 46A35, 46A45.

Let E be an l.c. TVS with a Schauder basis $\{x_i, f_i\}$ and λ be a perfect sequence space. Then

(1) $\{x_i, f_i\}$ is called a *semi-λ-basis* if for all $x \in E$ and $p \in \mathbb{D}_E$, $\{f_i(x)p(x_i)\} \in \lambda$.

(2) $\{x_i, f_i\}$ is called a *fully-λ-basis* if for each $p \in \mathbb{D}_E$ the mapping $\psi_p : E \to \lambda : x \to \{f_i(x)p(x_i)\}$ is continuous.

For $\lambda = \ell^1$, we get *semi-absolute basis* from (1), while (2) gives *absolute basis.*

Finally, we call $\{x_i, f_i\}$ a *strong basis* if for each $B \in \mathbb{B}_E$ and $p \in \mathbb{D}_E$, $\{p_B(f_i)p(x_i)\} \in \ell^1$.

Each strong basis in an l.c. TVS E is semi-absolute. Further, if E is a Mackey space with a weakly sequentially complete dual then each strong basis is an absolute basis.

Theorem 3.1 [3] indicates that an l.c. TVS with an equicontinuous basis is nuclear iff for each $p \in \mathbb{D}_E$ there corresponds a $q \in \mathbb{D}_E$ with $\{p(x_i)/q(x_i)\} \in \ell^1$. Using this result one can conclude immediately that an equicontinuous basis in a nuclear space is strong iff for each $B \in \mathbb{B}_E$ and $p \in \mathbb{D}_E$, $\{p_B(f_i)p(x_i)\} \in \ell^\infty$.

Note: A Banach space with a strong basis is finite dimensional. Indeed, a normed barrelled space with a strong basis is finite dimensional by the Corollary to Proposition 4, p. 647 [7]. This, serves to exhibit the significant difference between the two notions, namely; “absolute basis” and “strong basis”.

The discussion of this article is initiated by the characterization of strong basis, contained in

Proposition 1. *Let E be an l.c. TVS with a Schauder basis $\{x_i, f_i\}$. Suppose E^* is weakly sequentially complete. Then $\{x_i, f_i\}$ is a strong basis iff $\{x_i, f_i\}$ is a semi-absolute basis for E and $\{f_i, Jx_i\}$ is a semi-absolute basis for E^*_β.*

Proof. Suppose $\{x_i, f_i\}$ is strong. Then following the procedure laid down in Proposition 4, p. 647 [7] we find that $\{f_i, Jx_i\}$ is a semi-absolute basis for E^*_β.

Conversely, suppose $\{x_i, f_i\}$ is a semi-absolute basis for E while $\{f_i, Jx_i\}$ is a semi-absolute basis for E^*_β. Since $\{f_i, Jx_i\}$ is a semi-absolute basis, the sequence $\{P_B(f_i)f(x_i)\}$ is in ℓ^1 for each $f \in E^*$ and for every $B \in \mathbb{B}_E$. Now, take any $p \in \mathbb{D}_E$ and $B \in \mathbb{B}_E$. Then for $a = (a_i) = (1, 1, ...) \in \ell^\infty$ proceeding as in the Proof of Proposition 1 [6], we get $f \in E^*$ such that $p(x_i)a_i = f(x_i)$, for all i. Consequently, $\{P_B(f_i)p(x_i)\} \in \ell^1$. This completes the proof. ■

This immediately leads to

Corollary 2. *Let $\{x_i, f_i\}$ be a Schauder basis for a Mackey space E with a weakly sequentially complete dual. Suppose that $\{f_i, Jx_i\}$ is a semi-absolute basis for E^*_β. Then $\{x_i, f_i\}$ is semi-absolute iff weakly summable sequences in E are absolutely summable.*

Proof. It is just the application of Proposition 5 p. 647, [7] in view of Proposition 1. ■

Remarks. (i) Let E be a nuclear or a dual nuclear space having a Schauder basis $\{x_i, f_i\}$ such that $\{f_i, Jx_i\}$ is a semi-absolute basis for E^*_β. Then $\{x_i, f_i\}$ is a strong basis.

(ii) Suppose E is a Mackey space with a strong basis $\{x_i, f_i\}$ such that for some $B \in \mathbb{B}_E$, $p_B(f_i) \geq \varepsilon > 0$, for some ε. Further, if E^* is weakly sequentially complete and given $p \in \mathbb{D}_E$ there exists a $g \in \mathbb{D}_E$ with $p(x_i) \leq g^2(x_i)$, for all i, then E is nuclear. (Here the only thing to be observed is that $\{p(x_i)\} \in \ell^1$ for each $p \in \mathbb{D}_E$ and then apply Theorem 3.1 [3] as the basis is equicontinuous).

(iii) Since a barrelled space with an absolute basis is complete and quasi-complete nuclear spaces are semi-reflexive, by a result of Pietsch [11] one can infer that, each equicontinuous basis in a barrelled nuclear space is strong, in view of Proposition 1.

For a Frechet space E with a basis $\{x_i, f_i\}$ Mertins [10] proves that, E is nuclear iff $\{x_i, f_i\}$ is a strong basis. Turning to DF-spaces we have

Proposition 3. *Let E be DF-nuclear space with a Schauder basis $\{x_i, f_i\}$. If E is nuclear, then $\{x_i, f_i\}$ is a strong basis. Further the converse holds if E is barrelled.*

Proof. Since E is DF-nuclear, E^*_β is a Frechet nuclear space. Also DF-nuclear spaces are reflexive (cf. [11]) and hence $\{x_i, f_i\}$ is shrinking (cf. [1]). Consequently, $\{f_i, Jx_i\}$ is a semi-absolute (indeed, an absolute) basis for E^*_β. Since weakly summable sequences, are absolutely summable in a DF-nuclear space, (cf. [11]), Corollary 2 indicates that $\{x_i, f_i\}$ is a semi-absolute basis. Hence Proposition 1 concludes that $\{x_i, f_i\}$ is strong.

Conversely, if $\{x_i, f_i\}$ is a strong basis for E, then E becomes a complete space as a barrelled space with an absolute basis is always complete. Then invoking Theorem 10.1.4 [11], we can identify E topologically with the Köthe space $\lambda(P)$ where

$$P = \{p_B(f_i) : B \in \mathbb{B}_E\}.$$

Now the strong character of $\{e_i, e_i\}$ implies that bounded sets are simple in $\lambda(P)$, by Proposition 2, p. 650 [7]. Hence $\lambda(P)$ becomes a nuclear space by a result of Köthe [9]. ∎

This above result paves the way for

Corollary 4. *Let E be Frechet space with a Schauder basis $\{x_i, f_i\}$. Then $\{x_i, f_i\}$ is a strong basis iff $\{f_i, Jx_i\}$ is a strong basis for E^*_β.*

Proof. The proof follows from Proposition 3 and Mertins result in view of the fact that E is Frechet nuclear iff E^*_β is DF-nuclear. ∎

Note: A sequentially complete DF-space with a strong basis is strongly nuclear. This results from Proposition 3 as separable DF-spaces are infrabarrelled and nuclear DF-spaces are strongly nuclear (cf. [11]).

For the final result of this article we make use of

Lemma 5. *An l.c. TVS E with a fully-$\lambda(P)$-basis $\{x_i, f_i\}$ is always nuclear, for a nuclear G_∞-space $\lambda(P)$.*

Proof. Since $\lambda(P)$ is a nuclear G_∞-space, invoking Proposition 3.6.12 [12], we find a $b \in P$ with $\{1/b_i\} \in \ell^1$. Now, for any $p \in \mathbb{D}_E$ there exists $q \in \mathbb{D}_E$ such that

$$(*) \qquad \sum |f_i(x)| p(x_i) b_i \leq q(x).$$

In particular, we have $p(x_i)b_i \leq q(x_i)$ for every i, thereby giving $\{p(x_i)/q(x_i)\} \in \ell^1$. Also, $(*)$ yields that $\{x_i, f_i\}$ is an equicontinuous basis. Then nuclearity of E follows from Theorem 3.1 [3]. ∎

Lastly, we have a result in which the application of strong bases is prominently displayed;

Theorem 6. *Let $\{x_i, f_i\}$ be a Schauder basis for a barrelled space E and $\lambda(P)$ be a nuclear G_∞-space. Then $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis for E iff $\{f_i, Jx_i\}$ is a fully-$\lambda(P)$-basis for E^*_β.*

Proof. Suppose $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis for E. Then, if $B \in \mathbb{B}_E$, the set $\psi_p(B)$ is bounded in $\lambda(P)$ for each $p \in \mathbb{D}_E$ where $\psi_p : E \to \lambda(P)$ defined by $\psi_p(x) = \{f_i(x)p(x_i)\}$ is continuous. But in a nuclear Köthe space bounded sets are simple (cf. [9]) and hence we have

$$\{P_B(f_i)p(x_i)\} = \left\{ \sup_{x \in B} |f_i(x)| p(x_i) \right\} \in \lambda(P) \subset \ell^1.$$

So $\{x_i, f_i\}$ is a strong basis which gives the Montelness of E by Corollary to Proposition 1 in [7]. Now apply Corollary 1, p. 517 [6] to conclude that $\{f_i, Jx_i\}$ is a fully-$\lambda(P)$-basis for E^*_β.

Conversely, suppose $\{f_i, Jx_i\}$ is a fully-$\lambda(P)$-basis for E^*_β. Since an l.c. TVS with a fully-$\lambda(P)$-basis is always nuclear, it follows that E^*_β is nuclear. Now by Theorem 1, p. 649 [7], we infer that $\{x_i, f_i\}$ is strong and hence E is Montel by Corollary, p. 647 [7]. Consequently, $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis in view of Corollary 1, p. 517 [6]. ∎

<u>Note</u>: Compare this result with Corollary 4.7 [5] and Corollary 1, p. 517 [6].

Acknowledgements: We acknowledge with thanks the invaluable comments, remarks and constructive suggestions of the referee which led to the improvement and sharpening of this article considerably.

References

[1] Cook, T.A. : "Schauder Decompositions and semi-reflexive Spaces"; Math. Ann. 182(1969), 232-235.

[2] Horvath, J. : *"Topological Vector Spaces and Distributions I"*; Addison-Wesley (1966).

[3] Kamthan, P.K. : "A Nuclearity Criterion for Spaces having Schauder base"; Acta Math. Viet. 7(1), (1982), 41-45.

[4] Kamthan, P.K. and Gupta, M. : *"Sequence Spaces and Series"*; Lecture Notes 65, Marcel Dekker Inc. (1981).

[5] Kamthan, P.K., Gupta, M. and Sofi, M.A. : "λ-basis and their Applications"; Jour. Math. Anal. Appl. 88(1), (1982), 76-99.

[6] Kimpe, N. De Grande : "On Λ-Bases"; Jour. Math. Anal. Appl. 53(3), (1976), 508-520.

[7] Kimpe, N. De Grande : "Criteria for Nuclearity in Terms of Generalized Sequence Spaces"; Arch. Math. 28, (1977), 644-651.

[8] Köthe, G. : *"Topological Vector Spaces I"*; Springer Verlag, New York (1969).

[9] Köthe, G. : "Nukleare (F)- und (DF)-Folgenraume"; Theory of Sets and Topology (1972), 327-332.

[10] Mertins, U. : "Eigenschaften von Schauder Basen und Reflexivitat"; Studia Math. 46(1973), 223-231.

[11] Pietsch, A. : *"Nuclear Locally Convex Spaces"*; Springer Verlag (1972).

[12] Wong, Y.C. : "Schwartz Spaces, Nuclear Spaces and Tensor Products"; Springer Verlag Lect. Notes 726 (1979).

Department of Mathematics,
Sardar Patel University,
VALLABH VIDYANAGAR - 388 120 Gujarat,
INDIA