m-systems of Polar Spaces and Maximal Arcs in Projective Planes

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Abstract

Shult and Thas have shown in [13] that m-systems of certain finite classical polar spaces give rise to strongly regular graphs and two weight codes. The main result of this paper is to show that maximal arcs in symplectic translation planes may be obtained from certain m-systems of finite symplectic polar spaces. Many new examples of maximal arcs are then constructed. Examples of m-systems are also constructed in $Q^-(2n+1,q)$ and $W_{2n+1}(q)$. A method different from that of Shult and Thas is used to construct strongly regular graphs using "differences" of m-systems.

1 Introduction

We follow the definitions and notation of [13]. Let P be a finite classical polar space of rank $r \geq 2$. Then

$W_n(q)$ denotes the polar space arising from a symplectic polarity of $PG(n,q)$, n odd and $n \geq 3$,

$Q(2n,q)$ denotes the polar space arising from a non-singular parabolic quadric of $PG(2n,q)$, $n \geq 2$,

$Q^+(2n+1,q)$ denotes the polar space arising from a non-singular hyperbolic quadric of $PG(2n+1,q)$, $n \geq 1$,

$Q^-(2n+1,q)$ denotes the polar space arising from a non-singular elliptic quadric of $PG(2n+1,q)$, $n \geq 2$,

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$H(n, q^2)$ denotes the polar space arising from a non-singular hermitian polarity of $PG(n, q^2)$, $n \geq 3$.

In [13] E.E. Shult and J.A. Thas introduced the the following definitions of partial m-systems and m-systems of polar spaces.

Definition 1. *Let P be a finite classical polar space of rank r, $r \geq 2$. A* **partial m-system** *of P, with $0 \leq m \leq r-1$ is any set $\{\pi_1, \pi_2, \ldots, \pi_k\}$ of $k \neq 0$ totally singular m-spaces of P such that no generator containing π_i has a point in common with $(\pi_1 \cup \pi_2 \cup \ldots \cup \pi_k) - \pi_i$, $i = 1, \ldots, k$.*

Theorem 1. *[13, Theorem 4] Let M be a partial m-system of the finite classical polar space P. Then*

$$\begin{array}{ll} |M| \leq q^{n+1}+1 & \text{for } P = W_{2n+1}(q) \text{ or } P = Q^-(2n+1, q), \\ |M| \leq q^n+1 & \text{for } P = Q(2n, q) \text{ or } P = Q^+(2n+1, q), \\ |M| \leq q^{2n+1}+1 & \text{for } P = H(2n, q^2) \text{ or } P = H(2n+1, q^2). \end{array}$$

Definition 2. *Let M be a partial m-system of a finite classical polar space P. If for $|M|$ the upper bound in the previous Theorem is reached then M is called an* **m-system** *of P.*

In the same paper they gave several constructions of m-systems of polar spaces, and showed that m-systems of three of the polar spaces also gave rise to strongly regular graphs. In a later paper ([15]) bounds on the sizes of various partial m-systems were given and some non-existence results proved.

In a finite projective plane of order q, a $\{k; n\} - arc$ $\mathcal{K}$ is a non-empty proper subset of k points of the plane such that some line of the plane meets $\mathcal{K}$ in n points, but no line meets $\mathcal{K}$ in more than n points. For a given q and n, the size k can not exceed $q(n-1)+n$. If equality occurs the set is called a *maximal arc*. Equivalently, a maximal arc can be defined as a non-empty, proper subset of points of the plane such that every line meets the set in 0 or n points, for some n. The integer n is known as the *degree* of the maximal arc. For example, any point of a projective plane of order q is a maximal $\{1; 1\}-$arc in that plane, and the complement of any line is a maximal $\{q^2; q\}-$arc. These are known as *trivial* maximal arcs. See [8] for theorems and definitions.

If $\mathcal{K}$ is a maximal $\{q(n-1)+n; n\}-$arc, the set of lines external to $\mathcal{K}$ is a maximal $\{q(q-n+1)/n; q/n\}-$arc in the dual plane called the *dual* of $\mathcal{K}$. It follows that a necessary condition for the existence of a maximal $\{q(n-1)+n; n\}-$arc in a projective plane of order q is that n divides q. But it is not sufficient. In [1], S. Ball, A. Blokhuis and F. Mazzocca proved that no non-trivial maximal arcs exist in $PG(2, q)$ for q odd. In $PG(2, q)$, q even, R. H. F. Denniston has given a construction of maximal $\{k; n\}-$arcs for all n dividing q [4]. Hence the spectral problem for existence of maximal arcs in solved in Desarguesian projective planes. For non-Desarguesian planes few constructions are known. See [6][7] for known constructions.

The main result of the current paper is to show that under certain conditions m-systems of $W_{2n+1}(q)$ can be used to construct maximal arcs in symplectic translation planes (Section 4). Certain of the maximal arcs constructed in this way are shown to be new. In Section 2 constructions of m-systems of $Q^-(2n+1, q)$ and $W_{2n+1}(q)$

are given for certain values of m. In Section 3 it is shown that certain collections of m-systems of $W_{2n+1}(q)$, $Q^-(2n+1,q)$ and $H(2n,q^2)$ give strongly regular graphs.

2 Construction of m-systems

Recall that given a quadratic form Q on a projective space $PG(n,q)$ a bilinear form $B(x,y) = Q(x+y) - Q(x) - Q(y)$ may be defined on the projective space. Given a totally singular subspace S of a finite classical polar space in $PG(n,q)$ we may then define the tangent space $S^\perp$ to S as

$$S^\perp = \{x \in PG(n,q) | B(x,y) = 0 \text{ for all } y \in S\}.$$

Note that the tangent space to a totally singular subspace S contains every generator of the polar space that contains S. It follows easily that showing that a collection M of m-dimensional subspaces of a polar space is a partial m-system is equivalent to showing that the tangent space to any element of M meets no other element of M.

The following Theorems all use a "trace trick" to construct new m-systems of polar spaces from old. See [13] for other constructions of m-systems of polar spaces.

Theorem 2. *If $Q^-(2t-1,q^s)$ has an m-system then $Q^-(2st-1,q)$ has an $(s(m+1)-1)$-system.*

Proof: Let Q_{2t,q^s} be a non-degenerate quadratic form of elliptic type on the vector space $V(2t,q^s)$. Define a form on the vector space $V(2st,q)$ by $Q_{2st,q}(x) = Tr_{\frac{GF(q^s)}{GF(q)}}(Q_{2t,q^s}(x))$, where $Tr_{\frac{GF(q^s)}{GF(q)}}$ is the usual trace function from $GF(q^s)$ to $GF(q)$. The results of [14, Lemma 9.1] then show that $Q_{2st,q}$ is a non-degenerate quadratic form of elliptic type on $V(2st,q)$ (in fact much of the following proof is contained in that reference, but we include it for the completeness). Note that the points of the quadric Q_{2t,q^s} are a subset of the points of the quadric $Q_{2st,q}$.

Define the usual (alternating) bilinear forms on $V(2t,q^s)$ and $V(2st,q)$ respectively by

$$B_{2t,q^s}(x,y) = Q_{2t,q^s}(x+y) - Q_{2t,q^s}(x) - Q_{2t,q^s}(y),$$
$$B_{2st,q}(x,y) = Q_{2st,q}(x+y) - Q_{2st,q}(x) - Q_{2st,q}(y).$$

Suppose that M_{2t,q^s} is an m-system of Q_{2t,q^s}, i.e. is a collection of $q^{st}+1$ subspaces of Q_{2t,q^s} of (vector space) dimension $m+1$ such that the the tangent space to any element of M_{2t,q^s} does not meet any other element of M_{2t,q^s}. Each element of M_{2t,q^s} can then be considered as an $s(m+1)$-dimensional subspace of $V(2st,q)$ and is contained within $Q_{2st,q}$. Denote the collection of such subspaces by $M_{2st,q}$. We show that $M_{2st,q}$ is an $(s(m+1)-1)$-system of $Q_{2st,q}$.

Let x and y be non-zero points. Suppose that $x \in Q_{2t,q^s}$ and that y is in the tangent space to x with respect to the form B_{2t,q^s}. Then ([10, Lemma 22.3.1])

$$\begin{aligned} & B_{2t,q^s}(x,y) = 0 \\ \Leftrightarrow\ & Q_{2t,q^s}(x+y) - Q_{2t,q^s}(x) - Q_{2t,q^s}(y) = 0 \\ \Rightarrow\ & Tr_{\frac{GF(q^s)}{GF(q)}}\left(Q_{2t,q^s}(x+y) - Q_{2t,q^s}(x) - Q_{2t,q^s}(y)\right) = 0 \\ \Leftrightarrow\ & B_{2st,q}(x,y) = 0 \end{aligned}$$

Hence y is also in the tangent space to x with respect to the form $B_{2st,q}$.

Now consider an element $M \in M_{2t,q^s}$. It has dimension $m+1$ as a subspace of $V(2t,q^s)$, and its tangent space $M^\perp$ has dimension $2t-(m+1)$. As a subspace of $V(2st,q)$, M has dimension $s(m+1)$, and $M^\perp$ has dimension $s(2t-(m+1))$. Since M is contained within the quadric $Q_{2st,q}$ of $V(2st,q)$ it follows by the previous paragraph that $M^\perp$ is contained within the tangent space of M with respect to the form $B_{2st,q}$. But such a tangent space with respect to $B_{2st,q}$ has dimension $2st-s(m+1)$, the same as that of $M^\perp$. Hence $M^\perp$ is exactly the tangent space of M with respect to $B_{2st,q}$. Since M_{2t,q^s} is an m-system, $M^\perp$ does not contain any points of elements of the m-system apart from those of M. Hence the result. ∎

Shult and Thas ([13, Theorem 13*b*]) give a geometric proof that if $Q^-(2t-1,q^2)$, $t>1$, admits an m-system then $Q^-(4t-1,q)$ admits an $(2m+1)$-system. For q even, the Shult and Thas construction corresponds to $s=2$ in Theorem 2. The more general method of construction of m-systems of $Q^-(2st-1,q)$ given in the previous Theorem was first noted in [14, Remark, p.427].

For q even, $Q^-(2t-1,q^s)$ always has an $(t-2)$-system (see the discussion at the end of the next Section), which by Theorem 2 gives an $(st-s-1)$-system of $Q^-(2st-1,q)$. Shult and Thas give another construction in [13, Section 8].

Essentially the same construction as the previous Theorem can be applied for hyperbolic quadrics, for any q, but this only gives rise to partial m-systems. In this case an m-system of $Q^+(2t-1,q^s)$ gives rise to a partial $(s(m+1)-1)$-system of $Q^+(2st-1,q)$ with $q^{s(t-1)}+1$ elements. See also [14, Lemma 9.1].

Theorem 3. *If $W_{2t-1}(q^s)$ has an m-system then $W_{2st-1}(q)$ has an $(s(m+1)-1)$-system.*

Proof: Define a (non-degenerate) symplectic form on $GF(q^{st}) \oplus GF(q^{st})$ by $B((x_1,y_1),(x_2,y_2)) = x_1y_2 - x_2y_1$. Define the (non-degenerate) symplectic forms on $V(2t,q^s)$ and $V(2st,q)$ respectively by

$$B_{2t,q^s}((x_1,y_1),(x_2,y_2)) = Tr_{\frac{GF(q^{st})}{GF(q^s)}}(B((x_1,y_1),(x_2,y_2)))$$

$$B_{2st,q}((x_1,y_1),(x_2,y_2)) = Tr_{\frac{GF(q^s)}{GF(q)}}(B_{2t,q^s}((x_1,y_1),(x_2,y_2))).$$

Proceed as in the proof for the m-systems of elliptic quadrics in Theorem 2. ∎

The space $W_{2t-1}(q^s)$ always has an $(t-1)$-system (i.e. a symplectic spread of $PG(2t-1,q^s)$). Applying Theorem 3 gives a $(st-1)$-system of $W_{2st-1}(q)$, i.e. a spread again. This is a standard technique for constructing symplectic spreads.

For q even, the bilinear form from an elliptic quadric $Q^-(2n-1,q)$ is symplectic, and so any m-system of $Q^-(2n-1,q)$ is also an m-system of the associated $W_{2n-1}(q)$. Shult and Thas use this fact to construct m-systems of $W_{2n-1}(q)$ from those constructed for $Q^-(2n-1,q)$, q even.

It is worth noting that the Tits ovoid (see [9, p.46]) is an 0-system of $W_3(2^{2t+1})$, $t>0$. This example (and the result of applying Theorem 3 to it) give m-systems where the systems are not necessarily contained in some elliptic quadric.

3 Strongly regular graphs

In [13, Theorem 7] Shult and Thas show that the set of points of an m-system of $W_{2n+1}(q)$, $Q^-(2n+1,q)$ or $H(2n,q^2)$ has two intersection numbers with respect to hyperplanes of the underlying projective space, and hence gives rise to a strongly regular graph. In this section we show how m-systems that are "disjoint" or are "contained" within each other also give rise to strongly regular graphs.

For an m-system M of a finite classical polar space, denote the union of the points contained in elements of M by $\overline{M}$.

Let M_1 be an m_1-system and M_2 be an m_2-system of a finite classical polar space P. We say M_1 is *contained by* M_2 if every element of M_1 is a subspace of a unique element of M_2. We say M_1 and M_2 are *disjoint* if $\overline{M}_1 \cap \overline{M}_2$ is the empty set.

Theorem 4. *Let M_i be an m_i-system of $W_{2n+1}(q)$, $m_i \le n$, $i = 1 \dots k$, for some integer $k > 1$. For $i = 1 \dots k$ define*

$$a_i = \frac{(q^{m_i+1}-1)(q^n+1)}{(q-1)}.$$

(i) If for all $i \neq j$ M_i and M_j are disjoint, then the set $\overline{M}_1 \cup \overline{M}_2 \cup \dots \cup \overline{M}_k$ has two intersection numbers $a_1 + a_2 + \dots + a_k - q^n$ and $a_1 + a_2 + \dots + a_k$ with respect to hyperplanes in $PG(2n,q)$.
(ii) Suppose M_i is contained by M_{i+1}, $i = 1 \dots k-1$, then
(a) if k is even, the set $\mathcal{K} = (\overline{M}_k - \overline{M}_{k-1}) \cup (\overline{M}_{k-2} - \overline{M}_{k-3}) \cup \dots \cup (\overline{M}_2 - \overline{M}_1)$ has two intersection numbers $a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_2 - a_1 - q^n$ and $a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_2 - a_1$ with respect to hyperplanes in $PG(2n+1,q)$.
(b) if k is odd, the set $\mathcal{K} = (\overline{M}_k - \overline{M}_{k-1}) \cup (\overline{M}_{k-2} - \overline{M}_{k-3}) \cup \dots \cup (\overline{M}_3 - \overline{M}_2) \cup \overline{M}_1$ has two intersection numbers $a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_3 - a_2 + a_1 - q^n$ and $a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_3 - a_2 + a_1$ with respect to hyperplanes in $PG(2n+1,q)$.

Proof: Let $\perp$ be the polarity associated with the $W_{2n+1}(q)$. Then every hyperplane of $PG(2n+1,q)$ can be written as $p^\perp$ for some point $p \in PG(2n+1,q)$. Now [13, Theorem 7] shows that the sets $\overline{M}_i$ each have two intersection numbers with respect to hyperplanes. These intersection numbers are $a_i - q^n$ and a_i, $i = 1 \dots k$. Further, $p \in \overline{M}_i \iff |p^\perp \cap \overline{M}_i| = a_i - q^n$, and also $p \notin \overline{M}_i \iff |p^\perp \cap \overline{M}_i| = a_i$.

Proof of (i): If $p \notin \overline{M}_i$ for every $i = 1 \dots k$, then it follows immediately that $p^\perp$ meets each M_i in a_i points. If $p \in \overline{M}_i$ for (unique) $i \in \{1 \dots k\}$, then $p^\perp$ meets $\overline{M}_i$ in $a_i - q^n$ points, and $\overline{M}_j$ in a_j points, $j \neq i$.

Proof of (ii)(a): If $p \notin \overline{M}_i$ for every $i = 1 \dots k$, then it follows immediately that $p^\perp$ meets $\mathcal{K}$ in $a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_2 - a_1$ points.

Suppose $p \in \overline{M}_i$ and $p \notin M_{i-1}$ for some (unique) $i \in \{1 \dots k\}$. Then $p^\perp$ meets each M_j in a_j points for $j = 1 \dots i-1$, and $a_j - q^n$ points for $j = i \dots k$. If i is even, $p^\perp$ meets $\mathcal{K}$ in $((a_k - q^n) - (a_{k-1} - q^n)) + \dots + ((a_{i+2} - q^n) - (a_{i+1} - q^n)) + ((a_i - q^n) - a_{i-1}) + (a_{i-2} - a_{i-3}) + \dots + (a_2 - a_1) = a_k - a_{k-1} + a_{k-2} - a_{k-3} \dots + a_2 - a_1 - q^n$. For i odd, $p^\perp$ meets $\mathcal{K}$ in $((a_k - q^n) - (a_{k-1} - q^n)) + \dots + ((a_{i+3} - q^n) - (a_{i+2} - q^n)) + ((a_{i+1} - q^n) - (a_i - q^n)) + (a_{i-1} - a_{i-2}) + \dots + (a_2 - a_1) = a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_2 - a_1$.

Proof of (ii)(b) Similarly. ∎

Theorem 5. *Let M_i be an m_i-system of $Q^-(2n+1,q)$, $i = 1\dots k$, for some integer $k > 1$. For $i = 1\dots k$ define a_i as in Theorem 4, then the conclusions (i),(ii)(a) and (ii)(b) of Theorem 4 hold.*

Proof: As in Theorem 4. ∎

Theorem 6. *Let M_i be an m_i-system of $H(2n,q^2)$, $i = 1\dots k$, for some integer $k > 1$. For $i = 1\dots k$ define*

$$a_i = \frac{(q^{2m_i+2}-1)(q^{2n-1}+1)}{(q^2-1)}.$$

(i) If for all $i \neq j$ M_i and M_j are disjoint, then the set $\overline{M}_1 \cup \overline{M}_2 \cup \dots \cup \overline{M}_k$ has two intersection numbers $a_1 + a_2 + \dots + a_k - q^{2n-1}$ and $a_1 + a_2 + \dots + a_k$ with respect to hyperplanes in $PG(2n,q^2)$.
(ii) Suppose M_i is contained by M_{i+1}, $i = 1\dots k-1$, then
(a) if k is even, the set $\mathcal{K} = (\overline{M}_k - \overline{M}_{k-1}) \cup (\overline{M}_{k-2} - \overline{M}_{k-3}) \cup \dots \cup (\overline{M}_2 - \overline{M}_1)$ has two intersection numbers $a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_2 - a_1 - q^{2n-1}$ and $a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_2 - a_1$ with respect to hyperplanes in $PG(2n,q^2)$.
(b) if k is odd, the set $\mathcal{K} = (\overline{M}_k - \overline{M}_{k-1}) \cup (\overline{M}_{k-2} - \overline{M}_{k-3}) \cup \dots \cup (\overline{M}_3 - \overline{M}_2) \cup \overline{M}_1$ has two intersection numbers $a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_3 - a_2 + a_1 - q^{2n-1}$ and $a_k - a_{k-1} + a_{k-2} - a_{k-3} + \dots + a_3 - a_2 + a_1$ with respect to hyperplanes in $PG(2n,q^2)$.

Proof: As in Theorem 4 noting that for p some point in $PG(2n,q^2)$, $p \in \overline{M}_i$ if and only if $|p^\perp \cap \overline{M}_i| = a_i - q^{2n-1}$, and $p \notin \overline{M}_i$ if and only if $|p^\perp \cap \overline{M}_i| = a_i$. ∎

J.A. Thas notes that disjoint 0-systems of $H(3,q^2)$ may easily be constructed (personal communication). Take an hermitian curve $\mathcal{C}$ on $H(3,q^2)$ (the classical ovoid). Let V be a set of $q+1$ collinear points on $\mathcal{C}$; V is on a line L. Consider a second classical ovoid $\mathcal{C}'$ containing V. Let V' be the intersection of $H(3,q^2)$ with the polar line L' of L with respect to $H(3,q^2)$. Then $\mathcal{C}$ and $(\mathcal{C}' - V) \cup V'$ are disjoint ovoids of $H(3,q^2)$.

As far as the authors are aware there are no other known examples of disjoint m-systems of a finite classical polar space. However, a "chain" of m-systems of $Q^-(2ts-1,q)$, q even, where each m-system of the chain is contained by the next in the chain can be constructed as follows.

Suppose a field $GF(q^s)$, q even, contains a chain of subfields $GF(q^{s_0}) = GF(q) < GF(q^{s_1}) < \dots < GF(q^{s_k}) = GF(q^s)$, where each s_i necessarily divides s_{i+1}. Let $\xi^2 + \alpha\xi + 1$ be an irreducible polynomial over $GF(q^{st})$, q even. Define a quadratic form on $GF(q^{st}) \oplus GF(q^{st})$ by $f(x,y) = x^2 + \alpha xy + y^2$. Define $Q_i = Tr_{\frac{GF(q^{st})}{GF(q^{s_i})}}(f(x,y))$ for each $i = 0\dots k$. Then each Q_i is a non-degenerate elliptic quadratic form on the vector space $V(2(\frac{s}{s_i})t, q^{s_i})$. Further, $Q_i = Tr_{\frac{GF(q^{s_{i+1}})}{GF(q^{s_i})}}(Q_{i+1})$, hence considering each quadric as a subset of points of the quadric $Q_0 = Q^-(2st-1,q)$, the $(i+1)^{th}$ quadric is contained in the i^{th}.

Consider the set of subspaces

$$\{\{(a,ab) : a \in GF(q^{st})\} : b \in GF(q^{st})\} \cup \{(0,b) : b \in GF(q^{st})\}.$$

It is readily verified that the intersection of this set with a quadric Q_i induces a spread (an $((\frac{s}{s_i})t - 2)$-system projectively) of that quadric in $V(2(\frac{s}{s_i})t, q^{s_i})$, and so by Theorem 2 gives rise to an $(st - s_i - 1)$-system of $Q^-(2st - 1, q)$. Hence we get a chain of $(st - s_i - 1)$-systems of $Q^-(2st - 1, q)$, $i = 0 \dots k$ each element contained by the previous one in the chain.

4 Maximal Arcs in Symplectic Translation Planes

An n-**spread** $\mathcal{S}$ of $PG(2n + 1, q)$ is a set of projective spaces of dimension n such that every point of $PG(2n+1, q)$ lies in exactly one element of $\mathcal{S}$. Equivalently, it is a set of $q^{n+1} + 1$ pairwise disjoint n-dimensional projective subspaces of $PG(2n+1, q)$.

Let $PG(2n + 1, q)$ be embedded as a hyperplane Σ in $PG(2n + 2, q) = \Pi$, and let $\mathcal{S}$ be an n-spread of Σ. Construct a new incidence structure $\pi(\mathcal{S})$ as follows:

The *points* of $\pi(\mathcal{S})$ are the points of $\Pi\backslash\Sigma$, together with the elements of the spread. The *lines* of $\pi(\mathcal{S})$ are the $(n+1)$-dimensional subspaces of Π which intersect Σ in a member of $\mathcal{S}$, together with the line (often denoted L_∞) whose points are the elements of the spread.

The *incidence* relation of $\pi(\mathcal{S})$ is that induced by the incidence in Π.

Then $\pi(\mathcal{S})$ is a translation plane of order q^{n+1}, with translation line L_∞ [3]. The spread is called *symplectic* if there exists a non-degenerate symplectic space such that each element of $\mathcal{S}$ is a generator of the space.

In 1980 J.A. Thas gave the following construction of maximal arcs in certain translation planes of order q^d whose kernel contains $GF(q)$[17].

Theorem 7. *Let $Q^- = Q^-(2n + 1, q)$ be a non-singular elliptic quadric in $PG(2n + 1, q)$, $n > 0$, and let $\mathcal{S}^-$ be an $(n - 1)-$spread of Q^-. Suppose there exists an n-spread $\mathcal{S} = \{s_1, s_2, \dots, s_{q^{n+1}+1}\}$ of $PG(2n + 1, q)$ such that $\mathcal{S}^- = \{Q^- \cap s_1, \dots, Q^- \cap s_{q^{n+1}+1}\}$. Embed $PG(2n+1, q)$ as a hyperplane Σ in $PG(2n+2, q)$ and choose any point $x \in PG(2n + 2, q)\backslash\Sigma$. Let $\mathcal{K}$ be the affine points of the cone with vertex x and base Q^- i.e. the union of the points of $PG(2n+2, q)\backslash\Sigma$ on the lines of $PG(2n + 2, q)$ given by the span of x and each of the points of Q^-. Then $\mathcal{K}$ is the set of points of a degree q^n maximal arc in the translation plane $\pi(\mathcal{S})$ of order q^{n+1} determined by the spread $\mathcal{S}$.*

As Thas notes in [17], spreads of $PG(2n + 1, q)$ and $Q^-(2n + 1, q)$ of the form required for the construction are well known for q even, and the spreads of $PG(2n+1, q)$ are in fact symplectic with respect to the bilinear form induced by $Q^-(2n+1, q)$. In [2], for q odd, it is shown these spreads can not exist.

The point of interest to us here is that the $(n-1)$-spread of $Q^-(2n+1, q)$, q even, is in fact an $(n - 1)$-system of the associated symplectic space $W_{2n+1}(q)$. Hence, for q even, the Thas construction requires that an $(n - 1)$-system of $W_{2n+1}(q)$ is contained by an n-system (spread) of $W_{2n+1}(q)$. This leads us to the main Theorem of this paper.

Theorem 8. *Let M be an m-system of the symplectic polar space $W_{2n+1}(q)$ in $PG(2n + 1, q)$, $n > 0$. Suppose there exists an n-spread $\mathcal{S}$ of $W_{2n+1}(q)$ such that M is contained by $\mathcal{S}$ (considered as an n-system of $W_{2n+1}(q)$). Embed $PG(2n+1, q)$ in $PG(2n + 2, q)$ and choose some point $x \in PG(2n + 2, q)\backslash PG(2n + 1, q)$, and let $\mathcal{K}$ be the set of affine points on the cone with vertex x and base $\overline{M}$. Then $\mathcal{K}$ is the set* *of points of a degree q^{m+1} maximal arc in the translation plane of order q^{n+1} defined by $\mathcal{S}$.*

Proof: We show in $PG(2n+1,q)$ that every subspace of dimension $n+1$ containing an element $s \in \mathcal{S}$ meets the set of points of $\overline{M} - s$ in either 0 or q^{m+1} points. The result then follows by projection.

Let $\perp$ be the polarity of $W_{2n+1}(q)$. For $s \in \mathcal{S}$, let M_s be the (unique) element of the m-system contained in s, and let H be any subspace of dimension $n-1$ of s. Then $H^\perp$ is a subspace of dimension $n+1$ containing s. Restricting $\perp$ to $H^\perp$ gives a degenerate symplectic space with radical $H \cap H^\perp = H$, and the factor space $H^\perp/H$ is isomorphic to the projective line with a (non-degenerate) symplectic geometry on it. The space $H^\perp$ then contains $q+1$ generators $\mathcal{G}$ of $W_{2n+1}(q)$ all on H and partitioning the points of $H^\perp - H$. Note that $s \in \mathcal{G}$.

Now by [13, Theorem 8] every generator of $W_{2n+1}(q)$ meets $\overline{M}$ in $(q^{m+1} - 1)/(q-1) = |M_s|$ points. Suppose M_s is contained within H. Then each element of $\mathcal{G}$ contains M_s and so contains no other points of $\overline{M}$. Hence $H^\perp$ meets $\overline{M} - s$ in 0 points.

Suppose M_s is not contained within H. Then H is a hyperplane of s and so meets M_s in a subspace of dimension $m-1$. Hence H meets $\overline{M}$ in $(q^m-1)/(q-1)$ points. So each generator in $\mathcal{G}$ meets $\overline{M} - H$ in $((q^{m+1}-1)-(q^m-1))/(q-1)$ points. There are q such generators in $\mathcal{G}$ not equal to s (and partitioning $H^\perp - s$) and it then follows that $H^\perp - s$ meets $\overline{M}$ in $q((q^{m+1}-1)-(q^m-1))/(q-1) = q^{m+1}$ points.

It remains to be shown that any line of the projective plane defined by $\mathcal{S}$ meets $\mathcal{K}$ in 0 or q^{m+1} points. An $n+1$ dimensional subspace containing x and an element of $\mathcal{S}$ clearly contains q^{m+1} points of $\mathcal{K}$. Let N be an $n+1$ dimensional subspace containing an element $s \in \mathcal{S}$ but not containing x and not contained in $PG(2n+1,q)$. Then the projection of N through x onto $PG(2n+1,q)$ is of dimension $n+1$ an so meets $\overline{M} - s$ in either 0 or q^{m+1} points. It follows that N meets $\mathcal{K}$ in either 0 or q^{m+1} points. Finally, the line L_∞ does not meet $\mathcal{K}$. ∎

The above Theorem provides a method for constructing maximal arcs in symplectic translation planes if we can find m-systems of symplectic spaces of the right form. The question is then: can we find m-systems such that the resulting maximal arc is not one of those previously constructed by Thas.

The first thing that might be considered would be to take an $(n-1)$-system of $Q^-(2n+1,q^s)$ and an n-system (spread) of the associated $W_{2n+1}(q^s)$, q even, as per the Thas construction, and apply the results of Theorem 3. This would then give an $(sn-1)$-system contained by an $(sn+s-1)$-system (spread) of $W_{2sn+2s-1}(q)$, and hence maximal arcs by the previous Theorem. Unfortunately, the translation plane and the maximal arc obtained are then isomorphic to the original maximal arc and translation plane and so gives nothing new. But suppose we could "twist" the $(sn+s-1)$-system to find another $(sn+s-1)$-system that also contained the $(sn-1)$-system then we might get new maximal arcs. A method of "twisting" symplectic spreads is the subject of the next subsection.

4.1 Kantor's Cousins of Symplectic Spreads and Construction of Maximal Arcs

The following description of the construction of cousins of symplectic spreads follows that given in Kantor's Kerdock set papers [11][12]. See also R.H. Dye [5].

First we show how a spread of $Q^+(4n-1,q)$ may be "sliced" to obtain symplectic spreads of $PG(4n-3,q)$.

Let $Q^+(4n-1,q)$ be a non-degenerate hyperbolic quadric in $PG(4n-1,q)$, q even and $n>1$, with a spread $\mathcal{S}^+$ of totally singular subspaces of dimension $2n-1$. If y is some point not on the quadric, then $y^\perp$ is a $PG(4n-2,q)$ that meets $Q^+(4n-1,q)$ in a parabolic quadric $Q(4n-2,q)$. The point y is the nucleus of $Q(4n-2,q)$. The spread $\mathcal{S}^+$ induces a spread $\mathcal{S}$ of $Q(4n-2,q)$ by intersection. If $PG(4n-3,q)$ is a subspace of $PG(4n-2,q)$ not containing y, then $PG(4n-3,q)$ meets $Q(4n-2,q)$ in a non-degenerate quadric which is necessarily of elliptic or hyperbolic type. Then $PG(4n-3,q)$ provided with the projections of the totally singular subspaces of $Q(4n-2,q)$ is a symplectic space, and the projection through the nucleus of $Q(4n-2,q)$ of the spread $\mathcal{S}$ onto $PG(4n-3,q)$ is a symplectic spread of $PG(4n-3,q)$.

Conversely, for q even, if we start with a symplectic spread of $PG(4n-3,q)$ this can be "expanded" to a spread $\mathcal{S}^+$ of $Q^+(4n-1,q)$ as follows.

In the notation of the previous paragraphs, a symplectic spread of $PG(4n-3,q)$ can be "pulled back" to a spread $\mathcal{S}$ of $Q(4n-2,q)$ since there is a one to one correspondence between the points of $PG(4n-3,q)$ and $Q(4n-2,q)$. For $Q^+(4n-1,q)$ there are two types of maximal totally singular subspaces; two have the same type if and only if their intersection is a subspace of odd dimension. Each element of $\mathcal{S}$ is contained within two maximal totally singular subspaces of $Q^+(4n-1,q)$, one from each class. Choose either class of maximal totally singular subspaces in $Q^+(4n-1,q)$. Form the set $\mathcal{S}^+$ of the elements of that class that contain an element of $\mathcal{S}$; then $\mathcal{S}^+$ is a spread of $Q^+(4n-1,q)$.

Hence we get a correspondence between spreads of the symplectic space $W_{4n-3}(q)$ and $Q^+(4n-1,q)$, q even.

Suppose we take a Desarguesian spread of $W_{4n-3}(q)$, q even, and expand it to a spread $\mathcal{S}^+$ of $Q^+(4n-1,q)$ as above. Such a spread is called the *Desarguesian spread* of $Q^+(4n-1,q)$. Choosing a point $y'' \notin Q^+(4n-1,q)$ of $PG(4n-1,q)$, with $y'' \neq y$, another "slice" $y''^\perp$ of the Desarguesian spread of $Q^+(4n-1,q)$ may be taken which will also give rise to a symplectic spread of some $W_{4n-3}(q)$. Surprisingly such a slice may give rise to a non-Desarguesian spread of the symplectic space.

Kantor [11, Lemma 4.1] identifies four types of slice that can occur:
(I) $y''=y$.
(II) $y \neq y'' < y^\perp$.
(III) $\langle y, y''\rangle$ is a hyperbolic line, that is $\langle y'', y\rangle \cap Q^+(4n-1,q) = Q^+(1,q)$.
(IV) $\langle y, y''\rangle$ is an anisotropic line , that is $\langle y'', y\rangle \cap Q^+(4n-1,q) = Q^-(1,q)$.

These are known as the first, second, third and fourth cousins of the Desarguesian spread respectively [12, Section 4]. Spreads of $W_{4n-3}(q)$, and therefore $PG(4n-3,q)$, arising in different classes give rise to non-isomorphic projective planes, though spreads in the same class need not give rise to isomorphic planes.

More generally we can define the cousins of any symplectic spread of $PG(4n-3, q)$ in correspondence with cases (I) to (IV).

Theorem 9. *Let M be an m-system of a non-degenerate elliptic quadric $Q^-(4n-3, q)$ in $PG(4n-3, q)$, q even and $n > 1$. Suppose that the associated symplectic space $W_{4n-3}(q)$ admits a spread $\mathcal{S}$ such that M is contained by $\mathcal{S}$. Then the m-system gives rise to degree q^{m+1} maximal arcs in q of the projective planes arising from the fourth cousins of $\mathcal{S}$.*

Proof: We show that certain (symplectic) fourth cousin spreads also contain the m-system and so give rise to maximal arcs by Theorem 8.

Embed the elliptic quadric $Q^-(4n-3, q)$ into a parabolic quadric $Q(4n-2, q)$ in $PG(4n-2, q)$. Embed $Q(4n-2, q)$ into a hyperbolic quadric $Q^+(4n-1, q)$ in $PG(4n-1, q)$. Expand the symplectic spread $\mathcal{S}$ of $PG(4n-3, q)$ to a spread $\mathcal{S}^+$ of $Q^+(4n-1, q)$.

The spread $\mathcal{S}$ induces a spread $\mathcal{S}^-$ of $Q^-(4n-3, q)$ by intersection. By assumption, since M is a subset of $Q^-(4n-3, q)$ and $\mathcal{S}$ contains M there is a unique element of M contained within each element of $\mathcal{S}^-$, and so the m-system is contained by $\mathcal{S}^-$ considered as an m-system and an $(2n-3)$-system of $Q^-(4n-3, q)$. Note that $\mathcal{S}^-$ can also be viewed as the intersection of the spread of $Q(2n-2, q)$ (and so the spread $\mathcal{S}^+$) with $Q^-(4n-3, q)$.

Let $\perp$ be the polarity of $PG(4n-1, q)$ arising from $Q^+(4n-1, q)$. Let y be the point such that $y^\perp = PG(4n-2, q)$, i.e. gives rise to the slice of $Q^+(4n-1, q)$ that defines $Q(4n-2, q)$. Define the line l by $l = PG(4n-3, q)^\perp$. It follows that l is an anisotropic line containing y [10, Theorem 22.7.2].

Let $y'' \neq y$ be a point on l. Now $y''^\perp$ also contains $PG(4n-3, q)$ and meets $Q^+(4n-1, q)$ in a non-degenerate parabolic quadric $Q''(4n-2, q)$ with nucleus y''. The spread $\mathcal{S}''$ of $Q''(4n-2, q)$ induced by intersection with $\mathcal{S}^+$ has the property that each element of $\mathcal{S}''$ contains exactly one element of $\mathcal{S}^-$. Hence when $\mathcal{S}''$ is projected from y'' onto $PG(4n-3, q)$, to obtain a spread of $PG(4n-3, q)$, each element of the projected spread contains a unique element of M.

Hence this fourth cousin spread contains the m-system M of $Q^-(4n-3, q)$, and so gives rise to maximal arcs in the associated translation plane. There are q such cousins corresponding to the q points of $l - \{y\}$. ∎

Corollary 1. *Let s, t be positive integers such that $s.t$ is odd, $t > 1$. Then there exist degree $q^{s(t-1)}$ maximal arcs in (at least) q of the fourth cousins of the Desarguesian projective plane of order q^{st}, q even.*

Proof: Let $Q^-(2t-1, q^s)$ be a non-degenerate elliptic quadric in $PG(2t-1, q^s)$, q even. The discussion at the end of Section 3 gives a Desarguesian symplectic spread of $PG(2t-1, q^s)$ that induces a spread (an $(t-2)$-system) of $Q^-(2t-1, q^s)$. In fact these are just the structures used for the Thas 1980 construction of maximal arcs in Desarguesian planes.

Applying Theorems 2 and 3 gives rise to an $(st-s-1)$-system of $Q^-(2st-1, q)$ contained by a symplectic (Desarguesian) spread. The condition $s.t$ odd is required so that $2st-1 = 4n-3$ for some integer n. ∎

In the case that $s = 1$ these are Thas 1980 maximal arcs in the fourth cousins of the Desarguesian plane. However, for $s > 1$ the maximal arcs are new. This follows since the fourth cousins of the Desarguesian spread in $PG(2st-1,q)$ are symplectic with kernel $GF(q)$. But the only maximal arcs known in such planes are the Thas 1980 ones which have different degree to those of the Corollary.

Note that the procedure described in the Corollary can be applied to non-Desarguesian symplectic spreads of $PG(2t-1,q)$ to give maximal arcs. We restricted ourselves to the Desarguesian case since the isomorphism problem for fourth cousins had been solved by Kantor enabling us to identify the maximal arcs as new.

5 Conclusions

We have shown how an m-system of $W_{2n+1}(q)$ that is contained by a spread of $W_{2n+1}(q)$ gives rise to a maximal arc in the symplectic translation plane determined by the spread.

For q even, it would be interesting to have more examples. In particular do there exist m-systems contained by a Desarguesian spread of $W_{2n+1}(q)$ such that the resulting maximal arc is not a Thas 1980 maximal arc or another previously known. In [16], Thas constructs maximal arcs using the Tits ovoid and a symplectic spread of lines tangent to the ovoid in $PG(3,q)$, $q=2^e$, $e>2$, e odd. This can be seen as an 0-system contained by a spread of $W_3(q)$. So there is at least one class of examples that are not related to an elliptic quadric. Are there others?

For q odd, if the symplectic spread is Desarguesian then the results of [1] show that the spread contains no m-system. It would be interesting to know whether an m-system can be contained by a general symplectic spread for q odd.

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