Flag-transitive hyperplane complements in classical generalized quadrangles

A. Pasini S. Shpectorov

Abstract

Let H be a geometric hyperplane of a classical finite generalized quadrangle Q and let $C = Q \setminus H$ be its complement in Q, viewed as a point-line geometry. We shall prove that C admits a flag-transitive automorphism group if and only if H spans a hyperplane of the projective space in which Q is naturally embedded (but with Q viewed as $Q(4,q)$ when $Q = W(q)$, q even). Furthermore, if Q is the dual of $H(4,q^2)$ and H, C are as above, then C is flag-transitive if and only if $H = p^\perp$ for some point p of Q.

1 Introduction

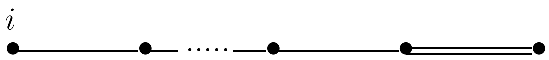
Recall that a *geometric hyperplane* H of a point-line geometry $\Gamma = (P, \mathcal{L})$ is a proper subset of P such that every line of Γ meets H in one or all points. Suppose H is a geometric hyperplane of Γ. Define a new (possibly disconnected) geometry $\mathcal{C}$ by taking as its set of points $P \setminus H$ and as its lines the intersections with $P \setminus H$ of those lines of Γ that are not fully contained in H. The most well-known example of this construction is the affine space wich is constructed in this way from a projective space of the same dimension. Because of this example we call the construction (generalized) *affinization*.

Let now Γ be a diagram geometry and i be one of the types of Γ. With Γ and i we can associate a point-line geometry Γ_i by taking as points of Γ_i all the elements of type i. The lines of Γ_i are defined as follows: each flag of Γ of cotype i defines a line, which is the set of points incident to that flag. Let H be a

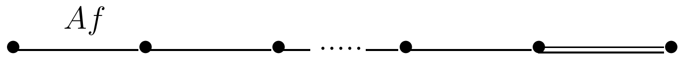
Received by the editors February 1998.
Communicated by J. Thas.
1991 *Mathematics Subject Classification :* 51E12, 51E24.
Key words and phrases : generalized quadrangles, ovoids, maximal subgroups.

geometric hyperplane in Γ_i. In [BP] it is discussed how the affinization of Γ_i by H can be unfolded into a full rank diagram geometry. We will not go into details of this, as they are in general somewhat complicated. It suffices to mention that the affinization for diagram geometries constitutes a natural way of constructing new diagram geometries from old ones. It is especially interesting to apply this construction to such classical geometries as spherical buildings.

There have been a number of papers studying the geometries obtained from thick spherical building geometries by affinization. For example, if Γ is a building of type B_n or C_n, with $n \geq 3$, then with the following choice of i

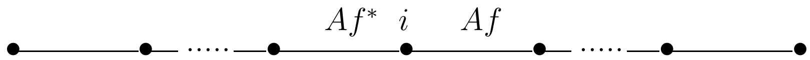


the affinization gives us a geometry called an affine polar space and belonging to the following diagram

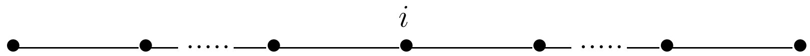


The affine polar spaces were characterized as point-line geometries in [CS].

As another example we can mention [C], in which the geometries with the diagram

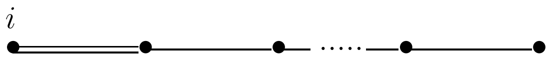


were classified modulo a minor assumption that there are enough points on each line. Notice that this diagram appears when affinization is applied to the A_n building

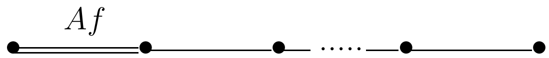


There are a few more results in the same spirit. However, in all the cases considered so far the effect of the affinization on the diagram was to replace one or more simple bonds by strokes $\bullet\!\overset{Af}{\text{———}}\!\bullet$, or $\bullet\!\overset{Af^*}{\text{———}}\!\bullet$, which is the dual of the first one.

Consider now the situation where Γ is a building of type B_n or C_n and i is chosen as follows:



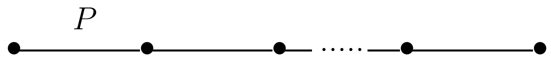
Then the geometry Γ_i is called a dual polar space. If we apply affinization in this case, we end up with a geometry which we call an *affine dual polar space* and which has a diagram of the following kind.



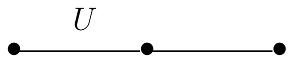
(†)

The leftmost stroke in the diagram is a new one and it represents a rank two geometry obtained from a generalized quadrangle by affinization. (This kind of rank two geometries was briefly mentioned in [P], Chapter 8, where they were called *affine generalized quadrangles*.)

The class of affine dual polar spaces is interesting on its own. However, it is even more remarkable that a number of sporadic geometries belong to the same kind of diagram. For example, one of the affine generalized quadrangles arriving from the generalized quadrangle $W(2)$ of order 2 is the Petersen graph geometry, usually denoted by P. (This geometry is the complement of the elliptic quadric, which is a geometric hyperplane of $W(2)$.) The geometries with the diagram



are called the Petersen type geometries. The flag-transitive Petersen type geometries were classified in [IS]. They are 8 geometries related to sporadic groups, ranging from the Mathieu group M_{22} to the Baby Monster BM and its non-split extension $3^{4371} \cdot BM$. As a second example, notice that the complete bipartite graph $K_{6,6}$ minus a matching (call this graph U) arises as a hyperplane complement in the generalized quadrangle $Q^-(5,2)$. The flag-transitive geometries with the diagram



are all known as a corollary to [PTs]. There are two such geometries, with the automorphism groups $2 \times \mathrm{Aut}\, M_{22}$ and $U_4(3).2^2$.

All this makes the classification of (flag-transitive) geometries with the diagram (†) a very interesting problem. As a first step, we classify in this paper all finite flag-transitive affine generalized quadrangles arising from classical generalized quadrangles. Our result is as follows.

Theorem 1. *Suppose an affine generalized quadrangle C is obtained by removing a geometric hyperplane H from a thick finite classical generalized quadrangle Q. Then C is flag-transitive if and only if one of the following holds.*

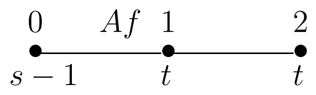
(1) *Q is embeddable and if V is the universal embedding space for Q then H is the intersection of Q with a hyperplane of V;*

(2) *Q is the dual of $H(4,q^2)$ and $H = p^\perp$ for some point p of Q.*

Notice that, according to (2), we consider the dual of the Hermitian quadrangle $H(4,q^2)$ (the notation of [PT]) a classical generalized quadrangle. Also, for a classical Q and a point $p \in Q$, one can show that $p^\perp$ is always a geometric hyperplane whose complement is flag-transitive (cf. Proposition 2.1 (1)). However, when Q is embeddable, the hyperplane $p^\perp$ is, in fact, included in (1).

Let Q be a generalized quadrangle of order (s,t). According to Lemma 2.2, every hyperplane H of Q, which is not of the type $p^\perp$ is either a subquadrangle of order (s,t') (a *full* subquadrangle), or an ovoid. Tables 1 and 2 give explicit lists of the hyperplanes from Theorem 1 of these two kinds.

It is, of course, interesting to list also all the flag-transitive groups of automorphisms of affine generalized quadrangles. However, in some cases a complete description would be somewhat tedious. Instead, we only consider the groups that are of primary interest to us, namely, those which might show up in the classification of flag-transitive geometries of rank at least 3, with the diagram (†). Without loss, we may restrict ourselves to the rank 3 case:



Q	H
$Q(5,q)$	$Q(4,q)$
$Q(4,q)$	$Q(3,q)$ (a grid)
$W(q)$, q odd	none
$H(4,q^2)$	$H(3,q^2)$
$H(3,q^2)$	none
dual of $H(4,q^2)$	none

Table 1: Full subquadrangles

Q	H
$Q(5,q)$	none
$Q(4,q)$	elliptic quadric
$W(q)$, q odd	none
$H(4,q^2)$	none
$H(3,q^2)$	hermitian unital
dual of $H(4,q^2)$	none

Table 2: Ovoids with flag-transitive complement

The stabilizer of a point (type 0) induces on the point residue a group containing $L_3(t)$, or one of the exceptional flag-transitive groups $Frob(7,3)$ (for $t=2$) and $Frob(73,9)$ (for $t=8$). In the principal case, where the induced group contains $L_3(t)$, the stabilizer of a $\{0,2\}$-flag induces on the $t+1$ elements of type 1 (lines) incident to that flag, a group containing $PGL(2,t)$.

Theorem 2. *Suppose (Q,H) is one of the pairs from the conclusion of Theorem* 1 *and $G = \mathrm{Aut}\, C$, $C = Q \setminus H$. Suppose $F \leq G$ has the property that, for every point $c \in C$, the stabilizer F_c induces on the set of lines on c at least the group $PGL_2(t)$. Then F contains G^∞. Furthermore, $Q \ncong H(4,s)$.*

As it will be clear from the proof of the above theorem (Section 5), in order to show that $F \geq G^\infty$ we only need to know that the action of F_c on the $t+1$ lines on c contains $L_2(t)$. Note also that, if $t=2$ or 3 then in some case G is solvable and, hence, the claim of the theorem trivializes.

2 Hyperplanes in generalized quadrangles

Let Q be a generalized quadrangle of classical type (including the dual of $H(4,q^2)$), of order (s,t) with $s,t>1$.

Let us first check that the hyperplanes H mentioned in (1) and (2) of Theorem 1 indeed lead to flag-transitive complements.

Proposition 2.1. (1) *For a point $p \in Q$, the complement of $H = p^{\perp}$ is flag-transitive.*

(2) *Suppose Q is embeddable, V is the universal embedding space of Q and H is the intersection of Q with a hyperplane of V. Then the complement of H is flag-transitive.*

Proof. For $p \in Q$, it is very well-known that $p^{\perp}$ is a geometric hyperplane in Q. (In fact, this is true for all generalized quadrangles Q, not only the classical ones.) If Q is classical then $\operatorname{Aut} Q$ acts transitively on the apartments, and the stabilizer of an apartment acts transitively on the flags inside the apartment. Let $\{q, L\}$ and $\{q_1, L_1\}$ be two flags of the complement C of $p^{\perp}$. Then there is an apartment A containing p, q and L, and, symmetrically, another apartment A_1 containing p, q_1 and L_1. By the above there is an automorphism taking A to A_1, q to q_1 and L to L_1. Since p is the opposite of q in A, and opposite of q_1 in A_1, this automorphism fixes p and hence it stabilizes H. Thus, C is flag-transitive.

Let now Q, V and H be as in (2). Then Q is defined by a quadratic, unitary or alternating form on V. In all the cases there is a bilinear (or sesquilinear, in the Hermitian case) form Φ defining the collinearity on Q. Let us first consider the case where the form Φ is non-degenerate. Then H is the intersection of Q with $h^{\perp}$ for some vector $h \in V$. (Here $\perp$ is taken with respect to Φ.) Choose two flags $\{p, L\}$ and $\{p_1, L_1\}$ in the complement C. Let $p = \langle v \rangle$ and $L = \langle v, u \rangle$. Similarly, $p_1 = \langle v_1 \rangle$ and $L_1 = \langle v_1, u_1 \rangle$. We may assume that u and u_1 are chosen in the intersection of L and L_1 with $h^{\perp}$, respectively. Thus, u is perpendicular to both h and v, and similarly, u_1 is perpendicular to h and v_1. On the other hand, v and v_1 are not perpendicular to h as v and v_1 are contained in C. Without loss of generality we may assume that $\Phi(h, v) = \Phi(h, v_1)$. Now all the assumptions of the Witt theorem (see [B], Chapter IX, 4.3) are satisfied for (h, v, u) and (h, v_1, u_1). (If Q is defined by a quadratic or unitary form then we also have to verify the values of that form on v and v_1, as well as, on u and u_1. However, all these vectors are singular, so that the values are all equal to 0.) In particular, there is an automorphism of Q that preserves $\langle h \rangle$ (and hence also H) and takes p to p_1, and L to L_1.

The only remaining case is where $Q = Q(4, q)$, q even. Then V is of dimension 5 and the form Φ is degenerate. However, the orbits of $O_5(q) \leq \operatorname{Aut} Q$ on the hyperplanes of V are known. In particular, H is one of the following: $p^{\perp}$ for a point $p \in Q$, or an elliptic quadric, or a hyperbolic quadric. The first case has been dealt with in the first paragraph of this proof. In the latter two cases, let us realize Q as a symplectic generalized quadrangle $W(q)$ embedded in a 4-dimensional vector space W endowed with a non-degenerate alternating form Ψ. Now H is simply the set of all singular points with respect to a quadratic form f compatible with Ψ. Choose two flags $\{p, L\}$ and $\{p_1, L_1\}$ in the complement C and assume that $p = \langle v \rangle$, $L = \langle v, u \rangle$, $p_1 = \langle v_1 \rangle$ and $L_1 = \langle v_1, u_1 \rangle$. Since q is even, every element in $GF(q)$ is a square, and hence, we may assume that $f(v) = f(v_1)$. Also u is perpendicular to v and u_1 is perpendicular to v_1. Without loss of generality we may choose u and u_1 to be singular. Again, all the assumptions of the Witt theorem are satisfied. Therefore, there exists a linear transformation of W preserving f (consequently also H; as well as Ψ and Q) and taking v to v_1 and u to u_1. ∎

This proposition establishes the "if" part of Theorem 1. For the "only if" part, we need to classify all the geometric hyperplanes H with a flag-transitive complement. We will do it case by case in Sections 3 and 4. As the first step, and also to indicate the main partition into cases, we present the following general result, which we borrow from [PT].

Lemma 2.2. *If H is a proper geometric hyperplane of Q then H is one of the following:*

(1) *$p^{\perp}$ for a point $p \in Q$;*

(2) *a subquadrangle of order (s, t'), $t' < t$;*

(3) *an ovoid.* ∎

We know already that in case (1) C is flag-transitive. In the next section we consider the (easier) case (2), where H is a full subquadrangle of Q. The last case will be treated in Section 4.

We close this section with the following lemma.

Lemma 2.3. *Let H be a geometric hyperplane of Q and $C = Q \setminus H$. Then* $\operatorname{Aut} C$ *coincides with the subgroup of* $\operatorname{Aut} Q$ *consisting of all automorphisms stabilizing H.*

Proof. It suffices to show that every automorphism of C extends in a unique way to an automorphism of Q. In turn, to prove that, we only need to show how to recover Q in terms of C. Namely, we need to recover the points in H, and the collinearity on H and between H and C.

We have two cases to consider:

CASE 1: *H does not have deep points.*

(Recall, that a *deep point* of a subset S of Q is a point of S that is not collinear to any point outside S.)

In this case every point of H is contained in a line of C. Since a line of C contains only one point from H, we can identify the points of H with the set of lines of C it belongs to. This defines a partition of the line set of C, and we need to show how to recover this partition solely in terms of C. This can be done as follows: two lines L_1 and L_2 of C contain the same point of H if and only if no point from $L_1 \cap C$ is collinear to a point from $L_2 \cap C$.

This gives us the points of H (the classes) as well as the collinearity between H and C. It remains to find out when two points of H are collinear.

Let p_1 and p_2 be two points of H (that is, two classes of lines of C). We claim that p_1 and p_2 are not collinear if and only if there is a point in C that is collinear with both of them. First of all, if p_1 and p_2 are collinear then the whole line through p_1 and p_2 is contained in H and hence no point of C is collinear with both of p_1 and p_2. Secondly, suppose p_1 and p_2 are not collinear. Since p_1 is not a deep point of H, there is a line L on p_1 not contained in H. Then p_1 is the only point of L that is in H. By the basic properties of generalized quadrangles p_2 is collinear with a point on L and that point is, clearly, not p_1. Hence there is a point in C (on L!) collinear with both p_1 and p_2.

Thus, we have also recovered the collinearity on H.

Case 2: *H has a deep point.*

This means we are in the subcase (1) of Lemma 2.2. In particular, the deep point is unique. As in the previous case, we can recover the non-deep points of H (as classes of lines from C), as well as the collinearity between the non-deep points of H and the points of C, and the collinearity on the set on non-deep points of H. However, this is essentially all we need, because the deep point is unique and it is collinear with all the non-deep points of H. ■

3 Subquadrangles

In the case (2) of Lemma 2.2 it follows from Chapter 2 of [PT], that s can never be greater than t, which excludes the cases $Q = H(3, q^2)$ and the dual of $H(4, q^2)$.

Since Q is classical and Q is not dual to $H(4, q^2)$, we can view it as embedded in the natural way in a projective space $P = PG(d, s)$. If $Q = W(q)$, q even, then we view it as $Q(4, q)$ and embed into $PG(4, q)$.

Let W be the vector space underlying P. This means that the dimension of W is $d+1$ and W is defined over $GF(s)$. The generalized quadrangle Q is then the set of all totally singular (isotropic) 1- and 2-spaces of W with respect to a non-degenerate quadratic (or alternating, or unitary) form ϕ.

Clearly, the embedding of Q into P induces also an embedding of a subquadrangle Q' of Q (as in (2)) into a subspace P' of P. It follows from Buekenhout and Lefèvre [BL] that Q' is classical itself. (Here we have to allow $t' = 1$, a grid.) Furthermore, from the same [BL], it follows that the embedding of Q' into P' is the natural one, that is, it is defined by a quadratic (or alternating, or unitary) form ϕ' on the subspace W' of W underlying P'. Since ϕ' is defined uniquely up to a scalar, it follows that ϕ' differs from the restriction of ϕ to W' by a scalar factor. (The restriction of ϕ cannot be null, because in that case Q' would be fully contained in a line of Q; then Q' is a point or a line, and one can easily see that neither is a hyperplane of Q.) In particular, $W' \neq W$ (otherwise $Q' = Q$!) So Q' is the intersection of Q with a proper subspace P' of P.

Table 1 shows all the possible Q' in the case where W' has codimension 1 in W. We now want to argue that this is always the case, so that all the full subquadrangles Q' are, indeed, shown in Table 1.

Suppose P' is not a hyperplane. Let P'' be a hyperplane of P that contains P' and one further point $p \in Q \setminus Q'$. Then $Q'' = Q \cap P''$ is a geometric hyperplane of Q properly containing Q'. It follows from Lemma 2.2 that Q'' is also a subquadrangle; namely, it is one of those listed in Table 1. Now also Q' is a full subquadrangle of Q''. If $Q' \neq Q''$ then we repeat the above and construct a hyperplane P''' of P'' and a subquadrangle $Q''' = Q \cap P'''$, such that Q' is contained in Q'''. However, as follows from Table 1, the only possibility for the sequence (Q, Q'', Q''') is $(Q(5, q), Q(4, q), Q(3, q))$ (cf. [PT]). However, $Q(3, q)$ is not a geometric hyperplane of $Q(5, q)$ and so Q' cannot be a geometric hyperplane of Q in that case, either. Thus, P' is indeed a hyperplane of P and, therefore, all the possibilities for Q' in Q are already listed in Table 1.

4 Ovoids

It remains to consider the case where our geometric hyperplane $H = O$ is an ovoid (cf. Lemma 2.2). The classification of all ovoids in classical generalized quadrangles is a hard problem, so, unlike the previous section, we will have to use the condition that the complement $C = Q \setminus O$ is flag-transitive.

We prove the following.

Proposition 4.1. *Suppose O is an ovoid in a classical generalized quadrangle Q, such that $C = Q \setminus O$ is flag-transitive. Then either $Q = Q(4, q)$ and O is the elliptic quadric, or $Q = H(3, q^2)$ and O is the Hermitian unital.*

Proof. In three cases, namely for $Q = Q(5, q)$, $W(q)$, q odd, and $H(4, q^2)$, it is known that no ovoid exists (cf. [PT], Section 3.4). Thus, we only need to handle the cases $Q = Q(4, q)$, $H(3, q^2)$ and the dual of $H(4, q^2)$. In all cases the size of an ovoid must be equal $st + 1$, so that in the above cases we obtain the size $q^2 + 1$, $q^3 + 1$ and $q^5 + 1$, respectively.

Let $\tilde{A} = \mathrm{Aut}\, Q$ and $G = \mathrm{Aut}\, C$. By Lemma 2.3, $G = Stab_{\tilde{A}}(O)$. Let $A_0 = F^*(\tilde{A})$ be the simple group involved in $\tilde{A}$ and define $A = A_0 G$. Let M be a maximal subgroup of A that contains G. Then M falls into one of several types, as described by Aschbacher [A] and detailized by Kleidman and Liebeck [KL]. We now plan to consider the possibilities for M case by case. Notice that, because of the flag-transitivity on C,

(∗) *G is transitive on the lines of Q and it has two orbits on the points of Q, of sizes $st + 1$ (the ovoid) and $s(st + 1)$ (the complement). Accordingly, either $M = G$ and then it has the same orbits, or $G < M$ and then M is transitive on the points and the lines of Q.*

(In fact, in the latter case M is even flag-transitive on Q; cf. [S].)

We now consider the cases. We follow the definitions and the notation of [KL]. The families of maximal subgroups $\mathcal{C}_i$, $i = 1, \ldots, 8$ and $\mathcal{S}$ are defined in [KL] in terms of certain substructures in the natural module V for A, that M is to stabilize. Notice that when Q is the dual of $H(4, q^2)$ then V is still the natural 5-dimensional module for $A_0 = U_5(q)$. Also, because of the way the tables in [KL] (we use Tables 3.5 B,C and D) are composed, when $Q = Q(4, q)$, q even, we have to take as V the natural 4-dimensional module for $PSp_4(q) \cong \Omega_5(q)$, the points (lines) of Q being 2–dimensional totally isotropic (resp. the 1–dimensional) subspaces of V.

Case 1: *$M \in \mathcal{C}_1$, i.e., M is the stabilizer of a proper subspace of V.*

If M is a maximal parabolic then it stabilizes a point or a line of Q which contradicts (∗). The remaining subcases are as follows:

(1.1) If $Q = Q(4, q)$, q odd, then M can be the stabilizer of a non-degenerate subspace $U < V$ of dimension 1 or 3. Clearly, M then also stabilizes $U^\perp$ which has dimension 4 and 2, respectively. If $\dim U = 3$ then U contains $q + 1$ points of Q. Hence M has an orbit of size at most $q + 1$, contradicting (∗). If $\dim U = 1$ and the form on $U^\perp$ has the plus type then $U^\perp$ contains $2(q+1)$ lines of Q, which again contradicts (∗). Finally, if that form is of the minus type then the singular points inside $U^\perp$ form a known ovoid—the elliptic quadric.

If q is even then recall that V is the natural module for $Sp_4(q)$, and this type of maximal subgroups does not occur (except for the parabolics, of course).

(1.2) If $Q = H(3, q^2)$ then M can be the stabilizer of a non-singular 1-dimensional subspace U. Then $U^\perp$ contains $q^3 + 1$ singular points and this is a known example of ovoid—the Hermitian unital.

(1.3) If Q is the dual of $H(4, q^2)$ then M stabilizes a non-degenerate subspace $U < V$ of dimension 1 or 2. Then $U^\perp$ contains some but not all of the singular points (which are lines of Q!), thus contradicting $(*)$.

Case 2: *$M \in \mathcal{C}_2$, i.e., M is the stabilizer of a direct sum decomposition of V with parts of equal dimension m.*

(2.1) If $Q = Q(4, q)$, q odd, we can only have 5 pairwise orthogonal non-singular 1-dimensional summands U_i. Table 3.5 D of [KL] indicates that here q must be prime. For a 3-subset $\{i, j, k\} \subset \{1, \ldots, 5\}$, define $U_{ijk} = U_i + U_j + U_k$. Then M permutes the 10 subspaces U_{ijk} and each of these subspaces contains exactly $q + 1$ points of Q. Hence M has an orbit of size at most $10(q + 1)$. Together with $(*)$ this implies that $q = 3$, 5 or 7. Furthermore, $q(q^2 + 1)$ (the size of C) divides the order of G, hence the order of M. The latter divides $(q - 1)^5 \cdot 5!$. This leaves the only possibility $q = 3$. However, if $q = 3$ then M (which is isomorphic to $2^4 : S_5$ or $2^4 : A_5$) is a known case of a flag-transitive action on Q (cf., [S]). Clearly, $G \neq M$. Since $q(q^2 + 1)$ divides the order of G, we see that G necessarily involves A_5 and, therefore, $G \cong A_5$ or S_5. Choose a basis u_i, $i = 1, \ldots, 5$ in V, such that $U_i = \langle u_i \rangle$. The full preimage of G in $GO(5, q)$ contains A_5 and so we can assume that $\{u_i\}$ is an orbit of that A_5. Then the singular points are represented by the vectors having support of size 3. The group G has two orbits on the singular points, represented by the vectors with coordinates $(1, 1, 1, 0, 0)$ (orbit of length 10) and $(1, 1, -1, 0, 0)$ (orbit of length 30). One can easily check that the singular points in the first orbit are pairwise non-collinear and hence they indeed form an ovoid. Moreover, these 10 points are all perpendicular to the point $(1, 1, 1, 1, 1)$, which means that this ovoid is the classical elliptic quadric (see Case 1). Since the full stabilizer of the elliptic quadric involves $A_6 \cong \Omega_4^-(3)$, we obtain a contradiction.

If q is even, then V is the 4-dimensional symplectic space. Here (cf. Table 3.5 C of [KL]) M stabilizes a decomposition $V = U_1 \oplus U_2$ where U_i are 2-spaces, which are either non-isotropic mutually orthogonal, or both totally isotropic. In either case, $U_1 \cup U_2$ contains some but not all 1-spaces, which are, in fact, lines of Q. Thus, M is not transitive on the lines of Q, in contradiction with $(*)$.

(2.2) If $Q = H(3, q^2)$ then there are two possibilities. If M stabilizes a decomposition $V = U_1 \oplus U_2$ (with U_i either mutually orthogonal non-degenerate 2-spaces, or disjoint totally singular 2-spaces), then $U_1 \cup U_2$ contains exactly $2(q+1)$ (respectively, $2(q^2+1)$) points of Q. However, this is always less then the size of the larger orbit of G, $q^2(q^3+1)$, and never equal to the size of the smaller orbit, q^3+1; a contradiction. In the other case, M stabilizes a decomposition $V = U_1 \oplus \ldots \oplus U_4$, where U_i are mutually orthogonal nonsingular 1-spaces. Each 2-subspace $U_{ij} = U_i + U_j$ contains $q+1$ points of Q. Therefore, the smaller orbit of G must have size at most $6(q+1)$. The inequality $q^3 + 1 \leq 6(q + 1)$ forces $q = 2$. However, if $q = 2$ then the two orbits must have size 9 and 36, which does not allow for an invariant set of $6(2+1) = 18$ points.

(2.3) If Q is the dual of $H(4, q^2)$ then the only possibility is that M stabilizes a decomposition $V = U_1 \oplus \ldots \oplus U_5$, with mutually orthogonal nonsingular 1-spaces U_i. The union of the subspaces U_{ij}, defined as above, give us then an invariant set of singular 1-spaces of size $\binom{5}{2}(q+1) = 10(q+1)$. This is less than the total number of singular 1-spaces and, thus, M cannot be transitive on the lines of Q.

Case 3: *$M \in \mathcal{C}_3$, i.e., the action of M on V is not absolutely irreducible and, hence, V possesses an M-invariant structure of an n/r-dimensional space over $GF(q^r)$ for some prime r.*

(3.1) If $Q = Q(4, q)$, q odd, then this case is impossible, since Table 3.5 D of [KL] forbids $r = n$. So we assume that q is even. Then V has dimension 4 and, therefore, $r = 2$. Furthermore, from Table 3.5 C of [KL] we read off that M stabilizes an alternating $GF(q^2)$-form f on V. The 1-dimensional spaces with respect to f are totally isotropic 2-spaces over $GF(q)$. They form a spread in the dual of Q, hence, an ovoid in Q. We claim this ovoid is the usual elliptic quadric.

We can always assume that f is expressed as $X_1Y_2 - X_2Y_1$. (Here the coordinates are taken in $GF(q^2)$.) Consequently, given an irreducible polynomial $t^2 - t - a$ over $GF(q)$, the alternating form g over $GF(q)$ left invariant by A admits the following expression with respect to a suitable basis:

$$x_1y_4 + x_2y_3 + x_2y_4 - x_3y_2 - x_4y_1 - x_4y_2.$$

The spread, say S, stabilized by M, consists of the 2-spaces of V spanned by the following pairs of vectors: (u_1, u_2, v_1, v_2) and $(au_2, u_1 + u_2, av_2, v_1 + v_2)$. Now it is a computational exercise to check that the points on the $q + 1$ lines of S meeting an arbitrary $GF(q)$ line $L \notin S$ satisfy a nontrivial homogeneous quadratic equation, so that they form a hyperbolic quadric. This is known as the regularity property of S, and by [PT], Section 3.4, S is an elliptic quadric in Q.

(3.2) If $Q = H(3, q^2)$ then this case does not occur, since according to Table 3.5 C of [KL] the prime r must be at least 3. Thus, assume that Q is the dual of $H(4, q^2)$. In this case, $r = 5$ is the only possibility. Hence M stabilizes the structure of a 1-dimensional $GF(q^{10})$-space on V. Furthermore, it preserves a non-degenerate unitary form on that space. Thus, the order of M is a divisor of $(q^5 + 1)10f$, where $q = p^f$, p prime. On the other hand, $G \leq M$ has to act transitively (by $(*)$) on $(q^5 + 1)q^3$ points of C. Hence q^3 has to divide $10f$, which is clearly impossible.

Case 4: *$M \in \mathcal{C}_4$, i.e., M stabilizes the factors of a decomposition of V into a tensor product, $V = U \otimes W$.*

Clearly, we must have $n = \dim V = mk$, where $m = \dim U$ and $k = \dim W$. Also, the cases $m = 1$ and $k = 1$ do not lead to proper subgroups. Thus, this type of maximal subgroups does not occur when $Q = Q(4, q)$, q odd, or the dual of $H(4, q^2)$. Let us consider the remaining cases.

(4.1) If $Q = Q(4, q)$ with q even, then $m = k = 2$. However, Table 3.5 C of [KL] requires also that one of these numbers be at least 3. So, no such maximal subgroup arises in this subcase.

(4.2) If $Q = U(3, q^2)$ then again $m = k = 2$. However, Table 3.5 B of [KL] prescribes now that one of these numbers be strictly less than $\sqrt{n} = 2$. So this is impossible, too.

CASE 5: *$M \in \mathcal{C}_5$, i.e., M is a group defined over a proper subfield $GF(q_0)$ of $GF(q)$ (or $GF(q^2)$ for the Hermitian generalized quadrangles).*

There are a variety of subcases in this case; however, in all of them except for one, M stabilizes a proper subquadrangle of Q. This makes it impossible for M to act transitively on the lines of Q.

The exceptional case is where $Q = H(3, q^2)$ and M stabilizes an elliptic quadric in a 4-dimensional $GF(q)$-subspace of V. (Recall that V is itself defined over $GF(q^2)$.) The points of this quadric number $q^2 + 1$, and they are points of Q. Thus, M has on the point set of Q an orbit of size at most $q^2 + 1$, which contradicts $(*)$.

CASE 6: *$M \in \mathcal{C}_6$, i.e., M originates from the normalizer of an extraspecial group acting on V faithfully and absolutely irreducibly.*

According to Tables 3.5 C and D of [KL], this case never occurs if $Q = Q(4, q)$.

(6.1) If $Q = H(3, q^2)$ then, by specializing to our subcase the conditions from Table 3.5 B of [KL], we obtain that M contains a normal subgroup $2^4.A_6$. Also, q must be the second power of an odd prime p. The order of M has to divide $2^2 \cdot 6! \cdot 2$. Also, it must be divisible by $q^2(q^3 + 1)$. This forces p^4 to divide $6!$, which is a contradiction.

(6.2) If Q is the dual of $H(4, q^2)$ then, similarly, we observe that M must normalize a subgroup $5^2.SL(2, 5)$; also that $q = p^f \equiv 1 \bmod 5$ with $f = 2$ or 4. Similarly to the previous case, on the one hand, $|M|$ must divide $5^2 \cdot 4 \cdot 5!$. On the other hand, it is divisible by $q^3(q^5 + 1)$. This forces q to be even. However, in that case $f = 4$ and 2^{12} must divide M, a contradiction.

CASE 7: *$M \in \mathcal{C}_7$, i.e., M stabilizes a decomposition of V as a tensor product $V = U_1 \otimes \ldots \otimes U_t$ with factors of the same dimension m.*

This is only possible when $n = 4$. Hence $Q = Q(4, q)$, q even, or $H(3, q^2)$. According to Tables 3.5 B and C of [KL], in no case this leads to a maximal subgroup.

CASE 8: *$M \in \mathcal{C}_8$, i.e., M is a classical group with V being its natural module.*

This case is nonempty only if $Q = Q(4, q)$, q even. Recall that in that case V is a 4-dimensional symplectic module. The group M is then an orthogonal group of dimension 4 and type plus or minus. Correspondingly, either M stabilizes a subquadrangle (a grid) in Q (plus type), or it stabilizes a set of $q^2 + 1$ lines (a spread) in Q (minus type). In both cases M is not transitive on the lines of Q.

CASE 9: *$M \in \mathcal{S}$, i.e., M is an almost simple group, and if $L = F^*(M)$ then V is an absolutely irreducible projective L-module over the field $F = GF(q)$ or $GF(q^2)$ (the latter for the unitary quadrangles Q). Furthermore, V cannot be realized over any proper subfield of F.*

In principle, the list of simple groups having low-dimensional (dimensions 4 and 5 is all we need) projective representations, is known. However, we could not find, wherever we looked, a complete list given in a clear and convenient form. The closest approximation to our needs is Chapter 5 of [K], where the complete lists of maximal subgroups of low-dimensional classical *simple* groups are given. Unfortunately, there seems to be no reason to assume that in our situation $M \cap A_0$ is necessarily a maximal subgroup of A_0. However, it is maximal in a maximal subgroup of a maximal subgroup of ... of A_0. Thus, by 'inductively' looking through the lists of subgroups of maximal subgroups [K], one can compile the list of the possible groups L: $L_2(q_0)$, $Sz(q_0)$, A_5, A_6, A_7, $L_2(7)$, $L_2(11)$, $L_3(4)$ and $U_4(2)$. In the first two cases q_0 is restricted by the condition that q^2 (respectively, q) is a power of q_0.

We now consider these possibilities case by case. Let $q = p^f$, p a prime.

(9.1) Suppose that $L \cong L_2(q_0)$ and q^2 is a power of q_0. Let $q_0 = p^m$, so that $2f = mr$ for some integer r. The order of M divides $|\mathrm{Aut}\, L| = q_0(q_0^2 - 1)m$. On the other hand, $|M|$ has to be divisible by the number of flags in Q, $s(st+1)(t+1)$, which, depending on the type of Q, is equal to $q(q^2+1)(q+1)$, $q^2(q^3+1)(q+1)$ or $q^3(q^5+1)(q^2+1)$. Clearly, $m \leq q_0$, so $|\mathrm{Aut}\, L| < q_0^4$. This, together with the divisibility requirement above, yields that $q_0 > q$, *i.e.*, $q_0 = q^2$. Still, the last case, where Q is the dual of $H(4, q^2)$, is impossible, because the order of $\mathrm{Aut}\, L$ is less than the number of flags. Furthermore, Q cannot be $H(3, q^2)$, either, because if $q^2(q^3+1)(q+1)$ divides $|\mathrm{Aut}\, L| = q^2(q^4-1)m$ then q^2-q+1 divides $(q-1)(q^2+1)m$. As $\gcd(q^2-q+1, (q-1)(q^2+1)) = 1$, we conclude that q^2-q+1 divides m. However, q^2-q+1 is odd, and the odd part of m divides f, which is no greater than $q = p^f$. This is a contradiction.

Thus, $Q = Q(4, q)$ and $L \cong L_2(q^2)$. Recall that in the case $Q = Q(4, q)$ we assume the dual point of view, so that V is a 4-dimensional module over $F = GF(q)$, and there is an invariant non-degenerate alternating form defined on V.

The irreducible projective modules of $L \cong L_2(q^2)$ in the defining characteristic are well-known. See, for example, [GLS], Examples 2.8.10ab. We read off from there that V is a $GF(q)$-realization of one of the following modules: For $p \geq 5$, there is a basic module of dimension 4; it can be constructed as the module on the space of homogeneous polynomials in two variables of degree 3. Any other irreducible 4-dimensional module is a tensor product of two natural modules for $SL(2, q^2)$, twisted by two different field automorphisms (the trivial automorphism included).

Suppose first that $p \geq 5$ and V is a $GF(q)$-realization of a basic module V_b. By a simple computation, the trace of $A \in SL(2, q^2)$ acting on V_b is equal to $t^3 - 2t$, where t is the trace of A as an element of $SL(2, q^2)$. Since t can be any element of $GF(q^2)$, the number of different values taken by the trace of A in the action on V_b is at least $q^2/3$ which is certainly greater than q. Therefore, V_b cannot be realized over $GF(q)$. This contradiction rules out the basic module.

Suppose now that V is a $GF(q)$-realization of the tensor product module $V_{ij} = U^{(p^i)} \otimes U^{(p^j)}$, where U is the natural module of $SL(2, q^2)$, and (p^i) and (p^j) are two different field automorphisms; namely, the ones defined by taking the p^ith and p^jth powers in $GF(q^2)$. Up to a field automorphism, we can assume that $i = 0$ and $j \leq \frac{2f}{2} = f$. Then the trace of $A \in SL(2, q^2)$ on $V_j = V_{0j}$ is equal to $t^{p^i + p^j} = t^{1+p^j}$, where t is, as above, the trace of A as an element of $SL(2, q^2)$. Since V is defined over $F = GF(q)$, every value of trace should be in F. The number of different values that the polynomial x^{1+p^j} takes on $GF(q^2)$ (recall that t may be any element of $GF(q^2)$) is at least $1 + \frac{p^{2f}-1}{p^j+1}$ which is less or equal than q only if $j = f$ (and then we have equality). Thus, V is defined uniquely up to conjugation by field automorphisms.

If $p = 2$ then V has to be the same as (a conjugate of) the natural module for $O_4^-(q) \cong L_2(q^2)$. This configuration has been considered and eliminated in Case 8. Thus, we only need to consider the case where q is odd. We claim that in that case no nontrivial alternating form on V is left invariant by L. Indeed, without loss of generality we may assume that V is a $GF(q)$-realization of the $GF(q^2)$-module $W = V_f = U \oplus U^{(q)}$. Clearly, if V possesses a non-degenerate alternating form then so also does W. Furthermore, since W is irreducible and absolutely irreducible, Schur lemma implies that, up to a scalar factor, there is a unique invariant bilinear form on W. Notice that $SL(2, q^2)$ is isomorphic to $Sp(2, q^2)$, so that U possesses an invariant alternating form Φ. Also, $\Phi^{(q)}$ is an invariant alternating form on $U^{(q)}$.

Define a four-linear mapping $U \times U^{(q)} \times U \times U^{(q)} \longrightarrow GF(q^2)$ as follows

$$(u_1, u_2, u_3, u_4) \mapsto \Phi(u_1, u_3)\Phi^{(q)}(u_2, u_4).$$

According to the general theory of tensor products, this defines a bilinear form Ψ on $W = U \otimes U^{(q)}$, via

$$\Psi(u_1 \otimes u_2, u_3 \otimes u_4) = \Phi(u_1, u_3)\Phi^{(q)}(u_2, u_4).$$

Clearly, Ψ is nontrivial and invariant. It remains to see that Ψ is symmetric, rather than alternating. Indeed, if $w_1 = u_1 \otimes u_2$ and $w_2 = u_3 \otimes u_4$ then $\Psi(w_2, w_1) = \Phi(u_3, u_1)\Phi^{(q)}(u_4, u_2) = (-\Phi(u_1, u_3))(-\Phi(u_2, u_4)) = \Psi(w_1, w_2)$. Hence, Ψ is symmetric, and, since q is odd, Ψ cannot be alternating. This is the final contradiction eliminating the case $L \cong L_2(q^2)$.

(9.2) Suppose now that $L \cong Sz(q_0)$ and q is a power of q_0. Here $q_0 = 2^m$ with m odd and greater than 1. Again, the number of flags in C must divide the order of M, which, in turn, divides $|\mathrm{Aut}\, L| = q_0^2(q_0 - 1)(q_0^2 + 1)m$. We now observe that, for all the three types of Q, $q + 1$ divides the number of flags in C. Since $\gcd(q + 1, q_0^2(q_0 - 1)) = 1$ (q and q_0 are even!), we conclude that $q + 1$ divides $(q_0^2 + 1)m$. However, q is a power of q_0 ($= 2^m$, $m > 1$). Hence $q = q_0^2$ and the number of flags of C is $q_0^2(q_0^4 + 1)(q_0^2 + 1)$, $q_0^4(q_0^6 + 1)(q_0^2 + 1)$ or $q_0^6(q_0^10 + 1)(q_0^4 + 1)$, according to the type of Q. The order $q_0^2(q_0 - 1)(q_0^2 + 1)m$ of $\mathrm{Aut}\, L$ is a multiple of one of those three numbers, but this is a contradiction. Thus, $L \ncong Sz(q_0)$.

Beside these two series, we only have a few small groups left. All the ordinary and Brauer character tables for these groups are known (cf. [MOD]). The relevant characteristics are restricted by the condition that p must be a divisor of $|M|$. (Indeed, G acts transitively on the point set of C, having size $s(st+1)$; hence, s divides $|G|$ and s is a power of p.) Thus, for each group L we only need to check the Brauer character tables. Furthermore, as V is absolutely irreducible, we are not interested in the characters of degree other than 4 or 5, depending on the case. Furthermore, V cannot be realized over a proper subfield of F. (Recall that $F = GF(q)$ or $GF(q^2)$, depending on the case.) This means that F is the minimal field of realization for V, and hence, F is simply the smallest field containing the values of the Brauer character of V, taken modulo p.

(9.3) If $L \cong A_5$ then the characteristics 2 and 5 are covered by (9.1), as $A_5 \cong L_2(4) \cong L_2(5)$. In characteristic 3, L has only one eligible irreducible projective module. The module has dimension 4 and is defined over $GF(3)$. (Hence $q = 3$ and $F = GF(3)$.) This restricts the situation to the case $Q = Q(4, 3)$. However, in that case the module must possess an invariant alternating form, which is not the case.

(9.4) For $L = A_6$, we can ignore the characteristic 3, because $A_6 \cong L_2(9)$. For characteristics 2 and 5, by browsing through the Brauer character tables we find the following modules of dimension 4 and 5:

(a) Two conjugate modules of dimension 4 over $GF(2)$. Each of them can be viewed as the natural module for $A_6 \cong Sp_4(2)'$. Correspondingly, $Q = Q(4, 2)$; however, M is then equal to A, a contradiction.

(b) Two conjugate 4-dimensional modules over $GF(5)$. Since the field is of prime order, this can only correspond to $Q = Q(4, 5)$. By the condition $(*)$, the order of M must be divisible by $5^2 + 1 = 2 \cdot 13$. This is impossible.

(c) Two conjugate 5-dimensional modules over $GF(5)$. Since the field is, again, of prime order, this module cannot be related to any of our quadrangles.

(9.5) For $L = A_7$ we find the following modules:

(a) Two conjugate 4-dimensional modules over $GF(2)$. Impossible, because Q has to be $Q(4, 2)$ whereas A_7 is not a subgroup of $Sp_4(2)$.

(b) Two conjugate modules of dimension 4 over $GF(9)$. Here Q cannot be $Q(4, q)$ because the Schur-Frobenius indicator of the character shows that there is no invariant alternating form. Therefore, we must have $Q = H(3, 3^2)$, and, indeed, A_7 is a maximal subgroup of $U_4(3)$. However, the $(*)$ test fails, as M cannot be transitive on lines. (Indeed, the stabilizer of a line would have to be a subgroup of order $3^2 \cdot 5$, and there is no such subgroup in A_7.)

(c) Two conjugate modules of dimension 4 over $GF(25)$. Again $Q = H(3, 5^2)$ is the only possibility. However, the order of $\mathrm{Aut}\, L$ is not divisible by the number of flags, $5^2(5^3 + 1)(5 + 1)$.

(d) A 4-dimensional module over $GF(7)$. This may only correspond to $Q = Q(4, q)$. However, in that case, $q^2 + 1 = 50$ should divide $|M|$, and this is not the case.

(e) A 5-dimensional module over $GF(7)$. This case does not correspond to any type of Q.

(9.6) For $L = L_2(7)$ we only need consider the characteristics 2 and 3. From [MOD] we see that L has modules of dimension 4 or 5 only if the characteristic is equal to 3. Namely, it has two modules of dimension 4, conjugate by the outer automorphism of L. Suppose V is one of these modules. Then, first of all, we observe from the character that $F = GF(9)$. Furthermore, after checking the Schur-Frobenius indicator, we find that there is no invariant alternating form on V. This forces $Q = H(3, 3^2)$. However, in that case, the number of points in C is equal to $3^2(3^3 + 1)$. Since 3^2 does not divide $|\mathrm{Aut}\, L|$, M cannot be transitive on the points of C.

(9.7) The group $L = L_2(11)$ has (outside the characteristic 11) no irreducible projective modules of dimension 4, and every irreducible module of dimension 5 arises by reduction modulo p of a unique (up to automorphisms) complex representation of dimension 5.

Since the dimension of the module is 5, we must have $Q = H(4, q^2)$. In particular,

$F = GF(q^2)$. In characteristics $p = 3$ or 5, all the values of the complex character reduced modulo p are in $GF(p)$. Hence, the only possible characteristic is $p = 2$.

In characteristic 2, $GF(4)$ is the smallest field over which the module can be realized. This implies that $q = 2$. Furthermore, the number of flags in Q is equal to $q^3(q^5+1)(q^2+1) = 1320$, which is exactly the order of $\operatorname{Aut} L \cong PGL(2,11)$. Therefore, $M \cong \operatorname{Aut} L$. On the other hand, the oval O consists in this case of q^5+1 points. Since M has to be transitive of the points of O, the stabilizer in M of a point $a \in O$ is a subgroup of order $2^3 \cdot 5$. It is easy to see that $PGL(2,11)$ has no such subgroup; a contradiction.

(9.8) For $L = L_3(4)$ there is only one configuration to consider: V can be one of two conjugate modules of dimension 4 defined over $GF(9)$. By checking the Schur-Frobenius indicator of the character of V, we establish that there is no invariant quadratic form on V. This eliminates the possibility of $Q = Q(4,9)$. Thus, Q must be $H(3,3^2)$. Upon checking [ATL], we see that, indeed, $L = L_3(4)$ is a subgroup of $U_4(3)$. In fact, the normalizer of L is flag-transitive on Q. (See [S]; also [M].) This means that G must be a proper subgroup of M. Since G has to be transitive on the flags of C, the order of G is divisible by $3^2(3^3+1)(3+1)$. By checking in [ATL] all the maximal subgroups of $L_3(4)$ (with possible outer automorphisms) we see that no subgroup satisfies this condition.

(9.9) For the group $L \cong U_4(2)$ we have to consider the following candidates for V:

(a) In characteristic 5, there is a pair of conjugated modules of dimension 4, and a pair of conjugated modules of dimension 5. For all these modules $F = GF(25)$. Furthermore, for the modules of dimension 4, the Schur-Frobenius indicator is equal 0, which means that there is no invariant alternating form. Therefore, $Q = H(3,5^2)$ for the modules of dimension 4, and $H(4,5^2)$ for the modules of dimension 5. In the first case, the number of flags in C is $5^2(5^3+1)(5+1)$; in the second case, it is $5^3(5^5+1)(5^2+1)$. Since $5^2+1 = 26$ does not divide the order of $\operatorname{Aut} L$, both cases lead to a contradiction.

(b) In characteristic 3, L has one module of dimension 4 and one of dimension 5. Both these modules are defined over $GF(5)$. Hence $Q = Q(4,3)$ and V is 4-dimensional. However, as $U_4(2) \cong S_4(3)$, the normalizer of L is the whole of A, rather than a maximal subgroup thereof.

(c) Similarly, in characteristic 2, L has a pair of conjugate modules of dimension 4, defined over $GF(4)$. By checking the Schur-Frobenius indicator, we exclude the case $Q = Q(4,4)$, leaving $Q = H(3,2^2)$ as the only possibility. However, again $L = U_4(2)$ is equal to A_0 and, hence, M is not a maximal subgroup.

This was the last case, and the proof of Theorem 1 is now complete. ∎

5 Proof of Theorem 2

In this section we prove Theorem 2. Let Q be a classical generalized quadrangle, H a geometric hyperplane as in the conclusion of Theorem 1, and $C = Q \setminus H$. Suppose $F \leq G = \mathrm{Aut}\, C$ has the property that, for any point $c \in C$, F_c induces at least $PGL(2,t)$ on the lines through c. This immediately excludes $Q \cong H(4,q^2)$, as in that case the action on $(t+1 = q^3+1)$ lines involves $U_3(q)$, which does not contain $L_2(q^3)$. In all the other cases the action on the star of a point involves $L_2(t)$ as a normal subgroup.

Let T be the set of $t+1$ points of H which are collinear with c. By assumption F_c induces on these points a group containing $PGL(2,t)$. We claim that $F_c^\infty = G_c^\infty \cong (S)L_2(t)$. For that we only need to notice that the kernel K of the action of G_c on T centralizes G_c^∞. (This follows from a case-by-case check.) Since it is also clear that $F_c^\infty K = G_c^\infty K$, it follows that $G_c^\infty = F_c^\infty$. Therefore, F contains the subgroup $\langle G_c^\infty \mid c \in C\rangle$, which is normal in G. It is now straightforward to check that in all cases G has at most one nonabelian composition factor. This implies that G^∞ is contained in every normal nonsolvable subgroup of G. In particular, if $t \geq 4$ then G^∞ is contained in F.

It remains to look at the possibilities $t = 2$ and 3. Clearly, if $G^\infty = 1$ then there is nothing to prove. The remaining cases are few: $Q = Q(4,2)$ and H is the elliptic quadric, $Q = Q(4,3)$ and H is again the elliptic quadric, and $Q = H(3,9)$ and H is the Hermitian unital. In all these cases, by looking through the list of subgroups of G one can see that any subgroup, whose order is divisible by the number of flags in C (equal to $s(st+1)(t+1)$ in all these cases), necessarily contains G^∞. This completes the proof of Theorem 2. ∎

References

[A] M. Aschbacher, *On the maximal subgroups of finite classical groups,* Invent. Math., **76** (1984), 469–514.

[BP] N. Batens and A. Pasini, *On certain hyperplanes and projective embeddings of D_4-buildings,* International Journal of Mathematics, Game Theory and Algebra, **8** (1998), 137-166.

[B] N. Bourbaki, *Algèbre*, Hermann, Paris, 1959.

[BL] F. Buekenhout and C. Lefèvre, *Generalized quadrangles in projective spaces,* Arch. Math., **25** (1974), 540–552.

[CS] A. Cohen and E. Shult, *Affine polar spaces,* Geom. Dedicata, **35** (1990), 43–76.

[ATL] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups,* Clarendon Press, Oxford, 1985.

[C] H. Cuypers, *Affine Grassmannians,* J. Combin. Theory, Series A, **70** (1995), 289–304.

[GLS] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups, Number* 3, AMS, 1998.

[IS] A.A. Ivanov and S. Shpectorov, *The flag-transitive tilde and Petersen-type geometries are all known,* Bull. A.M.S. **31** (1994), 173–184.

[MOD] Ch. Jansen, K. Lux, R. Parker and R. Wilson, *An Atlas of Brauer Characters,* Clarendon Press, Oxford, 1995.

[K] P. Kleidman, *The Subgroup Structure of Some Finite Simple Groups,* Ph. D. Thesis, Trinity College, Cambridge 1986.

[KL] P. Kleidman and M. Liebeck, *Subgroup Structure of the Finite Classical Groups,* London Math. Soc. Lect. Notes **129**, 1990.

[M] Th. Meixner, *Klassische Tits Kammersysteme mit einer transitiven Automorphismengruppe,* Mitt. Math. Sem Giessen **174**, 1986.

[P] A. Pasini, *Diagram Geometries,* Oxford Univ. Press, Oxford, 1994.

[PTs] A. Pasini and S. Tsaranov, *Flag-transitive geometries between A_3 and $\widetilde{A}_2$,* Simon Stevin **67** (1993), 107–120.

[PT] S.E. Payne and J.A. Thas, *Finite Generalized Quadrangles,* Pitman, Boston, 1984.

[S] G. Seitz, *Flag-transitive subgroups of Chevalley groups,* Ann. Math., **97** (1973), 27–56.

Antonio Pasini,
University of Siena,
Department of Mathematics,
Via del capitano 15,
I-53100 Siena, Italy.

Sergey Shpectorov,
Bowling Green State University,
Department of Mathematics and Statistics,
Bowling Green, OH 43403, U.S.A.