On different types of algebras contained in $CV(X)$

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Abstract

With every Nachbin family on a Hausdorff completely regular space X, we associate natural locally convex algebras of different types. Fundamental properties of these algebras are given. In particular every character of such an algebra E is shown to be an evaluation at some point of $\beta(X)$, the Stone-Čech compactification of X. Results are also furnished extending to general weighted algebras the relationship between the compact open, the strict and the uniform topologies on $C_b(X)$

Introduction

Let X be a Hausdorff completely regular space, V a Nachbin family on X and $CV(X)$ and $CV_0(X)$ the corresponding weighted locally convex spaces in the sense of [2] and [12]. In general these spaces need not be algebras. In [9], there are given necessary and sufficient conditions for $CV_0(X)$ and $CV(X)$ to be locally convex algebras of a certain type. In case these conditions are not satisfied, questions involving the algebra structure cannot be studied on the whole weighted space. In order to make such a study possible, at least on a large part of $CV(X)$ and $CV_0(X)$, we associate with every Nachbin family on X canonical locally convex (resp. locally A-convex, uniformly locally A-convex) algebras, contained respectively in $CV(X)$ and $CV_0(X)$. These algebras are maximal in some respect and offer a convenient framework for results of Buck stated for $C_b(X)$ in [3]. Finally we show that every character on a selfadjoint subalgebra E of $C(X)$ which is a $C_b(X)$-module (e.g. a weighted algebra)

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is an evaluation at some point of $\beta(X)$, the Stone-Čech compactification of X, giving the complex version of a result obtained, in the real case, in [1].

1 Preliminaries

Henceforth X will denote a Hausdorff completely regular space, $C(X)$ the algebra of all $\mathbb{K}$-valued continuous functions on X ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and V a Nachbin family on X, i.e. a family of upper semicontinuous (u.s.c.) non negative functions v on X such that for every $v_1, v_2 \in V$ and $\lambda > 0$, there exists $v \in V$ with $\max(\lambda v_1, \lambda v_2) \leq v$ and for every $x \in X$, $v(x) \neq 0$ for some $v \in V$. We will consider the so called weighted locally convex spaces $CV(X) := \{f \in C(X) : |f|v$ is bounded for every $v \in V\}$ and $CV_0(X) := \{f \in C(X) : fv$ vanishes at infinity for every $v \in V\}$, equipped with the (weighted) topology τ_V defined by the seminorms $(|\,|_v)_{v\in V}$, where $|f|_v := \sup\{v(t)|f(t)| : t \in X\}$, $f \in CV(X)$.

In all the sequel, unless the contrary is stated, all subspaces of $CV(X)$ we will consider are supplied with the topology induced by τ_V. For every $v \in V$, we will denote by $B_v(E)$ the unit ball of $|\,|_v$ in E. A subspace E of $CV(X)$ is said to be essential if, given $x \in X$, there is some $f \in E$ so that $f(x) \neq 0$.

A locally convex algebra (l. c. a.) is any (here commutative) algebra E endowed with a locally convex topology such that the multiplication of E is separately continuous. A l. c. a. is said to be locally A-convex (l. A-c.) if zero admits a fundamental system of neighbourhoods $(U_i)_{i\in I}$ consisting of A-convex sets (that is, for every $i \in I$, U_i is absolutely convex, absorbing and absorbs xU_i for every $x \in U_i$). Equivalently E is a locally A-convex algebra if and only if its topology can be given by a family $(P_i)_{i\in I}$ of A-seminorms; that is to say, for every $i \in I$ and $x \in E$, there is some $M(x,i) > 0$ such that $P_i(xy) \leq M(x,i)P_i(y)$, $y \in E$. If the constant $M(x,i)$ can be chosen depending only on x, but not on i, we say that E is a uniformly locally A-convex algebra (u. l. A-c. a.) (cf. [4]). For any A-convex set B, $\Im(B)$ will designate the idempotent kernel of B, where $\Im(B) := \{x \in B : xB \subset B\}$. It is clear that B and $\Im(B)$ generate the same linear space. Hence if E is a l. A-c. a., then for every $x \in E$ and every 0-neighbourhood θ, there is some $r > 0$ such that $\{(\frac{x}{r})^n, n \geq 1\} \subset \theta$. A locally m-convex algebra is a l. c. a. whose topology can be defined by a family $(P_i)_{i\in I}$ of submultiplicative seminorms (cf. [7]).

We assume familiarity with the book of Jarchow [6] for the notations or terminology not given here.

2 Locally convex algebras contained in $CV(X)$

Let us start with the following examples.

Examples 1: 1. Put $X = \mathbb{R}$, $V = \{\lambda v, \lambda > 0\}$, where $v(t) = e^{-|t|}$, $t \in \mathbb{R}$. Neither $CV(X)$ nor $CV_0(X)$ are algebras. But both contain, for example, the algebra $C_b(X)$ of all continuous and bounded functions and the algebra $P(\mathbb{R})$ of all polynomials. Here $C_b(X)$ is a l. c. a. while $P(\mathbb{R})$ is not.

2. Set $X = [0,1] \cup Q^+$, $v(t) = 1$ on $[0,1]$ and e^{-t} elsewhere and $V = \{\lambda v, \lambda > 0\}$. Then $CV_0(X)$ is isomorphic to the subalgebra of $C[0,1]$ consisting of those functions

vanishing at 1, with the uniform norm. This is a Banach algebra, while $CV(X)$ is not even an algebra.

Since $CV(X)$ and $CV_0(X)$ may fail to be algebras, one cannot deal with questions involving the algebra structure in general. We are going to introduce different types of locally convex algebras contained in $CV(X)$ or $CV_0(X)$. Let us first show the following lemma improving Theorem 2.1 of [12]. Our result is more general and the proof is much shorter. If E is a subspace of $CV(X)$, put $coZ(E) := \{x \in X : \exists f \in E$ with $f(x) \neq 0\}$.

Lemma 2. *Let E be a subspace of $CV(X)$ and $f \in C(X)$ such that $fE \subset CV(X)$. If E is a $C_b(X)$-module, then the mapping $I_f : g \mapsto fg$ is continuous from E into $CV(X)$ iff $|f|V \leq V$ on $coZ(E)$; i.e. for every $v \in V$, there is $v' \in V$ such that $|f|v \leq v'$ pointwise on $coZ(E)$.*

Proof : If I_f is continuous, then for every $v \in V$, there is $v' \in V$ so that for every $g \in E$, we have: $|fg|_v \leq |g|_{v'}$. We claim that $|f|v \leq v'$ pointwise on $coZ(E)$. Indeed, take $t \in coZ(E)$ and $g \in E$ so that $g(t) \neq 0$. Put $U_n := \{x \in X : v'(x) < v'(t) + \frac{1}{n}$ and $|g(x)| < |g(t)| + \frac{1}{n}\}$. This is an open neighbourhood of t. Let g_n be a continuous function such that $g_n(t) = 1$, $0 \leq g_n \leq 1$ and supp $g_n \subset U_n$. The function $h_n := gg_n$ belongs to E and then enjoys $v(t)|f(t)h_n(t)| \leq |h_n|_{v'}$. Hence $v(t)|f(t)g(t)| \leq (v'(t) + \frac{1}{n})(g(t) + \frac{1}{n})$. Since n is arbitrary and $g(t) \neq 0$, we get $|f(t)|v(t) \leq v'(t)$. The converse is trivial. ∎

Remark 3: 1. If $coZ(E) = X$ or $f \in E$, we get $|f|v \leq v'$ pointwise on the whole of X.

2. If, in the preceding proof, v' can be taken equal to v, the function f must be bounded on the set $N_v := \{x \in X : v(x) \neq 0\}$.

We are then led to consider the spaces $C_\ell V(X) := \{f \in CV(X) : |f|V \leq V\}$, $C_\ell V_0(X) := \{f \in CV_0(X) : |f|V \leq V\}$, $C_A V(X) := \{f \in CV(X) : f$ is bounded on each $N_v, v \in V\}$ and $C_A V_0(X) := \{f \in CV_0(X) : f$ is bounded on each N_v, $v \in V\}$. It is also worthwhile to take into account $C_A V_{00}(X) := \{f \in CV_0(X) : f$ vanishes at infinity on each $\overline{N_v}, v \in V\}$, $C_{uA}V(X) := CV(X) \cap C_b(X)$ and $C_{uA}V_0(X) := CV_0(X) \cap C_b(X)$. We clearly have:

$$\begin{array}{ccccccc} & & C_A V_{00}(X) & & & & \\ & & \cap & & & & \\ C_{uA}V_0(X) & \subset & C_A V_0(X) & \subset & C_\ell V_0(X) & \subset & CV_0(X) \\ \cap & & \cap & & \cap & & \cap \\ C_{uA}V(X) & \subset & C_A V(X) & \subset & C_\ell V(X) & \subset & CV(X). \end{array}$$

All the spaces here are solid and hence $C_b(X)$-modules. We will write $CV_{(0)}(X)$ to mean "$CV(X)$ (resp. $CV_0(X)$)". The same also holds for $C_\ell V_{(0)}(X)$, $C_A V_{(0)}(X)$ and $C_{uA}V_{(0)}(X)$. We summarise properties of these spaces in:

Theorem 4. *1. $C_{uA}V_{(0)}(X)$ is the largest u. l. A-c. a. contained in $CV_{(0)}(X)$. It is essential whenever $CV_{(0)}(X)$ is.*
2. $C_A V_{(0)}(X)$ is the largest locally A-convex algebra contained in $CV_{(0)}(X)$.
3. $C_\ell V_{(0)}(X)$ is the largest locally convex algebra which is both a $C_b(X)$-module and contained in $CV_{(0)}(X)$.
4. $C_A V_{00}(X)$ is essential whenever X is locally compact.

Proof : 1. It is clear that $C_{uA}V_{(0)}(X)$ is a uniformly locally A-convex algebra. Now, since every element of a u. l. A-c. a. is regular ([11]), if A is any such algebra contained in $CV_{(0)}(X)$, then for every $f \in A$, there is some $r > 0$ so that the sequence $((\frac{f}{r})^n)_n$ is bounded in A. This is true only if f is bounded. Now if $x \in X$ and $f \in CV_{(0)}(X)$ enjoy $f(x) \neq 0$, then the function $g := \min(|f(x)|, |f|)$ is in $C_{uA}V_{(0)}(X)$ and verifies $g(x) \neq 0$. Hence $C_{uA}V_{(0)}(X)$ is essential provided $CV_{(0)}(X)$ is.

2. By its very definition, $C_AV_{(0)}(X)$ is a locally A-convex algebra. Take any l. A-c. a. A contained in $CV_{(0)}(X)$. For every $v \in V$ and $f \in A$, there is some $r > 0$ such that $((\frac{f}{r})^n)_n \subset B_v(A)$. We then have, for every $t \in N_v$ and every $n \geq 1$, $|f(t)| \leq r(\frac{1}{v(t)})^{\frac{1}{n}}$. This is only true if $|f(t)| \leq r$. Hence $A \subset C_AV_{(0)}(X)$.

3. By Lemma 2, for every $f \in C_\ell V_{(0)}(X)$, the mapping $I_f : g \mapsto fg$ is continuous from $C_\ell V_{(0)}(X)$ into $CV_{(0)}(X)$. We then have to show that fg belongs to $C_\ell V_{(0)}(X)$. But this follows from the very definition of $C_\ell V_{(0)}(X)$. As to the largeness of $C_\ell V_{(0)}(X)$, it is also an immediate consequence of Lemma 2 and remark 3.1.

4. If X is locally compact, $C_AV_{00}(X)$ is essential since it contains the continuous functions with compact support. ■

Remark 5: 1. By Theorem 4, $CV_{(0)}(X)$ is a locally convex algebra (resp. a l. A-c. a. resp. a u. l. A-c. a.) iff it is equal to $C_\ell V_{(0)}(X)$ (resp. $C_AV_{(0)}(X)$, resp. $C_{uA}V_{(0)}(X)$) (compare [9], Proposition 2).

2. The algebra $C_\ell V_{(0)}(X)$ may fail to be the largest algebra contained in $CV_{(0)}(X)$. In example 1.1, the algebra $C_\ell V_{(0)}(X)$ coincides with $C_b(\mathbb{R})$ and $P(\mathbb{R})$ intersects $C_\ell V_{(0)}(X)$ only at $\{0\}$. Notice that $P(\mathbb{R})$ is not a l. c. a., the product by x is not continuous.

3. In general, the algebras in Theorem 4 may differ from each other. They will be referred to as weighted algebras. If, however, $V = \{\lambda v : \lambda > 0\}$ for some weight v on X, then $C_{uA}V(X) = C_\ell V(X)$ and $C_{uA}V_0(X) = C_\ell V_0(X)$.

To give an example where the nine (algebras or) spaces above all differ from each other, let us first recall a method, given in case of $n = 2$ in [10], of constructing new weighted spaces starting from given ones. For $i = 1, ..., n$, let X_i denote a Hausdorff completely regular space, and $X = \dot\cup_{1\leq i\leq n} X_i$ the disjoint union. This is the set $\cup_{1\leq i\leq n} X_i$ in which we distinguish any $x_i \in X_i$ from each $x_j \in X_j$, $i \neq j$. Equip X with the topology whose open sets are exactly the unions of open sets of the X_i's. Assume that, for every $i = 1, ..., n$, V_i is a Nachbin family on X_i and set $V := \prod_i^n V_i$. If, for $v = (v_i)_i \in V$ and $x \in X$, we put $v(x) = v_i(x)$ if $x \in X_i$, we then get a Nachbin family on the Hausdorff completely regular space X such that the following equalities hold algebraically and topologically: $CV_{(0)}(X) = \prod_{i=1}^n C(V_i)_{(0)}(X_i)$ and $C_\star V_{(0)}(X) = \prod_{i=1}^n C_\star(V_i)_{(0)}(X_i)$, where $\star$ stands for ℓ, A or uA; finally, also $C_AV_{00}(X) = \prod_{i=1}^n C_A(V_i)_{00}(X_i)$.

Example 6: For $i = 1, ..., 4$, take X_i to be the real line with its usual topology and $X = \dot\cup_i X_i$. Assume $V_1 := \{\lambda 1_K,\ K \subset X_1 \text{ compact}, \lambda > 0\}$, $V_2 := \{\lambda 1, \lambda > 0\}$, where 1_K denotes the characteristic function of K and 1 the constant function with value 1, $V_3 := \{\lambda e^{-\frac{1}{n}|\cdot|}, n \geq 1, \lambda > 0\}$ and $V_4 := \{\lambda e^{-|\cdot|}, \lambda > 0\}$. If we put $V := \prod_1^4 V_i$, then, by the remark above, we get that the (algebras and) spaces $CV(X)$, $CV_0(X)$, $C_\ell V(X)$, $C_\ell V_0(X)$, $C_AV(X)$, $C_AV_0(X)$, $C_{uA}V(X)$, $C_{uA}V_0(X)$, and

$C_AV_{00}(X)$ are pairewise different since $CV(V_4)(X_4)$ is not an algebra, $C(V_2)(X_2) \neq C(V_2)_0(X_2)$, $C_\ell(V_2)(X_2) \neq C_\ell(V_2)_0(X_2)$, $C_A(V_3)(X_3) \neq C_\ell(V_3)(X_3)$, $C_A(V_3)_0(X_3) \neq C_\ell(V_3)_0(X_3)$, $C_A(V_2)_0(X_2) \neq C_A(V_2)(X_2)$, $C_{uA}(V_1)(X_1) \neq C_A(V_1)(X_1)$, $C_{uA}(V_1)_0(X_1) \neq C_A(V_1)_0(X_1)$, $C_{uA}(V_1)_0(X_1) \neq C_A(V_1)_{00}(X_1)$, $C_{uA}(V_1)(X_1) \neq C_A(V_1)_{00}(X_1)$ and $C_A(V_3)_{00}(X_3) \neq C_A(V_3)_0(X_3)$.

Actually such an example may also be obtained by considering, as X, the boundary of any complex rectangle without its extreme points, with an appropriate Nachbin family.

Let E be a l. c. a. over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) and $x \in E$. By the spectrum of x relatively to E (even in the real case), we mean the set $\mathrm{sp}_E x := \{\lambda \in \mathbb{K} : x - \lambda e$ is not invertible$\}$ if E has a unit e and $\mathrm{sp}_E x := \{0\} \cup \{\lambda \in \mathbb{K}\backslash\{0\} : \frac{x}{\lambda}$ is not quasi-invertible$\}$ if E does not have a unit, where x is said to be quasi-invertible if there is $y \in E$ so that $xy = x + y$. The radius of boundedness $\beta(x)$ and the spectral radius $\rho(x)$ are defined as $\beta(x) = \inf\{\alpha > 0 : ((\frac{x}{\alpha})^n)_n$ is bounded$\}$ and $\rho(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_E x\}$. We then obtain:

Proposition 7. *Let E be a subalgebra of $C_\ell V(X)$. If E is a $C_b(X)$-module, then for every $f \in E$, one has $\rho(f) = \beta(f) = ||f||$, where $||f|| := \sup\{|f(t)|, t \in X\}$.*

Proof : Clearly $||f|| \leq \rho(f)$. For the converse, we only have to show that if $|\lambda| > ||f||$, then $\lambda \notin \mathrm{sp}_E f$. For such a λ, the function $g := f - \lambda$ is continuous and bounded away from 0 (i.e. $|f - \lambda| > \delta$ for some $\delta > 0$). If $1 \in E$, then $C_b(X)$ is contained in E and then $\frac{1}{g} \in E$. Hence $\lambda \notin \mathrm{sp}_E f$. Now if $1 \notin E$, the function $\frac{f}{g}$ belongs to E and the equality $\frac{f^2}{\lambda g} = \frac{f}{g} + \frac{f}{\lambda}$ shows that $\lambda \notin \mathrm{sp}_E f$. As to β, if $|f(t)| > \alpha$ for some $t \in X$, then $((\frac{f}{\alpha})^n)_n$ is unbounded even in $CV(X)$. Hence $||f|| \leq \beta(f)$. In particular, if f is unbounded, $||f|| = \beta(f) = +\infty$. Now if $f \in C_b(X) \cap E$ and $\alpha > ||f||$ are given, then for every $v \in V$ and $n \geq 1$, we have $|(\frac{f}{\alpha})^n|_v \leq |\frac{f}{\alpha}|_v$. Hence $\beta(f) \leq \alpha$ and then $\beta(f) = ||f||$. ∎

The equality $\beta(f) = ||f||$ actually holds in every subalgebra of $C_\ell V(X)$. Moreover, with a similar proof as in the first part above, one shows easily that if f belongs to a subalgebra E of $C_\ell V(X)$ which is a $C_b(X)$-module, then $f(X) \subset \mathrm{sp}_E f \subset \overline{f(X)}$.

A locally convex algebra E is said to be strongly sequential if the set $\{x \in E : (x^n)_n$ is bounded$\}$ is a 0-neighbourhood. It is a Q-algebra if the set of all its (quasi-) invertible elements is open. Equivalently, E is strongly sequential (resp. Q) iff β (resp. ρ) is continuous at 0 (cf. [5]). We then have:

Corollary 8. *Let E be a subalgebra of $C_\ell V(X)$. If E is a $C_b(X)$-module, the following assertions are equivalent: i) E is strongly sequential, ii) E is a Q-algebra, and iii) $||\ ||$ is a continuous norm on E.*

This corollary provides a large class of non complete normed Q-algebras. Take any $C_{uA}V_{(0)}(X)$ and endow it with the uniform norm.

3 M-convex topologies in subalgebras of $C_\ell V(X)$

Notice first that, with the same proof as for Lemma 2.1 of [10], one shows

Lemma 9. *Let E be a subalgebra of $C_\ell V(X)$ which is a $C_b(X)$-module. For every $v \in V$ and $t \in coZ(E)$, one has $\frac{1}{v(t)} = \sup\{|f(t)| : f \in B_v(E)\}$ with $\frac{1}{0} = +\infty$.*

For every locally A-convex topology τ on an algebra E, we will write $M(\tau)$ for the weakest locally m-convex topology on E stronger than τ (cf. [8]). In what follows, we deal with $M(\tau_V)$ on $C_AV(X)$ and its subalgebras. For this purpose, we need some additional notations from [10]. For every $v \in V$ and every $\epsilon > 0$, we consider $N_{v,\epsilon} := \{x \in X : v(x) \geq \epsilon\}$, $N_v := \{x \in X : v(x) > 0\}$ and the mappings $u_{v,\epsilon} := \max(v, \epsilon)$ on $\overline{N_v}$ and 0 otherwise and $w_{v,\epsilon} = v$ on $N_{v,\epsilon}$ and 0 elsewhere. Both $u_{v,\epsilon}$ and $w_{v,\epsilon}$ are weights on X and verify $u_{v,\epsilon} = \epsilon u_{\frac{v}{\epsilon},1}$ and $w_{v,\epsilon} = \epsilon w_{\frac{v}{\epsilon},1}$. Hence $U := \{\lambda u_{v,1}, \lambda > 0\}$ and $W := \{\lambda w_{v,1}, \lambda > 0\}$ are Nachbin families on X. Let $CU_{(0)}(X)$ and $CW_{(0)}(X)$ be the corresponding weighted spaces. By Proposition 2 of [9] these are locally m-convex algebras and, by construction, $W \leq V \leq U$. Hence $CU_{(0)}(X) \subset CV_{(0)}(X) \subset CW_{(0)}(X)$ and on $CU_{(0)}(X)$, one has $\tau_W \leq \tau_V \leq \tau_U$. Actually we have:

Theorem 10. *The equalities $CU(X) = C_AV(X)$ and $CU_0(X) = C_AV_{00}(X)$ hold algebraically. Moreover on every $C_b(X)$-module E which is a subalgebra of $C_AV(X)$, τ_U coincides with $M(\tau_V)$ and τ_W is the strongest weighted locally m-convex topology coarser than τ_V.*

Proof : Let f be in $CU(X)$ and $v \in V$. Then $f \in CV(X)$ and for every $t \in N_v$, one has $|f(t)| \leq |f(t)|u_{v,1}(t) \leq |f|_{u_{v,1}}$. Hence f belongs to $C_AV(X)$. Assume now f is in $C_AV(X)$ and v in V. We have that $|f|_{u_{v,1}}$ is less than the maximum of $|f|_v$ and $||f||_{N_v}$. Then f belongs to $CU(X)$. As to $CU_0(X) = C_AV_{00}(X)$, let $f \in CU_0(X)$, $v \in V$ and $\epsilon > 0$ be given. The set $\{t \in \overline{N_v} : |f(t)| \geq \epsilon\}$ is closed and contained in the compact set $\{t \in X : |f(t)|u_{v,1}(t) \geq \epsilon\}$. Hence $f \in C_AV_{00}(X)$. Conversely, take $f \in C_AV_{00}(X)$, $v \in V$ and $\epsilon > 0$. The sets $A := \{t \in N_{v,1} : |f(t)|v(t) \geq \epsilon\}$ and $B := \{t \in \overline{N_v} : |f(t)| \geq \epsilon\}$ are compact. Since $\{t \in X : |f(t)|u_{v,1}(t) \geq \epsilon\}$ is closed and contained in $A \cup B$, f belongs to $CU_0(X)$. Now let E be a $C_b(X)$-module which is a subalgebra of $C_AV(X)$ and let us show that $\tau_U = M(\tau_V)$ on E. Since τ_U is m-convex, $M(\tau_V)$ is coarser than τ_U (cf. [8]). To get the equality, we only have to show that for every $v \in V$, $B_{u_{v,1}}(E)$ contains the idempotent kernel $\Im(B_v(E))$ of $B_v(E)$. But for $f \in \Im(B_v(E))$, we have $f \in B_v(E)$ and $fB_v(E) \subset B_v(E)$. This means that $|f(t)|v(t) \leq 1$ and $|f(t)g(t)|v(t) \leq 1$ for $t \in N_v$ and $g \in B_v(E)$. Now if $t \notin coZ(E)$, obviously $|f(t)|u_{v,1}(t) \leq 1$. If $t \in \overline{coZ(E)}$, by Lemma 9, $\frac{1}{v(t)} = \sup\{|g(t)| : g \in B_v(E)\}$, whence $|f(t)| \leq 1$ for every $t \in \overline{N_v}$. This leads to $|f(t)|u_{v,1}(t) \leq 1$, $t \in N_v$ and then $f \in B_{u_{v,1}}(E)$. Now take a Nachbin family V' on X so that $(E, \tau_{V'})$ is locally m-convex. From Proposition 2. 4) of [9], we may assume that, for every $v' \in V'$, $\epsilon := \inf\{v'(t) : t \in N_{v'} \cap coZ(E)\} > 0$. If, in addition, $\tau_{V'} \leq \tau_V$ on E, then for such a v' there is some $v \in V$ so that $|f|_{v'} \leq |f|_v$, $f \in E$. This leads to $v' \leq v$ pointwise and then $v' \leq w_{v,\epsilon}$ on $coZ(E)$. Since v' is arbitrary in V', we get the required result. ∎

Actually, on every subalgebra of $C_\ell V(X)$, τ_W is the strongest weighted locally m-convex topology coarser than τ_V. We do not know whether τ_W coincides with the strongest locally m-convex topology coarser than τ_V.

We now give a result connecting the bounded sets of the topologies τ_W, τ_V and τ_U, extending the corresponding results of [3] from $C_b(X)$ to general weighted algebras. By an m-bounded set, we mean any set absorbed by an idempotent bounded disc.

Proposition 11. *Let E be a subalgebra of $C_AV(X)$.*
1. τ_W and τ_U always have the same m-bounded sets in E.
2. If E is a $C_b(X)$-module, then every completing τ_V-bounded disc in E is τ_U-bounded. In particular τ_U and τ_V have the same bounded sets whenever (E, τ_V) is locally complete.
3. Given $(f_n)_n \subset E$. Then $(f_n)_n$ τ_V-converges to $f \in E$ whenever it is τ_U-bounded and τ_W-convergent. The converse is true if E is a $C_b(X)$-module and locally complete.

Proof : 1. Let B be an idempotent τ_W-bounded disc of E and v in V. Then, for every $\epsilon > 0$, there is $M_\epsilon > 0$ so that $|f^n(t)|v(t) \leq M_\epsilon$, $t \in N_{v,\epsilon}$ and $n \geq 1$. This gives $|f(t)| \leq 1$ on $N_{v,\epsilon}$. Since ϵ is arbitrary, we get $|f(t)|_{u_{v,1}} \leq \max(1, M_1)$ and B is τ_U-bounded.

2. derives from the well known Banach-Mackey theorem and the fact that every 0-neighbourhood for $M(\tau_V)$ contains a τ_V-barrel.

3. Assume $(f_n)_n$ τ_U-bounded and τ_W-convergent to $f \in C_AV(X)$. Then for every $v \in V$, there is $M > 1$ with $|f_n(t)| \leq M$ and $|f(t)| \leq M$ for every $t \in N_v$ and $n \in \mathbb{N}$. Since $(f_n)_n$ τ_W-converges to f, for a given $\epsilon > 0$, consider $\epsilon' = \frac{\epsilon}{2M}$ and $v' = v/\epsilon'$; then $|f_n - f|_{w_{v',1}} \leq 1$ for sufficiently large n. This leads, for such an n, to $|f_n - f|_v \leq \epsilon$ and then $(f_n)_n$ τ_V-converges to f. The converse is a consequence of 2.

4 Scalar homomorphisms on weighted algebras

In this section we are interested in the characters of a weighted algebra E. By a character, we mean a non zero algebra morphism from E onto $\mathbb{K}$. The set of all characters will be denoted by $M^*(E)$ while $M(E)$ will designate the continuous ones. We show that every character of E is an evaluation at some point of $\beta(X)$, the Stone-Čech compactification of X. Actually we get this result for more general subalgebras of $C(X)$, providing the complex version of Lemma 2 of [1].

From now on, let E denote an arbitrary subalgebra of $C(X)$ and for every $f \in E$, let $\bar{f}$ be the complex-conjugate of f. We will say that E is selfadjoint if $\bar{f} \in E$ whenever $f \in E$. E is said to be hermitian if it is selfadjoint and for every character χ of E and every $f \in E$, the equality $\chi(\bar{f}) = \overline{\chi(f)}$ holds. The character χ will be referred to as verifying property (P) if for every $f \in E$, $\chi(f)$ belongs to the closure $\mathrm{cl}(f(X))$ of $f(X)$.

Lemma 12. *If χ is a character on E with property (P), then $\chi(\bar{g}) = \overline{\chi(g)}$ whenever g and $\bar{g}$ belong to E. This holds, in particular, when E is a $C_b(X)$-module.*

Proof : If $\mathbb{K} = \mathbb{R}$, there is nothing to show. In the complex case, let g and $\bar{g}$ belong to E. Then $\mathrm{Re}g = (g + \bar{g})/2$ and $\mathrm{Im}g = (g - \bar{g})/2i$ also belong to E. The hypothesis on χ ensures that $\chi(\mathrm{Re}g)$ and $\chi(\mathrm{Im}g)$ are real. The linearity of χ then leads to the required equality. Now assume that E is a $C_b(X)$-module, $\chi \in M^*(E)$, $f \in E$ and

$\chi(f) \notin \mathrm{cl}(f(X))$. Then there is $\epsilon > 0$ so that $|f(x) - \chi(f)| > \epsilon$, $x \in X$. In case $\chi(f) \neq 0$, the function $g := \frac{f}{f-\chi(f)}$ belongs to E and then $f = gf - \chi(f)g$. This leads to the contradiction $\chi(f) = 0$. Now $\chi(f) = 0$ cannot occur since then $1/f$ would belong to E. ∎

According to Lemma 12, if E is a $C_b(X)$-module, then the algebra $E \cap \bar{E}$ is hermitian. In particular, the weighted algebras (involved in Theorem 4) are all hermitian.

In the sequel, for $f \in C(X)$, $\tilde{f}$ will designate the (unique) extension of f to $\beta(X)$ with values in the one point compactification $\mathbb{K} \cup \{\infty\}$ of $\mathbb{K}$. If E fails to be essential, its unitization E_1 (consisting of all the functions of the form $f + \lambda$, $f \in E$ and $\lambda \in \mathbb{K}$) is always essential. Moreover $\chi \in M^*(E)$ satisfies (P) with respect to E if and only if this also holds for $\chi_1 : f + \lambda \mapsto \chi(f) + \lambda$ with respect to E_1. We then get the complex version of Lemma 2 of [1]:

Theorem 13. *Let E be a selfadjoint subalgebra of $C(X)$. If $\chi \in M^*(E)$ satisfies (P), then there is some $z \in \beta(X)$ such that $\chi(f) = \tilde{f}(z)$ for each $f \in E$.*

Proof : According to the comment above, we (may) assume E essential. For $f \in E$ and $\epsilon > 0$, set $F(f,\epsilon) := \{x \in X : |f(x) - \chi(f)| \leq \epsilon\}$ and $G(f,\epsilon) := \{x \in \beta(X) : |\tilde{f}(x) - \chi(f)| \leq \epsilon\}$. We then have $F(f,\epsilon) \subset G(f,\epsilon)$ and the hypothesis on χ gives $F(f,\epsilon) \neq \emptyset$. By compactness of $\beta(X)$, the set $I_f := \cap_{\epsilon>0} G(f,\epsilon)$ cannot be empty for any $f \in E$. Moreover for every $x \in I_f$, one has $\chi(f) = \tilde{f}(x)$. Furthermore if $f_1, f_2, ..., f_n$ are elements of E and $\epsilon > 0$, since E is selfadjoint, the function $h := \sum_{i=1}^n f_i\bar{f}_i - \sum_{i=1}^n \overline{\chi(f_i)} f_i - \sum_{i=1}^n \chi(f_i)\bar{f}_i$ belongs to E. By Lemma 12, $F(h,\epsilon^2) \subset \cap_{i=1}^n G(f_i,\epsilon)$. Once again by compactness, $I = \cap_{f\in E} I_f$ is not empty and χ is the evaluation at any point of I. ∎

If E is selfadjoint and a $C_b(X)$-module, then every character on E is an evaluation at some point of $\beta(X)$. This holds in particular for any weighted algebra. Another consequence of Theorem 13 is, if E is a $C_b(X)$-module and $\chi \in M^*(E)$ does not vanish identically on $E \cap \bar{E}$, then χ is an evaluation at a point of $\beta(X)$.

Henceforth, as in [9], we will consider: $N(E) := \{x \in \beta(X) : \tilde{f}(x) \neq 0$ for some $f \in E\}$, $F(E) := \{x \in \beta(X) : \tilde{f}(x) \neq \infty$ for every $f \in E\}$ and $S^*(E) := N(E) \cap F(E)$. We then give a precise description of $M^*(E)$:

Corollary 14. *Let E be selfadjoint and a $C_b(X)$-module. Then $M^*(E)$ is homeomorphic to $S^*(E)$.*

Proof : Let δ be the mapping which assigns to every $x \in S^*(E)$ the evaluation δ_x at x. It is one to one, since E is a $C_b(X)$-module and $\beta(X)$ compact, and onto by Theorem 13. Now if $x_0 \in S^*(E)$ and $f_1, ..., f_n \in E$ are given, then

$$\delta(\{x \in S^*(E) : |\tilde{h}(x) - \tilde{h}(x_0)| < \epsilon^2\}) \subset \{\delta_x \in M^*(E) : |\delta_x(f_i) - \delta_{x_0}(f_i)| < \epsilon,$$
$$i = 1, ..., n\}$$

where $h := \sum_{i=1}^n f_i\bar{f}_i - \sum_{i=1}^n \overline{\tilde{f}_i(x_0)} f_i - \sum_{i=1}^n \tilde{f}_i(x_0)\bar{f}_i$. This shows the continuity of δ at x_0. To show that δ is open, let U be an open subset of $S^*(E)$ and $x_0 \in U$. Then $U = W \cap S^*(E)$ for some open set W in $\beta(X)$. Take $f \in C(\beta(X))$ and $g \in E$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and supp $f \subset W$ and $\tilde{g}(x_0) = 1$. Then $k := f_{|X} g \in E$ and $\{\chi \in M^*(E) : |\chi(k) - \tilde{k}(x_0)| < \frac{1}{2}\} \subset \delta(U)$, whence the result. ∎

Remark 15: 1. It is not difficult to show that $S^*(E)$ is actually contained in $\overline{coZ(E)}^{\beta X}$. But the latter is a continuous image of $\beta(coZ(E))$. Hence every character on E is an evaluation at some point of the Stone-Čech compactification of $coZ(E)$. This can be directly seen using a slight modification of the proof of Theorem 13.

2. If u is an u.s.c. function on a subset A of a Hausdorff completely regular space Y with values in $\mathbb{R}_+$ and if w is an u.s.c. extension of u to Y with values in $\mathbb{R}_+ \cup \{+\infty\}$, we will consider $\tilde{u} := \inf\{w : Y \longrightarrow \mathbb{R}_+ \cup \{+\infty\}, w \text{ u.s.c. and } w_{|X} \geq u\}$. This is the minimal u.s.c. extension of u to Y. If u happens to be bounded, then so is also $\tilde{u}$ and $||u||_A = ||\tilde{u}||_Y$. In case $E \subset CV(X)$, $Y = S^*(E)$ and $A = coZ(E)$, every $v \in V$ admits a minimal u.s.c. extension $\tilde{v}$ to $S^*(E)$. For, by Lemma 9, $1/w$ is u.s.c. and extends v to $S^*(E)$, where $w(x) := \sup\{|\tilde{f}(x)|, f \in B_v(E)\}$. Furthermore if $f \geq 0$ belongs to $CV(X)$, then $\tilde{f}\tilde{v}$ is u.s.c. and extends fv. Hence $\widetilde{fv} \leq \tilde{f}\tilde{v}$. Conversely if we define w on $S^*(E)$ by $w(x) = \frac{\widetilde{fv}(x)}{\tilde{f}(x)}$ if $\tilde{f}(x) \neq 0$ and $w(x) = +\infty$ elsewhere, we get an u.s.c. function on $S^*(E)$ whose restriction to $coZ(E)$ is larger than v. Hence $\tilde{v} \leq w$ and then $\tilde{v}\tilde{f} = \widetilde{fv}$.

In what follows, for every $v \in V$, set $\tilde{N}_v := \{x \in S^*(E) : \tilde{v}(x) \neq 0\}$ and let $\tilde{N}(V)$ be the union of all the $\tilde{N}_v$'s. We then obtain:

Theorem 16. *Let $E \subset C_\ell V(X)$ be selfadjoint and a $C_b(X)$-module. Then $M(E)$ is homeomorphic to $\tilde{N}(V)$.*

Proof : We only have to show the algebraic equality $\delta(\tilde{N}(V)) = M(E)$. Let $\chi = \delta_x$ be a continuous character on E. There is some $v \in V$ such that, for every $f \in E$, $|\tilde{f}(x)| \leq |f|_v$. We claim that $\tilde{v}(x) \neq 0$. If not, take $f \in E$ and an open subset W_n of $\beta(X)$ such that $\tilde{f}(x) = 1$ and $U_n = W_n \cap S^*(E)$, where $U_n := \{t \in S^*(E) : 1 - \frac{1}{n} < |\tilde{f}(t)| < 1 + \frac{1}{n} \text{ and } v(t) < \frac{1}{n}\}$. Let $g_n \in C(\beta(X))$ enjoy $0 \leq g_n \leq 1$, $g_n(x) = 1$ and supp $g_n \subset W_n$. Then $h_n := (g_n)_{|X} f$ belongs to E and $|\tilde{h}_n(x)| \leq |h_n|$, $n \in \mathbb{N}$. Hence

$$\begin{aligned} |g_n(x)\tilde{f}(x)| = 1 &\leq \sup_{t \in U_n} \tilde{v}(t)|\tilde{f}(t)g_n(t)| \\ &\leq \frac{1}{n}(1 + \frac{1}{n}). \end{aligned}$$

This is impossible since n is arbitrary. Conversely if $\tilde{v}(x) \neq 0$, then for every $f \in E$, one has $|\tilde{f}(x)| \leq \frac{1}{\tilde{v}(x)}|f|_v$. ∎

Corollary 17. *If $E \subset CV_0(X)$ is selfadjoint and a $C_b(X)$-module, then $M(E)$ is homeomorphic to $coZ(E)$.*

Proof : Assume that $\delta_x \in M(E)$, $\tilde{v}(x) \neq 0$ and $x \notin coZ(E)$. From the equality $\tilde{v}(x)\tilde{f}(x) = \widetilde{fv}(x) = 0$, for every $f \in E$, one derives $x \notin S^*(E)$. This is a contradiction. ∎

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